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A generalized F-expansion method with symbolic computation exactly solving Broer–Kaup equations

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Abstract

In this paper, a generalized F-expansion method is applied to construct exact solutions of the $(2+1)$-dimensional Broer–Kaup equations. As a result, many general exact non-travelling wave and coefficient function solutions are obtained including single and combined non-degenerate Jacobi elliptic function solutions, soliton-like solutions and trigonometric function solutions, each of which contains two arbitrary functions. Compared with most existing F-expansion methods, the proposed method gives new and more general exact solutions. More importantly, with the aid of symbolic computation, the method provides a powerful mathematical tool to solve a large many nonlinear partial differential equations.

Keywords: Generalized F-expansion method; Jacobi elliptic function solutions; Soliton-like solutions; Trigonometric function solutions

1. Introduction

In the past several decades, many significant methods for solving nonlinear partial differential equations (NLPDEs) have been presented, such as inverse scattering method [1], Hirota's bilinear method [2], Bäcklund transformation [3], Painlevè expansion [4], sine–cosine method [5], Adomian Pade approximation [6], homogenous balance method [7], homotopy perturbation method [8–10], variational method [11–14], asymptotic methods [15], non-perturbative methods [16], exp-function method [17], algebraic method [18–21], Weierstrass semi-rational expansion method [22], unified rational expansion method [23] and so on.

Recently, F-expansion method [24–26] was proposed to construct periodic wave solutions of NLPDEs, which can be thought of as an over-all generalization of Jacobi elliptic function expansion method [27–29]. F-expansion method was later extended in different manners [30–36]. Very recently, Zhang and Xia [37] proposed a generalized F-expansion method to construct more general exact solutions which contain not only the results obtained by using the method [24–26,32,36] but also a series of new and more general exact solutions, in which the restriction on $\omega(x_1, x_2, \ldots, t)$ as merely a linear function of $x_1, x_2, \ldots, t$ and the restriction on

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the coefficients being constants are removed. With the aid of symbolic computation, the method provides a powerful mathematical tool to solve a large many NLPDEs.

In this paper, we apply the generalized F-expansion method [37] to construct exact solutions of the $(2+1)$-dimensional Broer–Kaup (BK) equations. As a result, many new and more exact solutions with two arbitrary functions are obtained, which include single and combined non-degenerate Jacobi elliptic function solutions, soliton-like solutions and trigonometric function solutions.

The rest of this paper is organized as follows: in Section 2, we give the description of the generalized F-expansion method; in Section 3, we apply this method to the $(2+1)$-dimensional BK equations; in Section 4, some conclusions are given.

2. The generalized F-expansion method

In this section, we give the detailed description of our method.

For a given NLPDE with independent variables $x = (t, x_1, x_2, \ldots, x_m)$ and dependent variable u:

$$F(u, u_t, u_{x_1}, u_{x_2}, \ldots, u_{x_m}, u_{x_1 t}, u_{x_2 t}, \ldots, u_{x_m t}, u_{tt}, u_{x_1 x_1}, u_{x_2 x_2}, \ldots, u_{x_m x_m}, \ldots) = 0, \tag{1}$$

we seek its solutions in the more general form:

$$u = a_0 + \sum_{i=1}^{n} \left\{ a_i F^i(\omega) + b_i F^{-i}(\omega) + c_i F^{i-1}(\omega) F'(\omega) + d_i F^{-i}(\omega) F'(\omega) \right\}, \tag{2}$$

where $a_0 = a_0(x)$, $a_i = a_i(x)$, $b_i = b_i(x)$, $c_i = c_i(x)$, $d_i = d_i(x)$ $(i = 1, 2, \ldots, n)$ and $\omega = \omega(x)$ are all functions to be determined, $F(\omega)$ and $F'(\omega)$ in (2) satisfy

$$F'^2(\omega) = PF^4(\omega) + QF^2(\omega) + R, \tag{3}$$

and hence holds for $F(\omega)$ and $F'(\omega)$

$$\begin{cases} F''(\omega) = 2PF^3(\omega) + QF(\omega), \\ F^{(3)}(\omega) = \left(6PF^2(\omega) + Q\right)F'(\omega), \\ F^{(4)}(\omega) = 24P^2F^5(\omega) + 20PQF^3(\omega) + (Q^2 + 12PR)F(\omega), \\ \ldots \end{cases} \tag{4}$$

where P, Q and R are all parameters, the prime denotes $\mathrm{d}/\mathrm{d}\omega$. Given different values of P, Q and R, the different Jacobi elliptic function solutions $F(\omega)$ can be obtained from Eq. (3) (see Appendix A). To determine u explicitly, we take the following four steps:

Step 1. Determine the integer n by balancing the highest order nonlinear term(s) and the highest order partial derivative of u in Eq. (1).

Step 2. Substitute (2) along with (3) and (4) into Eq. (1) and collect coefficients of $F'^i(\omega)F^j(\omega)$ $(i = 0, 1; j = 0, \pm 1, \pm 2, \ldots)$, then set each coefficient to zero to derive a set of over-determined partial differential equations for a_0, a_i, b_i, c_i, d_i $(i = 1, 2, \ldots, n)$ and ω.

Step 3. Solve the system of over-determined partial differential equations obtained in Step 2 for a_0, a_i, b_i, c_i, d_i and ω by use of *Mathematica*.

Step 4. Select P, Q, R and $F(\omega)$ from Appendix A and substitute them along with a_0, a_i, b_i, c_i, d_i and ω into (2) to obtain Jacobi elliptic function solutions of Eq. (1) (see Appendix B for $F'(\omega)$), from which hyperbolic function solutions and trigonometric function solutions can be obtained in the limit cases when $m \to 1$ and $m \to 0$ (see Appendix C).

3. Exact solutions of the (2 + 1)-dimensional BK equations

Let us consider the well known $(2+1)$-dimensional BK equations:

$$H_{yt} + 2G_{xx} + 2(HH_x)_y - H_{xxy} = 0, \tag{5}$$

$$G_t + 2(HG)_x + G_{xx} = 0, \tag{6}$$

which can be obtained from the Darbox transformation related symmetry constrains of the Kasomtsev–Petviashvili (KP) equation [38], play an important role in many scientific fields such as statistics physics, plasma, nonlinear optical fiber, etc. Recently, the Painlevé properties, infinitely many symmetries, the localized coherent structures, non-travelling wave solutions and other types of solutions can be found in Refs. [39–50].

According to Step 1, we can get $n = 1$ for H and $n = 2$ for G. In order to search for explicit solutions, we suppose that Eqs. (5) and (6) have the formal solutions:

$$H = a_0 + a_1F(\omega) + b_1F^{-1}(\omega) + c_1F'(\omega) + d_1F'(\omega)F^{-1}(\omega), \tag{7}$$

$$\begin{aligned} G = {} & A_0 + A_1F(\omega) + A_2F^2(\omega) + B_1F^{-1}(\omega) + B_2F^{-2}(\omega) + C_1F'(\omega) \\ & + C_2F'(\omega)F(\omega) + D_1F'(\omega)F^{-1}(\omega) + D_2F'(\omega)F^{-2}(\omega), \end{aligned} \tag{8}$$

where $a_0 = a_0(y,t)$, $a_1 = a_1(y,t)$, $b_1 = b_1(y,t)$, $c_1 = c_1(y,t)$, $d_1 = d_1(y,t)$, $A_0 = A_0(y,t)$, $A_1 = A_1(y,t)$, $A_2 = A_2(y,t)$, $B_1 = B_1(y,t)$, $B_2 = B_2(y,t)$, $C_1 = C_1(y,t)$, $C_2 = C_2(y,t)$, $D_1 = D_1(y,t)$, $D_2 = D_2(y,t)$, $\eta = \eta(y,t)$, $\omega = kx + \eta$, k is a non-zero constant.

With the aid of *Mathematica*, substituting (7) and (8) along with Eqs. (3) and (4) into Eqs. (5) and (6), the left-hand sides of Eqs. (5) and (6) are converted into two polynomials of $F'^i(\omega)F^j(\omega)$ $(i = 0, 1; j = 0, \pm1, \pm2, \cdots)$, then setting each coefficient to zero, we get a set of over-determined partial differential equations for a_0, a_1, b_1, c_1, d_1, A_0, A_1, A_2, B_1, B_2, C_1, C_2, D_1, D_2 and η. Solving these over-determined partial differential equations by use of *Mathematica*, we get the following results:

Case 1

$$a_0 = -\frac{1}{2k}f_2'(t), \quad a_1 = \pm k\sqrt{P}, \quad b_1 = \pm k\sqrt{R}, \quad A_0 = \frac{1}{4}(-kQ \pm 2k\sqrt{PR})f_1(y), \tag{9}$$

$$A_2 = -\frac{1}{2}kPf_1(y), \quad B_2 = -\frac{1}{2}kRf_1(y), \quad C_1 = \pm\frac{1}{2}k\sqrt{P}f_1(y), \quad D_2 = \mp\frac{1}{2}k\sqrt{R}f_1(y), \tag{10}$$

$$c_1 = d_1 = A_1 = B_1 = C_2 = D_1 = 0, \quad \eta = \int f_1(y)\,\mathrm{d}y + f_2(t), \tag{11}$$

where $f_1(y)$, $f_2(t)$ are arbitrary functions of y and t respectively, $f_2'(t) = \mathrm{d}f_2(t)/\mathrm{d}t$. The signs "$\pm$" and "$\mp$" in (9) and (10) mean that the same sign must be used in a_1 and C_1, and different signs must be used in b_1 and D_2. If "+" is used in b_1, then the same sign must be used in a_1 and A_0. If "−" is used in b_1, then different signs must be used in a_1 and A_0. Hereafter, the signs "$\pm$" and "$\mp$" always stand for this meaning in the similar circumstances.

Case 2

$$a_0 = -\frac{1}{2k}f_2'(t), \quad a_1 = \pm\frac{1}{2}k\sqrt{P}, \quad b_1 = \pm\frac{1}{2}k\sqrt{R}, \quad d_1 = \frac{1}{2}k, \tag{12}$$

$$A_0 = \pm\frac{1}{2}k\sqrt{PR}f_1(y), \quad B_2 = -\frac{1}{2}kRf_1(y), \quad D_2 = \mp\frac{1}{2}k\sqrt{R}f_1(y), \tag{13}$$

$$c_1 = A_1 = A_2 = B_1 = C_1 = C_2 = D_1 = 0, \quad \eta = \int f_1(y)\,\mathrm{d}y + f_2(t), \tag{14}$$

where $f_1(y)$, $f_2(t)$ are arbitrary functions of y and t respectively, $f_2'(t) = \mathrm{d}f_2(t)/\mathrm{d}t$. The signs "$\pm$" and "$\mp$" in (12) and (13) mean that different signs must be used in b_1 and D_2. If "+" is used in b_1, then the same sign mused be used in a_1 and A_0. If "−" is used in b_1, then different signs must be used in a_1 and A_0. Hereafter, the signs "$\pm$" and "$\mp$" always stand for this meaning in the similar circumstances.

Case 3

$$a_0 = -\frac{1}{2k}f_2'(t), \quad a_1 = \pm\frac{1}{2}k\sqrt{P}, \quad b_1 = \pm\frac{1}{2}k\sqrt{R}, \quad d_1 = -\frac{1}{2}k, \tag{15}$$

$$A_0 = \pm\frac{1}{2}k\sqrt{PR}f_1(y), \quad A_2 = -\frac{1}{2}kPf_1(y), \quad C_1 = \pm\frac{1}{2}k\sqrt{P}f_1(y), \tag{16}$$

$$c_1 = A_1 = B_1 = B_2 = C_2 = D_1 = D_2 = 0, \quad \eta = \int f_1(y)\,\mathrm{d}y + f_2(t), \tag{17}$$

where $f_1(y)$, $f_2(t)$ are arbitrary functions of y and t respectively, $f_2'(t) = \mathrm{d}f_2(t)/\mathrm{d}t$. The sign "±" in (15) and (16) means that the same sign must be used in a_1 and C_1. If "+" is used in b_1, then the same sign must be used in a_1 and A_0. If "−" is used in b_1, then different signs must be used in a_1 and A_0. Hereafter, the signs "±" always stands for this meaning in the similar circumstances.

Case 4

$$a_0 = -\frac{1}{2k}f_2'(t), \quad b_1 = \pm k\sqrt{R}, \quad A_0 = -\frac{1}{4}kQf_1(y), \quad B_2 = -\frac{1}{2}kRf_1(y), \quad D_2 = \mp\frac{1}{2}k\sqrt{R}f_1(y), \tag{18}$$

$$a_1 = c_1 = d_1 = A_1 = A_2 = B_1 = C_1 = C_2 = D_1 = 0, \quad \eta = \int f_1(y)\,\mathrm{d}y + f_2(t), \tag{19}$$

where $f_1(y)$, $f_2(t)$ are arbitrary functions of y and t respectively, $f_2'(t) = \mathrm{d}f_2(t)/\mathrm{d}t$.

Case 5

$$a_0 = -\frac{1}{2k}f_2'(t), \quad d_1 = k, \quad B_2 = -kRf_1(y), \quad \eta = \int f_1(y)\,\mathrm{d}y + f_2(t), \tag{20}$$

$$a_1 = b_1 = c_1 = A_0 = A_1 = A_2 = B_1 = C_1 = C_2 = D_1 = D_2 = 0, \tag{21}$$

where $f_1(y)$, $f_2(t)$ are arbitrary functions of y and t respectively, $f_2'(t) = \mathrm{d}f_2(t)/\mathrm{d}t$.

Case 6

$$a_0 = -\frac{1}{2k}f_2'(t), \quad d_1 = -k, \quad A_2 = -kPf_1(y), \quad \eta = \int f_1(y)\,\mathrm{d}y + f_2(t), \tag{22}$$

$$a_1 = b_1 = c_1 = A_0 = A_1 = B_1 = B_2 = C_1 = C_2 = D_1 = D_2 = 0, \tag{23}$$

where $f_1(y)$, $f_2(t)$ are arbitrary functions of y and t respectively, $f_2'(t) = \mathrm{d}f_2(t)/\mathrm{d}t$.

Substituting Cases 1–6 into (7) and (8) respectively, we have six kinds of formal solutions of Eqs. (5) and (6):

$$H = -\frac{1}{2k}f_2'(t) \pm k\sqrt{P}F(\omega) \pm k\sqrt{R}F^{-1}(\omega), \tag{24}$$

$$G = \frac{1}{4}(-kQ \pm 2k\sqrt{PR})f_1(y) - \frac{1}{2}kPf_1(y)F^2(\omega) - \frac{1}{2}kRf_1(y)F^{-2}(\omega) \pm \frac{1}{2}k\sqrt{P}f_1(y)F'(\omega) \mp \frac{1}{2}k\sqrt{R}f_1(y)F'(\omega)F^{-2}(\omega), \tag{25}$$

where $\omega = kx + \int f_1(y)\,\mathrm{d}y + f_2(t)$.

$$H = -\frac{1}{2k}f_2'(t) \pm \frac{1}{2}k\sqrt{P}F(\omega) \pm \frac{1}{2}k\sqrt{R}F^{-1}(\omega) + \frac{1}{2}kF'(\omega)F^{-1}(\omega), \tag{26}$$

$$G = \pm\frac{1}{2}k\sqrt{PR}f_1(y) - \frac{1}{2}kRf_1(y)F^{-2}(\omega) \mp \frac{1}{2}k\sqrt{R}f_1(y)F'(\omega)F^{-2}(\omega), \tag{27}$$

where $\omega = kx + \int f_1(y)\,\mathrm{d}y + f_2(t)$.

$$H = -\frac{1}{2k}f_2'(t) \pm \frac{1}{2}k\sqrt{P}F(\omega) \pm \frac{1}{2}k\sqrt{R}F^{-1}(\omega) - \frac{1}{2}kF'(\omega)F^{-1}(\omega), \tag{28}$$

$$G = \pm\frac{1}{2}k\sqrt{PR}f_1(y) - \frac{1}{2}kPf_1(y)F^2(\omega) \pm \frac{1}{2}k\sqrt{P}f_1(y)F'(\omega), \tag{29}$$

where $\omega = kx + \int f_1(y)\,\mathrm{d}y + f_2(t)$.

$$H = -\frac{1}{2k}f_2'(t) \pm k\sqrt{R}F^{-1}(\omega), \tag{30}$$

$$G = -\frac{1}{4}kQf_1(y) - \frac{1}{2}kRf_1(y)F^{-2}(\omega) \mp \frac{1}{2}k\sqrt{R}f_1(y)F'(\omega)F^{-2}(\omega), \tag{31}$$

where $\omega = kx + \int f_1(y)\,\mathrm{d}y + f_2(t)$.

$$H = -\frac{1}{2k}f_2'(t) + kF'(\omega)F^{-1}(\omega), \tag{32}$$

$$G = -kRf_1(y)F^{-2}(\omega), \tag{33}$$

where $\omega = kx + \int f_1(y)\,\mathrm{d}y + f_2(t)$.

$$H = -\frac{1}{2k}f_2'(t) - kF'(\omega)F^{-1}(\omega), \tag{34}$$

$$G = -kPf_1(y)F^{2}(\omega), \tag{35}$$

where $\omega = kx + \int f_1(y)\,\mathrm{d}y + f_2(t)$.

With the aid of Appendices A, B and C, from (24)–(35), we can obtain many exact solutions of Eqs. (5) and (6).

For example, from Appendix A, choosing $F(\omega) = \mathrm{ns}\,\omega$, $P = 1$, $Q = -(1 + m^2)$, $R = m^2$, inserting them into (24) and (25), and using Appendix B, we obtain combined non-degenerate Jacobi elliptic function solutions of Eqs. (5) and (6):

$$H = -\frac{1}{2k}f_2'(t) \pm k\mathrm{ns}\,\omega \pm km\mathrm{sn}\,\omega, \tag{36}$$

$$G = \frac{1}{4}(k(1+m^2) \pm 2km)f_1(y) - \frac{1}{2}kf_1(y)\mathrm{ns}^2\,\omega - \frac{1}{2}km^2f_1(y)\mathrm{sn}^2\,\omega \mp \frac{1}{2}kf_1(y)\,\mathrm{cs}\,\omega\mathrm{ds}\,\omega \pm \frac{1}{2}kmf_1(y)\,\mathrm{cn}\,\omega\,\mathrm{dn}\,\omega, \tag{37}$$

where $\omega = kx + \int f_1(y)\,\mathrm{d}y + f_2(t)$. In the limit case when $m \to 1$, from (36) and (37) we get soliton-like solutions of Eqs. (5) and (6):

$$H = -\frac{1}{2k}f_2'(t) \pm k\coth\omega \pm k\tanh\omega, \tag{38}$$

$$G = \frac{1}{4}(2k \pm 2k)f_1(y) - \frac{1}{2}kf_1(y)\coth^2\omega - \frac{1}{2}kf_1(y)\tanh^2\omega \mp \frac{1}{2}kf_1(y)\,\mathrm{csch}^2\,\omega \pm \frac{1}{2}kf_1(y)\,\mathrm{sech}^2\,\omega, \tag{39}$$

where $\omega = kx + \int f_1(y)\,\mathrm{d}y + f_2(t)$.

When $m \to 0$, from (36) and (37) we get trigonometric function solutions of Eqs. (5) and (6):

$$H = -\frac{1}{2k}f_2'(t) \pm k\csc\omega, \tag{40}$$

$$G = \frac{1}{4}kf_1(y) - \frac{1}{2}kf_1(y)\csc^2\omega \pm \frac{1}{2}kf_1(y)\cot\omega\csc\omega, \tag{41}$$

where $\omega = kx + \int f_1(y)\,\mathrm{d}y + f_2(t)$.

Choosing again $F(\omega) = \mathrm{ns}\,\omega \pm \mathrm{cs}\,\omega$, $P = 1/4$, $Q = (1 - 2m^2)/2$, $R = 1/4$, inserting them into (24) and (25), and using Appendix B, we obtain combined non-degenerate Jacobi elliptic function solutions of Eqs. (5) and (6):

$$H = -\frac{1}{2}f_2'(t) \pm \frac{1}{2}k\,(\mathrm{ns}\,\omega \pm \mathrm{cs}\,\omega) \pm \frac{1}{2}k\frac{1}{\mathrm{ns}\,\omega \pm \mathrm{cs}\,\omega}, \tag{42}$$

$$\begin{aligned} G = {} & \frac{1}{8}(-k(1-2m^2) \pm k)f_1(y) - \frac{1}{8}kf_1(y)\,(\mathrm{ns}\,\omega \pm \mathrm{cs}\,\omega)^2 - \frac{1}{8}kf_1(y)\frac{1}{(\mathrm{ns}\,\omega \pm \mathrm{cs}\,\omega)^2} \\ & \mp \frac{1}{4}kf_1(y)\,(\mathrm{cs}\,\omega\,\mathrm{ds}\,\omega \pm \mathrm{ns}\,\omega\,\mathrm{ds}\,\omega) \mp \frac{1}{4}kf_1(y)\frac{\mp\mathrm{ds}\,\omega}{\mathrm{ns}\,\omega \pm \mathrm{cs}\,\omega}, \end{aligned} \tag{43}$$

where $\xi = kx + \int f_1(y)\,\mathrm{d}y + f_2(t)$.

In the limit case when $m \to 1$, from (42) and (43) we get soliton-like solutions of Eqs. (5) and (6):

$$H = -\frac{1}{2}f_2'(t) \pm \frac{1}{2}k(\coth\omega \pm \mathrm{csch}\,\omega) \pm \frac{1}{2}k\frac{1}{\coth\,\omega \pm \mathrm{csch}\,\omega}, \tag{44}$$

$$\begin{aligned} G = {} & \frac{1}{8}(k \pm k)f_1(y) - \frac{1}{8}kf_1(y)(\coth\omega \pm \mathrm{csch}\,\omega)^2 - \frac{1}{8}kf_1(y)\frac{1}{(\coth\omega \pm \mathrm{csch}\,\omega)^2} \\ & \mp \frac{1}{4}kf_1(y)(\mathrm{csch}^2\,\omega \pm \coth\omega\mathrm{csch}\,\omega) \mp \frac{1}{4}kf_1(y)\frac{\mp\mathrm{csch}\,\omega}{\coth\,\omega \pm \mathrm{csch}\,\omega}, \end{aligned} \tag{45}$$

where $\xi = kx + \int f_1(y)\,\mathrm{d}y + f_2(t)$.

When $m \to 0$, from (42) and (43) we get trigonometric function solutions of Eqs. (5) and (6):

$$H = -\frac{1}{2}f_2'(t) \pm \frac{1}{2}k(\csc\omega \pm \cot\omega) \pm \frac{1}{2}k\frac{1}{\csc\omega \pm \cot\omega}, \tag{46}$$

$$G = \frac{1}{8}(-k \pm k)f_1(y) - \frac{1}{8}kf_1(y)(\csc\omega \pm \cot\omega)^2 - \frac{1}{8}kf_1(y)\frac{1}{(\csc\omega \pm \cot\omega)^2} \mp \frac{1}{4}kf_1(y)(\cot\omega\csc\omega \pm \csc^2\omega) \mp \frac{1}{4}kf_1(y)\frac{\mp\csc\omega}{\csc\omega \pm \cot\omega}, \tag{47}$$

where $\xi = kx + \int f_1(y)\,\mathrm{d}y + f_2(t)$.

With the aid of Appendices A, B and C, from (24)–(35) we can also obtain other Jacobi elliptic function solutions, soliton-like solutions and trigonometric function solutions of Eqs. (5) and (6), here we omit them for simplify.

Remark 1. All the solutions obtained in this paper cannot be obtained by the existing F-expansion methods [24–26,30–36], which are new and have been not reported yet.

4. Conclusion

A generalized F-expansion method has been applied to construct exact solutions of the $(2+1)$-dimensional BK equations. With the aid of *Mathematica*, we have obtained many new and more general exact non-travelling wave and coefficient function solutions including single and combined non-degenerate Jacobi elliptic function solutions, soliton-like solutions, trigonometric function solutions. All the solutions presented in this paper have been checked with *Mathematica* with respect to the original Eqs. (5) and (6). Compared with the existing F-expansion methods [24–26,32,36], the generalized F-expansion method is more powerful in searching for exact solutions of NLPDEs. The arbitrary functions in the obtained solutions imply that these solutions have rich local structures. It may be important to explain some physical phenomena.

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Appendix A

Relations between values of (P, Q, R) and corresponding $F(\omega)$ in Eq. (3)

P	Q	R	$F(\omega)$
m^2	$-(1+m^2)$	1	$\mathrm{sn}\,\omega, \mathrm{cd}\,\omega = \frac{\mathrm{cn}\,\omega}{\mathrm{dn}\,\omega}$
$-m^2$	$2m^2-1$	$1-m^2$	$\mathrm{cn}\,\omega$
-1	$2-m^2$	m^2-1	$\mathrm{dn}\,\omega$
1	$-(1+m^2)$	m^2	$\mathrm{ns}\,\omega = (\mathrm{sn}\,\omega)^{-1}, \mathrm{dc}\,\omega = \frac{\mathrm{dn}\,\omega}{\mathrm{cn}\,\omega}$
$1-m^2$	$2m^2-1$	$-m^2$	$\mathrm{nc}\,\omega = (\mathrm{cn}\,\omega)^{-1}$
m^2-1	$2-m^2$	-1	$\mathrm{nd}\,\omega = (\mathrm{dn}\,\omega)^{-1}$
$1-m^2$	$2-m^2$	1	$\mathrm{sc}\,\omega = \frac{\mathrm{sn}\,\omega}{\mathrm{cn}\,\omega}$
$-m^2(1-m^2)$	$2m^2-1$	1	$\mathrm{sd}\,\omega = \frac{\mathrm{sn}\,\omega}{\mathrm{dn}\,\omega}$
1	$2-m^2$	$1-m^2$	$\mathrm{cs}\,\omega = \frac{\mathrm{cn}\,\omega}{\mathrm{sn}\,\omega}$
1	$2m^2-1$	$-m^2(1-m^2)$	$\mathrm{ds}\,\omega = \frac{\mathrm{dn}\,\omega}{\mathrm{sn}\,\omega}$
$\frac{1}{4}$	$\frac{1-2m^2}{2}$	$\frac{1}{4}$	$\mathrm{ns}\,\omega \pm \mathrm{cs}\,\omega$
$\frac{1-m^2}{4}$	$\frac{1+m^2}{2}$	$\frac{1-m^2}{4}$	$\mathrm{nc}\,\omega \pm \mathrm{sc}\,\omega$
$\frac{1}{4}$	$\frac{m^2-2}{2}$	$\frac{m^2}{4}$	$\mathrm{ns}\,\omega \pm \mathrm{ds}\,\omega$
$\frac{m^2}{4}$	$\frac{m^2-2}{2}$	$\frac{m^2}{4}$	$\mathrm{sn}\,\omega \pm \mathrm{icn}\,\omega$

Appendix B

Derivatives of the Jacobi elliptic functions

$$\mathrm{sn}'\omega = \mathrm{cn}\,\omega\mathrm{dn}\,\omega,\quad \mathrm{cd}'\omega = -(1-m^2)\mathrm{sd}\,\omega\mathrm{nd}\,\omega,\quad \mathrm{cn}'\omega = -\mathrm{sn}\,\omega\mathrm{dn}\,\omega,\quad \mathrm{dn}'\omega = -m^2\mathrm{sn}\,\omega\mathrm{cn}\,\omega,$$
$$\mathrm{ns}'\omega = -\mathrm{cs}\,\omega\mathrm{ds}\,\omega,\quad \mathrm{dc}'\omega = (1-m^2)\mathrm{nc}\,\omega\mathrm{sc}\,\omega,\quad \mathrm{nc}'\omega = \mathrm{sc}\,\omega\mathrm{dc}\,\omega,\quad \mathrm{nd}'\omega = m^2\mathrm{cd}\,\omega\mathrm{sd}\,\omega,$$
$$\mathrm{sc}'\omega = \mathrm{dc}\,\omega\mathrm{nc}\,\omega,\quad \mathrm{cs}'\omega = -\mathrm{ns}\,\omega\mathrm{ds}\,\omega,\quad \mathrm{ds}'\omega = -\mathrm{cs}\,\omega\mathrm{ns}\,\omega,\quad \mathrm{sd}'\omega = \mathrm{nd}\,\omega\mathrm{cd}\,\omega.$$

Appendix C

The Jacobi elliptic functions degenerate into hyperbolic functions when $m \to 1$:

$$\mathrm{sn}\,\omega \to \tanh\omega,\quad \mathrm{cn}\,\omega \to \mathrm{sech}\,\omega,\quad \mathrm{dn}\,\omega \to \mathrm{sech}\,\omega,\quad \mathrm{sc}\,\omega \to \sinh\omega,\quad \mathrm{sd}\,\omega \to \sinh\omega,\quad \mathrm{cd}\,\omega \to 1,$$
$$\mathrm{ns}\,\omega \to \coth\omega,\quad \mathrm{nc}\,\omega \to \cosh\omega,\quad \mathrm{nd}\,\omega \to \cosh\omega,\quad \mathrm{cs}\,\omega \to \mathrm{csch}\,\omega,\quad \mathrm{ds}\,\omega \to \mathrm{csch}\,\omega,\quad \mathrm{dc}\,\omega \to 1.$$

The Jacobi elliptic functions degenerate into trigonometric functions when $m \to 0$:

$$\mathrm{sn}\,\omega \to \sin\omega,\quad \mathrm{cn}\,\omega \to \cos\omega,\quad \mathrm{dn}\,\omega \to 1,\quad \mathrm{sc}\,\omega \to \tan\omega,\quad \mathrm{sd}\,\omega \to \sin\omega,\quad \mathrm{cd}\,\omega \to \cos\omega,$$
$$\mathrm{ns}\,\omega \to \csc\omega,\quad \mathrm{nc}\,\omega \to \sec\omega,\quad \mathrm{nd}\,\omega \to 1,\quad \mathrm{cs}\,\omega \to \cot\omega,\quad \mathrm{ds}\,\omega \to \csc\omega,\quad \mathrm{dc}\,\omega \to \sec\omega.$$

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