ESAIM: COCV
DOI: 10.1051/cocv/2011189

ESAIM: Control, Optimisation and Calculus of Variations
www.esaim-cocv.org

MULTIPLICITY OF SOLUTIONS FOR THE NONCOOPERATIVE p-LAPLACIAN OPERATOR ELLIPTIC SYSTEM WITH NONLINEAR BOUNDARY CONDITIONS

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Abstract. In this paper, we study the multiplicity of solutions for a class of noncooperative p-Laplacian operator elliptic system. Under suitable assumptions, we obtain a sequence of solutions by using the limit index theory.

Mathematics Subject Classification. 35J70, 35B20.

Received May 5, 2011.

1. Introduction

In this paper we deal with the existence and multiplicity of solutions to the following p-Laplacian operator elliptic system with nonlinear boundary conditions.

$$\begin{cases} \Delta_p u - |u|^{p-2}u = F_u(x,u,v), & \text{in } \Omega, \\ -\Delta_p v + |v|^{p-2}v = F_v(x,u,v), & \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = |u|^{p^*-2}u, \quad |\nabla v|^{p-2}\frac{\partial v}{\partial \nu} = |v|^{p^*-2}v, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $1 < p < N$, $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded domain with smooth boundary, $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is a p-Laplacian operator and $\frac{\partial}{\partial \nu}$ is the outer normal derivative, $F = F(x,u,v)$, $F_u = \frac{\partial F}{\partial u}$, $F_v = \frac{\partial F}{\partial v}$, $p^* = Np/(N-p)$ is the critical exponent according to the Sobolev embedding.

In recent years, the existence and multiplicity of solutions for a noncooperative elliptic system have been obtained by many papers. In [1], Benci assumed X is a Hilbert space, f satisfies (PS)-condition and is the form

$$f(u) = \frac{1}{2}\langle Lu, u\rangle + \Phi(u),$$

where L is bounded self-adjoint operator and Φ' is compact.

Keywords and phrases. p-Laplacian operator, limit index, critical growth, concentration-compactness principle.

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When $p = 2$ (a constant) with Dirichlet boundary condition, Lin and Li [9] considered the following system

$$\begin{cases} \Delta u = |u|^{2^*-2}u + F_u(x,u,v), & \text{in } \Omega, \\ -\Delta v = |v|^{2^*-2}v + F_v(x,u,v), & \text{in } \Omega, \\ u = 0, \quad v = 0, & \text{on } \partial\Omega, \end{cases}$$

by applying the Limit Index Theory, they obtained the existence of multiple solutions under some assumptions on nonlinear part.

When $p \neq 2$, Huang and Li [6] considered the following the system of elliptic equations involving the p-Laplacian in the unbounded domain of $\mathbb{R}^N$ by applying the Limit Index Theory,

$$\begin{cases} \Delta_p u - |u|^{p-2}u = F_u(x,u,v), & \text{in } \mathbb{R}^N, \\ -\Delta_p v + |v|^{p-2}v = F_v(x,u,v), & \text{in } \mathbb{R}^N, \\ u, v \in W^{1,p}(\mathbb{R}^N), \end{cases}$$

where $1 < p < N$ and they extended some results of [8].

We note that these papers deal with Dirichlet boundary condition [2,7]. However, nonlinear boundary conditions have only been considered in recent years. For the Laplace operator with nonlinear boundary conditions see for example [3,14]. For elliptic systems with nonlinear boundary conditions see [5]. For previous work for the p-Laplacian with nonlinear boundary conditions of different type see [4,13].

Motivated by papers above, a natural question arises whether the existence and multiplicity of solutions to the p-Laplacian operator elliptic system with nonlinear boundary conditions (1.1) can be obtained. In this paper we deal with the problem (1.1). Throughout this paper, we assume that $F(x,u,v)$ satisfies the following conditions:

(H_1) $F \in C(\overline{\Omega} \times R^2, R^+)$ and $F(x,s,t) = F(x,-s,-t)$ for all $(x,s,t) \in \Omega \times \in \mathbb{R}^2$;
(H_2) $\lim_{|t|\to\infty} \frac{F_t(x,s,t)}{|t|^{p-1}} = 0$ uniformly for $x \in \Omega$;
(H_3) $sF_s(x,s,t) \geq 0$ for all $(x,s,t) \in \overline{\Omega} \times R^2$.

Under assumptions (H_1) and (H_2), we have

$$F_v(x,u,v)v = o\left(|v|^p\right),$$

which means that, for all $\varepsilon > 0$, there exist $a(\varepsilon)$, $b(\varepsilon) > 0$ such that

$$|F(x,0,v)v| \leq a(\varepsilon) + \varepsilon|v|^p \tag{1.2}$$

and

$$|F_v(x,u,v)v| \leq b(\varepsilon) + \varepsilon|v|^p. \tag{1.3}$$

Hence, together with condition (1.2), (1.3) and the mean value theorem for any constants β and fixed u we have

$$|F(x,u,v) - \beta F_v(x,u,v)v| \leq c(\varepsilon) + \varepsilon|v|^p, \tag{1.4}$$

for some $c(\varepsilon) > 0$.

Furthermore, we assume that $F(x,u,v)$ satisfies condition:

(H_4) There exist $L > 0$ (where L will be determined later) and

$$\xi < |\Omega|^{-1} \min\left\{0, \frac{1}{N}S^{p^*/(p^*-p)} - c\left(\frac{1}{2N}\right)|\Omega|\right\}$$

such that $F(x,s,t)t \geq L|t|^p - \xi$, for every $(x,s,t) \in \overline{\Omega} \times \mathbb{R}^2$.

Notation. Weak (resp. strong) convergence is denoted by $\rightharpoonup$ (resp., $\to$). $|\cdot|_p$ is the usual norm in $L^p(\Omega)$. $L_2^p(\Omega) = L^p(\Omega) \times L^p(\Omega)$ with the norm $|(u,v)|_p := (|u|_p^p + |v|_p^p)^{\frac{1}{p}}$. $E := W^{1,p}(\Omega)$ with the norm $\|u\|_p := \int_\Omega (|\nabla u|^p + |u|^p)\mathrm{d}x$. $Y = E \times E$, $X_n = E \times E_n$. c_i denote a positive constant and can be determined in concrete conditions.

According to [15], there exists a Schauder basis ~~Schauder basis~~ $\{e_n\}_{n=1}^\infty$ for E. Furthermore, since E is reflexive, $\{e_n^*\}_{n=1}^\infty$ the biorthogonal functionals associated to the basis $\{e_n\}_{n=1}^\infty$ which are characterized by the relations

$$e_n^*(e_m) = \delta_{n,m} = \begin{cases} 1, & \text{if } n = m, \\ 0, & \text{if } n \neq m, \end{cases}$$

form a basis for E^* with the following properties (*cf.* [10] Prop. 1.b.1 and Thm. 1.b.5). Denote

$$E_n = \mathrm{span}\{e_1, \ldots, e_n\}, \quad E_n^\perp = \overline{\mathrm{span}\{e_{n+1}, \ldots\}}$$

and

$$E_n^* = \mathrm{span}\{e_1^*, \ldots, e_n^*\}.$$

Let $P_n : E \to E_n$ be the projector corresponding to decomposition $E = E_n \oplus E_n^\perp$ and $P_n^* : E^* \to E_n^*$ be the projector corresponding to the decomposition $E^* = E_n^* \oplus (E_n^*)^\perp$. Then $P_n u \to u$, $P_n^* v^* \to v^*$ for any $u \in E$, $v^* \in E^*$ as $n \to \infty$ and $\langle P_n^* v^*, u\rangle = \langle v^*, P_n u\rangle$. Let $\tau : E \to E^*$ be the mapping given by

$$\langle \tau u, \widetilde{u}\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla \widetilde{u}\mathrm{d}x, \quad \text{for all } u, \widetilde{u} \in E.$$

It is easy to check that the operator τ is bounded, continuous. And if $u_n \rightharpoonup \widetilde{u}$ in E and $\langle \tau u_n - \tau\widetilde{u}, u_n - \widetilde{u}\rangle \to 0$, then $u_n \to \widetilde{u}$ in E (see [6,8])

The energy functional corresponding to problem (1.1) is defined as follows,

$$\begin{aligned} J(u,v) = &-\frac{1}{p}\int_\Omega (|\nabla u|^p + |u|^p)\mathrm{d}x + \frac{1}{p}\int_\Omega (|\nabla v|^p + |v|^p)\mathrm{d}x \\ &-\frac{1}{p^*}\int_{\partial\Omega} |u|^{p^*}\mathrm{d}\sigma - \frac{1}{p^*}\int_{\partial\Omega} |v|^{p^*}\mathrm{d}\sigma - \int_\Omega F(x,u,v)\mathrm{d}x. \end{aligned} \tag{1.5}$$

The main result of this paper is as follows.

Theorem 1.1. *Suppose that $F(x,u,v)$ satisfies conditions (H_1)–(H_4). Then there exists $k_0 > 1$ such that (1.1) possesses at least $k_0 - 1$ pairs weak nontrivial solutions.*

Remark 1.2. There are two difficulties in considering the elliptic problem (1.1). One is the functional $J(u,v)$ is strongly indefinite. Therefore one cannot apply the symmetric Mountain pass theorem of the functional $J(u,v)$. The other one in solving the problem is a lack of compactness which can be illustrated by the fact that the embedding of $W^{1,p}(\Omega)$ into $L^{p^*}(\partial\Omega)$ is no longer compact.

Remark 1.3. Theorem 1.1 is new as far as we know. We mainly follow the way in [8] to prove our main result.

2. Preliminaries and lemmas

First of all, we recall the limit index theory due to Li [8]. In order to do that, we introduce the following definitions.

Definition 2.1. [8,16] The action of a topological group G on a normed space Z is a continuous map

$$G \times Z \to Z : [g,z] \mapsto gz$$

such that

$$1 \cdot z = z, \quad (gh)z = g(hz) \quad z \mapsto gz \text{ is linear}, \quad \forall\, g, h \in G.$$

The action is isometric if

$$\|gz\| = \|z\|, \quad \forall\, g \in G, \quad z \in Z.$$

And in this case Z is called G-space.

The set of invariant points is defined by

$$\mathrm{Fix}G := \{z \in Z : gz = z, \ \forall\, g \in G\}.$$

A set $A \subset Z$ is invariant if $gA = A$ for every $g \in G$. A function $\varphi : Z \to R$ is invariant $\varphi \circ g = \varphi$ for every $g \in G$, $z \in Z$. A map $f : Z \to Z$ is equivariant if $g \circ f = f \circ g$ for every $g \in G$.

Suppose Z is a G-Banach space, that is, there is a G isometric action on Z. Let

$$\Sigma := \{A \subset Z : A \text{ is closed and } gA = A, \forall\, g \in G\}$$

be a family of all G-invariant closed subset of Z, and let

$$\Gamma := \left\{h \in C^0(Z, Z) : h(gu) = g(hu), \quad \forall\, g \in G\right\}$$

be the class of all G-equivariant mapping of Z. Finally, we call the set

$$O(u) := \{gu : g \in G\}$$

G-orbit of u.

Definition 2.2. [8] An index for (G, Σ, Γ) is a mapping $i : \Sigma \to \mathcal{Z}_+ \cup \{+\infty\}$ (where $\mathcal{Z}_+$ is the set of all nonnegative integers) such that for all $A, B \in \Sigma$, $h \in \Gamma$, the following conditions are satisfied:

(1) $i(A) = 0 \Leftrightarrow A = \emptyset$;
(2) (monotonicity) $A \subset B \Rightarrow i(A) \le i(B)$;
(3) (subadditivity) $i(A \cup B) \le i(A) + i(B)$;
(4) (supervariance) $i(A) \le i(\overline{h(A)}), \forall\, h \in \Gamma$;
(5) (continuity) If A is compact and $A \cap \mathrm{Fix}G = \emptyset$, then $i(A) < +\infty$ and there is a G-invariant neighbourhood N of A such that $i(\overline{N}) = i(A)$;
(6) (normalization) If $x \notin \mathrm{Fix}G$, then $i(O(x)) = 1$.

Definition 2.3. [1] An index theory is said to satisfy the d-dimension property if there is a positive integer d such that

$$i(V^{dk} \cap S_1) = k$$

for all dk-dimensional subspaces $V^{dk} \in \Sigma$ such that $V^{dk} \cap \mathrm{Fix}G = \{0\}$, where S_1 is the unit sphere in Z.

Suppose U and V are G-invariant closed subspaces of Z such that

$$Z = U \oplus V,$$

where V is infinite dimensional and

$$V = \overline{\bigcup_{j=1}^{\infty} V_j},$$

where V_j is a dn_j-dimensional G-invariant subspace of V, $j = 1, 2, \ldots$, and $V_1 \subset V_2 \subset \ldots \subset V_n \subset \ldots$ Let

$$Z_j = U \bigoplus V_j,$$

and $\forall\ A \in \Sigma$, let

$$A_j = A \bigoplus Z_j.$$

Definition 2.4. [8] Let i be an index theory satisfying the d-dimension property. A limit index with respect to (Z_j) induced by i is a mapping

$$i^\infty : \Sigma \to \mathcal{Z} \cup \{-\infty, +\infty\}$$

given by

$$i^\infty(A) = \limsup_{j\to\infty} (i(A_j) - n_j).$$

Proposition 2.5. [8] *Let $A, B \in \Sigma$. Then i^∞ satisfies:*
(1) $A = \emptyset \Rightarrow i^\infty = -\infty$;
(2) *(monotonicity)* $A \subset B \Rightarrow i^\infty(A) \le i^\infty(B)$;
(3) *(subadditivity)* $i^\infty(A \cup B) \le i^\infty(A) + i^\infty(B)$;
(4) *If* $V \cap \mathrm{Fix}G = \{0\}$, *then* $i^\infty(S_\rho \cap V) = 0$, *where* $S_\rho = \{z \in Z : \|z\| = \rho\}$;
(5) *If* Y_0 *and* $\widetilde{Y_0}$ *are G-invariant closed subspaces of V such that* $V = Y_0 \oplus \widetilde{Y_0}$, $\widetilde{Y_0} \subset V_{j_0}$ *for some* j_0 *and* $\dim\widetilde{Y_0} = dm$, *then* $i^\infty(S_\rho \cap Y_0) \ge -m$.

Definition 2.6. [16] A functional $J \in C^1(Z, R)$ is said to satisfy the condition $(PS)^*_c$ if any sequence $\{u_{n_k}\}$, $u_{n_k} \in Z_{n_k}$ such that

$$J(u_{n_k}) \to c, \quad \mathrm{d}J_{n_k}(u_{nk}) \to 0, \quad \text{as } k \to \infty$$

possesses a convergent subsequence, where Z_{n_k} is the n_k-dimension subspace of Z, $J_{n_k} = J|_{Z_{n_k}}$.

Theorem 2.7. [8] *Assume that*

(B_1) *$J \in C^1(Z, R)$ is G-invariant;*
(B_2) *there are G-invariant closed subspaces U and V such that V is infinite dimensional and $Z = U \oplus V$;*
(B_3) *there is a sequence of G-invariant finite dimensional subspaces*

$$V_1 \subset V_2 \subset \cdots \subset V_j \subset \cdots, \quad \dim V_j = dn_j,$$

such that $V = \overline{\cup_{j=1}^\infty V_j}$;
(B_4) *there is an index theory i on Z satisfying the d-dimension property;*
(B_5) *there are G-invariant subspaces* Y_0, $\widetilde{Y_0}$, Y_1 *of V such that* $V = Y_0 \oplus \widetilde{Y_0}$, $Y_1, \widetilde{Y_0} \subset V_{j_0}$ *for some* j_0 *and* $\dim\widetilde{Y_0} = \mathrm{d}m < \mathrm{d}k = \dim Y_1$;
(B_6) *there are α and β, $\alpha < \beta$ such that f satisfies* $(PS)^*_c$, $\forall\ c \in [\alpha, \beta]$;
(B_7)

$$\begin{cases} (a)\ \textit{either}\ \mathrm{Fix}G \subset U \oplus Y_1, \quad \textit{or} \quad \mathrm{Fix}G \cap V = \{0\}, \\ (b)\ \textit{there is}\ \rho > 0\ \textit{such that}\ \forall\ u \in Y_0 \cap S_\rho, f(z) \ge \alpha, \\ (c)\ \forall\ z \in U \oplus Y_1,\ f(z) \le \beta, \end{cases}$$

if i^∞ is the limit index corresponding to i, then the numbers

$$c_j = \inf_{i^\infty(A) \ge j} \sup_{z \in A} f(u), \quad -k+1 \le j \le -m,$$

are critical values of f, and $\alpha \le c_{-k+1} \le \cdots \le c_{-m} \le \beta$. *Moreover, if* $c = c_l = \cdots = c_{l+r}$, $r \ge 0$, *then* $i(\mathbb{K}_c) \ge r + 1$, *where* $\mathbb{K}_c = \{z \in Z : \mathrm{d}f(z) = 0, f(z) = c\}$.

3. Local Palais-Smale condition

To prove ~~our existence result,~~ ~~since we have lost the compactness~~ in the inclusion $W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\partial\Omega)$, we can no longer expect the Palais-Smale condition to hold. Anyway we can prove a local Palais-Smale condition that will hold for $J(u,v)$ below a certain value of energy. Let u_n be a bounded sequence in $W^{1,p}(\Omega)$ then there exists a subsequence that we still denote u_n such that

$$\begin{aligned} &u_n \rightharpoonup u \quad \text{weakly in } W^{1,p}(\Omega),\\ &u_n \to u \quad \text{strongly in } L^r(\Omega),\ 1 \le r < p^*,\\ &|\nabla u_n|^p \rightharpoonup \mathrm{d}\mu, \quad |u_n|_{\partial\Omega}|^{p^*} \rightharpoonup \mathrm{d}\eta, \end{aligned}$$

weakly-* in the sense of measures. Observe that $\mathrm{d}\eta$ is a measure supported on $\partial\Omega$.

If we consider $\phi \in C^\infty(\overline{\Omega})$, from the Sobolev trace inequality we obtain, passing to the limit

$$\left(\int_{\partial\Omega} |\phi|^{p^*}\mathrm{d}\eta\right)^{\frac{1}{p^*}} S^{\frac{1}{p}} \le \left(\int_\Omega |\phi|^p \mathrm{d}\mu + \int_\Omega |u|^p|\nabla\phi|^p\mathrm{d}x + \int_\Omega |\phi u|^p \mathrm{d}x\right)^{\frac{1}{p}}, \tag{3.1}$$

where S is the best constant in the Sobolev trace embedding theorem. From (3.1) we observe that if $u = 0$ we get a reverse Hölder-type inequality (but it involves one integral over Ω) between the two measures μ and η.

Similar to the proof of [11,12], we have the following lemma.

Lemma 3.1. [4] *Let u_j be a weakly convergent sequence in $W^{1,p}(\Omega)$ with weak limit u such that*

$$|\nabla u_j|^p \rightharpoonup \mathrm{d}\mu, \quad \text{and} \quad |u_j|_{\partial\Omega}|^{p^*} \rightharpoonup \mathrm{d}\eta,$$

weakly- in the sense of measures. Then there exists $x_1, \dots, x_l \in \partial\Omega$ such that*

(i) $\mathrm{d}\eta = |u|^{p^*} + \sum_{j=1}^l \eta_j\delta_{x_j}$, $\eta_j > 0$;

(ii) $\mathrm{d}\mu \ge |\nabla u|^p + \sum_{j=1}^l \mu_j\delta_{x_j}$, $\mu_j > 0$;

(iii) $(\eta_j)^{\frac{p}{p^*}} \le \frac{\mu_j}{S}$.

Similar to [6,16], it is easy to obtain the following lemma.

Lemma 3.2. *Assume $1 \le \theta_1, \theta_2, \theta < \infty$, $I \in C(\overline{\Omega} \times R^2, R)$ and*

$$I(x,u,v) \le C\left(|u|^{\frac{\theta_1}{\theta}} + |v|^{\frac{\theta_2}{\theta}}\right).$$

Then for every $(u,v) \in L^{\theta_1}(\Omega) \times L^{\theta_2}(\Omega)$, $I(\cdot,u,v) \in L^\theta(\Omega)$ and the operator

$$T : (u,v) \mapsto I(x,u,v)$$

is a continuous map from $L^{\theta_1}(\Omega) \times L^{\theta_2}(\Omega)$ to $L^\theta(\Omega)$.

Lemma 3.3. *Suppose that $F(x,u,v)$ satisfies conditions (H_1)–(H_3). Then*

(i) $J \in C^1(X,R)$;

(ii)

$$\begin{aligned} \langle \mathrm{d}J(u,v), (\widehat{u},\widehat{v})\rangle = &-\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\widehat{u} + |u|^{p-2}u\widehat{u}\mathrm{d}x - \int_{\partial\Omega} |u|^{p^*-2}u\widehat{u}\mathrm{d}\sigma\\ &+ \int_\Omega |\nabla v|^{p-2}\nabla v \cdot \nabla\widehat{v} + \ll |v|^{p-2}v\widehat{v}\mathrm{d}x - \int_{\partial\Omega} |v|^{p^*-2}v\widehat{v}\mathrm{d}\sigma\\ &- \int_\Omega F_u(x,u,v)\widehat{u}\mathrm{d}x - \int_\Omega F_v(x,u,v)\widehat{v}\mathrm{d}x; \end{aligned}$$

(iii) A critical point of J is a weak solution of system (1.1).

Now set

$$X = U \oplus V, \quad U = E \times \{0\}, \quad V = \{0\} \times E,$$

$$Y_0 = \{0\} \times E_1^{\perp}, \quad V = Y_0 \oplus \widetilde{Y_0},$$

$$Y_1 = \{0\} \times E_{k_0}, \quad E_{k_0} = \operatorname{span}\{e_1, \ldots, e_{k_0}\},$$

then $\dim \widetilde{Y_0} = 1$, $\dim Y_1 = k_0$.

Define a group action $G_2 = \{1, \tau\} \cong \mathcal{Z}_2$ by setting $\tau(u, v) = (-u, -v)$, then $\mathrm{Fix}G = \{0\} \times \{0\}$ (also denote $\{0\}$). It is clear that U and V are G-invariant closed subspaces of X, and Y_0, $\widetilde{Y_0}$ and Y_1 are G-invariant subspace of V. Set

$$\Sigma := \{A \subset X \setminus \{0\} : \ A \text{ is closed in } X \text{ and } (u, v) \in A \Rightarrow (-u, -v) \in A\}.$$

Define an index γ on Σ by:

$$\gamma(A) = \begin{cases} \min\{N \in \mathcal{Z} : \exists\, h \in C(A, \mathbb{R}^N \backslash \{0\} \text{ such that } h(-u, -v) = h(u, v))\}, \\ 0, \quad \text{if } A = \emptyset, \\ +\infty, \quad \text{if such } h \text{ does not exist.} \end{cases}$$

Then we have the following proposition from [6]: γ is an index satisfying the properties given in Definition 2.2. Moreover, γ satisfies the one-dimension property. According to Definition 2.4 we can obtain a limit index γ^{∞} with respect to (X_n) from γ.

Now we turn to prove local Palais-Smale condition.

Lemma 3.4. *Assume condition (H_1)–(H_3) hold, Then the functional J satisfies the local $(PS)_c$ condition in*

$$c \in \left(-\infty, \frac{1}{N} S^{p^*/(p^*-p)} - c\left(\frac{1}{2N}\right)|\Omega|\right),$$

in the following sense: if

$$J(u_{n_k}, v_{n_k}) \to c \in \left(-\infty, \frac{1}{N} S^{p^*/(p^*-p)} - c\left(\frac{1}{2N}\right)|\Omega|\right), \quad \mathrm{d}J_{n_k}(u_{n_k}, v_{n_k}) \to 0, \quad \text{as } k \to \infty,$$

where $J_{n_k} = J|_{X_{n_k}}$. Then $\{(u_{n_k}, v_{n_k})\}$ contains a subsequence converging strongly in X.

Proof. First, we show that $\{(u_{n_k}, v_{n_k})\}$ is bounded in X.

We note that by condition (H_3),

$$\begin{aligned} o(1)\|u_{n_k}\|_p &\geq \langle -\mathrm{d}J_{n_k}(u_{n_k}, v_{n_k}), (u_{n_k}, 0)\rangle \\ &= \int_\Omega |\nabla u_{n_k}|^p + |u_{n_k}|^p \mathrm{d}x + \int_{\partial\Omega} |u_{n_k}|^{p^*} \mathrm{d}\sigma + \int_\Omega F_u(x, u_{n_k}, v_{n_k}) u_{n_k} \mathrm{d}x \\ &\geq \int_\Omega |\nabla u_{n_k}|^p + |u_{n_k}|^p \mathrm{d}x + \int_{\partial\Omega} |u_{n_k}|^{p^*} \mathrm{d}x \\ &\geq \|u_{n_k}\|_p^p, \end{aligned} \tag{3.2}$$

since $p > 1$, from (3.2), we know that $\|u_{n_k}\|_p$ is bounded. On the one hand, we have

$$\begin{aligned} &J_{n_k}(0, v_{n_k}) - \frac{1}{p^*}\langle \mathrm{d}J_{n_k}(u_{n_k}, v_{n_k}), (0, v_{n_k})\rangle \\ &= \left(\frac{1}{p} - \frac{1}{p^*}\right) \int_\Omega \left(|\nabla v_{n_k}|^p + |v_{n_k}|^p\right) \mathrm{d}x - \int_\Omega \left[F(x, u_{n_k}, v_{n_k}) - \frac{1}{p^*} F_v(x, u_{n_k}, v_{n_k}) v_{n_k}\right] \mathrm{d}x \\ &= c + o(1)\|v_n\|_p, \end{aligned}$$

i.e.

$$\frac{1}{N}\int_\Omega (|\nabla v_{n_k}|^p + |v_{n_k}|^p)\,\mathrm{d}x = \int_\Omega \left[F(x,u_{n_k},v_{n_k}) - \frac{1}{p^*}F_v(x,u_{n_k},v_{n_k})v_{n_k}\right]\mathrm{d}x$$
$$+ c + o(1)\|v_{n_k}\|_p.$$

Then by (1.4), we have

$$\left(\frac{1}{N} - \varepsilon\right)\|v_{n_k}\|_p^p \le c(\varepsilon)|\Omega| + c + o(1)\|v_{n_k}\|_p,$$

where $|\cdot|$ denote by Lebesgue measure. Setting $\varepsilon = 1/2N$, we get

$$\|v_{n_k}\|_p^p \le M + o(1)\|v_{n_k}\|_p, \tag{3.3}$$

where $o(1) \to 0$ and M is a some positive number. Thus (3.3) implies that $\{v_{n_k}\}$ is bounded in $W^{1,p}(\Omega)$. This implies $\|u_{n_k}\|_p + \|v_{n_k}\|_p$ is bounded in X.

Next, we prove that $\{(u_{n_k}, v_{n_k})\}$ contains a subsequence converging strongly in X.

We note that $\{u_{n_k}\}$ is bounded in E. Hence, up to a subsequence, $u_{n_k} \rightharpoonup u$ weakly in E and $u_{n_k}(x) \to u(x)$, a.e. in $\mathbb{R}^N$. We claim that $u_{n_k} \to u$ strongly in E. In fact, note that

$$\int_\Omega |\nabla u_{n_k} - \nabla u|^p + |u_{n_k} - u|^p \mathrm{d}x + \int_{\partial\Omega} |u_{n_k} - u|^{p^*}\mathrm{d}\sigma + \int_\Omega F_u(x, u_{n_k} - u, v_{n_k})(u_{n_k} - u)\mathrm{d}x$$
$$= \langle -\mathrm{d}J_{n_k}(u_{n_k} - u, v_{n_k}), (u_{n_k} - u, 0)\rangle \to 0 \quad \text{as } n \to \infty,$$

and condition (H_3) imply that

$$u_{n_k} \to u \quad \text{strongly in } E. \tag{3.4}$$

In the following we will prove that there exists $v \in E$ such that

$$v_{n_k} \to v \quad \text{strongly in } E. \tag{3.5}$$

By Lemma 3.1 and (3.3) there exists a subsequence, there exists a subsequence, that we still denote v_{n_k} such that

$$\begin{aligned}
&v_{n_k} \rightharpoonup v \quad \text{weakly in } W^{1,p}(\Omega),\\
&v_{n_k} \to v \quad \text{strongly in } L^r(\Omega),\ 1 \le r < p^*,\ \text{and a.e. in } \Omega,\\
&|\nabla v_{n_k}|^p \rightharpoonup \mathrm{d}\mu \ge |\nabla v|^p + \sum_{k=1}^{l} \mu_k \delta_{x_k},\\
&|v_{n_k}|_{\partial\Omega}|^{p^*} \rightharpoonup \mathrm{d}\eta = |v|_{\partial\Omega}|^{p^*} + \sum_{k=1}^{l} \eta_k \delta_{x_k}.
\end{aligned}$$

Let $\phi(x) \in C^\infty(\Omega)$ such that $\phi(x) \equiv 1$ in $B(x_k, \varepsilon)$, $\phi(x) \equiv 0$ in $\Omega \setminus (x_k, 2\varepsilon)$ and $|\nabla\phi| \le 2/\varepsilon$, where x_k belongs to the support of $\mathrm{d}\eta$. Consider Then $\{\phi v_{n_k}\}$ is bounded in E, Obviously, $\langle \mathrm{d}J_{n_k}(u_{n_k}, v_{n_k}), (0, v_{n_k}\phi)\rangle \to 0$, *i.e.*

$$\begin{aligned}
&\lim_{n\to\infty}\left[\int_\Omega (|\nabla v_{n_k}|^p + |v_{n_k}|^p)\,\phi\mathrm{d}x - \int_{\partial\Omega} |v_{n_k}|^{p^*}\phi\mathrm{d}\sigma - \int_\Omega F_v(x, u_{n_k}, v_{n_k})v_{n_k}\phi\mathrm{d}x\right]\\
&= -\lim_{n\to\infty}\int_\Omega \left(v_{n_k}|\nabla v_{n_k}|^{p-2}\nabla v_{n_k}\nabla\phi\right)\mathrm{d}x.
\end{aligned} \tag{3.6}$$

On the other hand, by Hölder inequality and weak convergence, we obtain

$$\begin{aligned}
0 &\le \lim_{\varepsilon\to 0}\lim_{n\to\infty}\left|\int_\Omega v_{n_k}|\nabla v_{n_k}|^{p-2}\nabla v_{n_k}\nabla\phi \mathrm{d}x\right| \\
&\le \lim_{\varepsilon\to 0}\lim_{n\to\infty}\left(\int_\Omega |v_{n_k}|^p|\nabla\phi|^p\mathrm{d}x\right)^{\frac{1}{p}}\left(\int_\Omega |\nabla v_{n_k}|^q\mathrm{d}x\right)^{\frac{p-1}{p}} \\
&\le C\lim_{\varepsilon\to 0}\left(\int_\Omega |v|^p|\nabla\phi|^p\mathrm{d}x\right)^{\frac{1}{p}} \\
&\le C\lim_{\varepsilon\to 0}\left(\int_{B(x_j,\varepsilon)} |\nabla\phi|^N\mathrm{d}x\right)^{\frac{1}{N}}\left(\int_{B(x_j,\varepsilon)} |v|^{p^*}\mathrm{d}x\right)^{\frac{1}{p^*}} \\
&\le C\lim_{\varepsilon\to 0}\left(\int_{B(x_j,\varepsilon)} |v|^{p^*}\mathrm{d}x\right)^{\frac{1}{p^*}} = 0. \qquad (3.7)
\end{aligned}$$

~~From (3.6) and (3.7), we have~~

$$0 = \lim_{\varepsilon\to 0}\left[\int_{\partial\Omega}\phi\mathrm{d}\eta - \int_\Omega \phi\mathrm{d}\mu - \int_\Omega |v|^p\phi\mathrm{d}x - \int_\Omega F_v(x,u,v)v\phi\mathrm{d}x\right] = \eta_k - \mu_k. \qquad (3.8)$$

Combing this with Lemma 3.1, we obtain $(\mu_k)^{p/p^*}S \le \mu_k$. This result implies that

$$\mu_k = 0 \quad \text{or} \quad \mu_k \ge S^{p^*/(p^*-p)}.$$

If the second case $\mu_k \ge S^{p^*/(p^*-p)}$ holds, for some $k \in J$, then by using Lemma 3.1 and the Hölder inequality, we have ~~that~~

$$\begin{aligned}
c &= \lim_{n\to\infty}\left(J_{n_k}(0,v_{n_k}) - \frac{1}{p^*}\langle \mathrm{d}J_{n_k}(u_{n_k},v_{n_k}),(0,v_{n_k})\rangle\right) \\
&= \left(\frac{1}{p}-\frac{1}{p^*}\right)\int_\Omega \left(|\nabla v_{n_k}|^p + |v_{n_k}|^p\right)\mathrm{d}x - \int_\Omega\left[F(x,u_{n_k},v_{n_k}) - \frac{1}{p^*}F_v(x,u_{n_k},v_{n_k})v_{n_k}\right]\mathrm{d}x \\
&\ge \frac{1}{N}\int_\Omega \mathrm{d}\mu - c\left(\frac{1}{2N}\right)|\Omega| \\
&\ge \frac{1}{N}\int_\Omega |\nabla v_{n_k}|^p\mathrm{d}x + \frac{1}{N}S^{p^*/(p^*-p)} - c\left(\frac{1}{2N}\right)|\Omega| \\
&\ge \frac{1}{N}S^{p^*/(p^*-p)} - c\left(\frac{1}{2N}\right)|\Omega|,
\end{aligned}$$

where $\varepsilon = 1/2N$. This is impossible. Consequently, $\mu_k = 0$ for all $k \in J$. From (3.8) we know that $\eta_k = 0$ for all $k \in J$ and hence

$$\int_{\partial\Omega}|v_{n_k}|^{p^*}\mathrm{d}\sigma \to \int_{\partial\Omega}|v|^{p^*}\mathrm{d}\sigma.$$

Now $v_{n_k} \rightharpoonup v$ in E and Brezis-Lieb lemma [2] implies that

$$\lim_{n\to\infty}\int_{\partial\Omega}|v_{n_k} - v|^{q^*}\mathrm{d}\sigma = 0.$$

Thus, we have

$$
\begin{aligned}
o(1)\|v_{n_k}\|_p &= \|v_{n_k}\|_p^p - \int_\Omega |v_{n_k}|^{p^*}\mathrm{d}\sigma - \int_\Omega F_v(x,u_{n_k},v_{n_k})v_{n_k}\mathrm{d}x\\
&= \|v_{n_k}-v\|_p^p + \|v\|_p^p - \int_\Omega |v|^{p^*}\mathrm{d}\sigma - \int_\Omega F_v(x,u,v)v\mathrm{d}x\\
&= \|v_{n_k}-v\|_p^p + o(1)\|v\|_p,
\end{aligned}
$$

since $\mathrm{d}J_{n_k}(0,v)=0$. Thus we prove that $\{v_{n_k}\}$ strongly converges to v in E. Thus (3.5) holds. (3.4) and (3.5) imply the conclusion of Lemma 3.4 follows. □

4. Proof of Theorem 1.1

Proof of Theorem 1.1. Now we shall verify the conditions of Theorem 2.7. Obviously, (B_1), (B_2), (B_4) in Theorem 2.7 are satisfied. Set $V_j = E_j = \mathrm{span}\{e_1,e_2,\ldots,e_j\}$, then (B_3) is also satisfied. Since $1=\dim\widetilde{Y_0} < k_0 = \dim Y_1$, (B_5) is satisfied. In the following we verify the conditions in (B_7). Since $\mathrm{Fix}G\cap V=0$, that is (a) of (B_7) holds. It remains to verify (b), (c) of (B_7). Choose a number α such that

$$
\alpha < \min\left\{0, \frac{1}{N}S^{p^*/(p^*-p)} - c\left(\frac{1}{2N}\right)|\Omega|, \frac{1}{N}2^{\frac{p^*}{p-p^*}}S^{\frac{pp^*}{p-p^*}} - b\left(\frac{1}{2p}\right)|\Omega|\right\}. \tag{4.1}
$$

(i) If $(0,v)\in Y_0\cap S_\rho$ (where ρ is to be determined) then by (H_2),

$$
\begin{aligned}
J(0,v) &= \frac{1}{p}\int_\Omega |\nabla v|^p + |v|^p\mathrm{d}x - \frac{1}{p^*}\int_{\partial\Omega}|v|^{p^*}\mathrm{d}\sigma - \int_\Omega F(x,0,v)\mathrm{d}x\\
&\ge \left(\frac{1}{p}-\varepsilon\right)\cdot\int_\Omega |\nabla v|^p + |v|^p\mathrm{d}x - \frac{1}{p^*}\int_{\partial\Omega}|v|^{p^*}\mathrm{d}\sigma - b(\varepsilon)|\Omega|\\
&\ge \frac{1}{2p}\|v\|_p^p - \frac{1}{p^*}S^{p^*}\|v\|_p^{p^*} - b\left(\frac{1}{2p}\right)|\Omega|,
\end{aligned}
\tag{4.2}
$$

where $\varepsilon = \frac{1}{2p}$. Since

$$
\max_{t\in\mathbb{R}}\left(\frac{1}{2p}t^p - \frac{1}{p^*}S^{p^*}t^{p^*} - b\left(\frac{1}{2p}\right)|\Omega|\right) = \frac{1}{N}2^{\frac{p^*}{p-p^*}}S^{\frac{pp^*}{p-p^*}} - b\left(\frac{1}{2p}\right)|\Omega|,
$$

~~T~~herefore, there exists $\rho>0$ such that $J(0,v)\ge\alpha$ for every $\|v\|_p=\rho$, that is (b) of (B_7) holds.

(ii) For each $(u,v)\in U\oplus Y_1$, by condition (H_4), we ~~have~~

$$
\begin{aligned}
J(u,v) &= -\frac{1}{p}\int_\Omega(|\nabla u|^p+|u|^p)\mathrm{d}x + \frac{1}{p}\int_\Omega(|\nabla v|^p+|v|^p)\mathrm{d}x\\
&\quad -\frac{1}{p^*}\int_{\partial\Omega}|u|^{p^*}\mathrm{d}\sigma - \frac{1}{p^*}\int_{\partial\Omega}|v|^{p^*}\mathrm{d}\sigma - \int_\Omega F(x,u,v)\mathrm{d}x\\
&\le \frac{1}{p}\|v\|_p^p - L|v|_p^p + \xi|\Omega|\\
&\le \max_{v\in E_{k_0}}\left(\frac{1}{p}\|v\|_p^p - L|v|_p^p\right) + \xi|\Omega|\\
&= \max_{\{t\ge0, v\in\partial B_1(0)\cap E_{k_0}\}}\left[t^p\left(\frac{1}{p}-L|v|_p^p\right)\right] + \xi|\Omega|.
\end{aligned}
\tag{4.3}
$$

Let $r = \min\{\int_\Omega |v|^p \mathrm{d}x : v \in \partial B_1(0) \cap E_{k_0}\}$. By taking $L \geq \frac{1}{pr}$, we have

$$\frac{1}{p} - L|v|_p^p \leq \frac{1}{p} - Lr \leq 0. \tag{4.4}$$

It follows from (4.3), (4.4) and (H_4) that

$$J(u,v) \leq \xi|\Omega| \leq \min\left\{0, \frac{1}{N} S^{p^*/(p^*-p)} - c\left(\frac{1}{2N}\right)|\Omega|\right\}.$$

Let $\beta = \xi|\Omega|$, so we get (c) in (B_7). By Lemma 3.4, for any $c \in [\alpha, \beta]$, $J(u,v)$ satisfies the condition of $(PS)_c^*$, then (B_6) in Theorem 2.7 holds. So according to Theorem 2.7,

$$c_j = \inf_{i^\infty(A) \geq j} \sup_{z \in A} f(u), \quad -k_0 + 1 \leq j \leq -1\cancel{,}$$

are critical values of J, $\alpha \leq c_{-k_0+1} \leq \cdots \leq c_{-1} \leq \beta < 0$ and J has at least $k_0 - 1$ pairs critical points. □

Acknowledgements. The Project is supported by NSFC (10871096), Research Fundation during the 12st Five-Year Plan Period of Department of Education of Jilin Province, China (Grant [2011] No. 196), Natural Science Foundation of Changchun Normal University.

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