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An extended replacement policy for a deteriorating system with multi-failure modes ☆

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ABSTRACT

In this paper, the maintenance problem for a deteriorating system with $k+1$ failure modes, including an unrepairable failure (catastrophic failure) mode and k repairable failure (non-catastrophic failure) modes, is studied. Assume that the system after repair is not "as good as new" and its deterioration is stochastic. Under these assumptions, an extended replacement policy N is considered: the system will be replaced whenever the number of repairable failures reaches N or the unrepairable failure occurs, whichever occurs first. Our purpose is to determine an optimal extended policy N^* such that the average cost rate (i.e. the long-run average cost per unit time) of the system is minimized. The explicit expression of the average cost rate is derived, and the corresponding optimal extended policy N^* can be determined analytically or numerically. Finally, a numerical example is given to illustrate some theoretical results of the repair model proposed in this paper.

1. Introduction

In the earliest maintenance models, most research works usually assumed that a system after repair can be restored to "as good as new". This is a perfect repair model. However, this assumption is not always realistic. In practice, most repairable systems are deteriorating because of the aging effect and accumulative wear or the degeneration of repair technology. Consequently, Barlow and Hunter [1] first presented a minimal repair model in which a failed system after repair will function again, but with the same failure rate and the same effective age as at the time of failure. Later on, Brown and Proschan [2] first considered an imperfect repair model in which the repair will be perfect repair with probability p or minimal repair with probability $1-p$. Many research works on the minimal repair model and the imperfect repair model have been done by Park [16], Kijima [5], Makis and Jardine [14], Moustafa et al. [15], Sheu et al. [18] and others. In some scenarios, for a deteriorating system, it seems more reasonable to assume that the successive working times of the system after repair will become shorter and shorter, while the consecutive repair times of the system after failure will become longer and longer. Ultimately, it cannot work any longer, neither can it be repaired. To consider deteriorating systems with such characteristic, Lam [6,7] first introduced a geometric process repair model and studied two replacement policies: one based on the working age T of the system and the other based on the failure number N of the system. The objective is to choose optimal replacement policies T^* and N^* respectively such that the average cost rate is minimized. Finkelstein [4] presented a general repair model based on a scale transformation after each repair to generalize Lam's work. Zhang [23] generalized Lam's work by a bivariate replacement policy (T,N) under which the system is replaced at the working age T or at the time of the Nth failure, whichever occurs first. Later on, Zhang [24] applied the geometric process repair model to a two-component cold standby repairable system with one repairman and considered a replacement policy N based on the repair number of component 1. The problem is to determine

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the optimal replacement policy N^* such that the long-run expected reward per unit time is maximized. Further, Zhang and Wang [26] applied the geometric process repair model to a deteriorating two-component cold standby repairable system with priority in use, they not only studied some important reliability indices, but also considered a replacement policy based on the working age T of component 1. An optimal replacement policy T^* for minimizing the average cost rate of the system can be found. Many research works on the geometric process repair model see e.g., [8–10,19,20,12,13,25,21,22,3,27].

In most existing models (including the geometric process repair model) for maintenance problems, it is usually assumed that a system has only one failure mode. However, in practice, a system will probably have more than one failure mode. For example, a mechanical component can have slight failure and serious failure, and a switch component such as a relay or a diode may fail due to short or open circuit. For a deteriorating system with k failure modes, when the system cannot be repaired "as good as new", Zhang et al. [28] considered a replacement policy N based on the number of failures of the system. They determined the optimal replacement policy N^* such that the long-run expected profit per unit time is maximized and showed that the repair model for such a system forms a general monotone process model which includes the geometric process repair model as a special case. Lam et al. [11] also studied a monotone process repair model for such system with k failure modes, and showed that this model is equivalent to a geometric process repair model for a simple system with one failure mode such that both systems have the same average cost rate and optimal replacement policy. Note that they all assumed k failure modes are repairable. However, in many practical situations, some failures are unrepairable (i.e. catastrophic failures) whereas some are repairable (i.e. non-catastrophic failures). Considering such facts, this paper considers a deteriorating system with $k + 1$ failure modes, including an unrepairable failure (catastrophic failure) mode and k repairable failure (non-catastrophic failure) modes. When a repairable failure occurs, the system cannot be repaired "as good as new", and the deterioration of the system is stochastic. If the system fails due to the catastrophic failure, it can be only replaced by a new and identical one. Under these assumptions, we introduce a new repair-replacement policy N, or called an extended replacement policy N under which the system will be replaced whenever the number of repairable failures reaches N or the unrepairable failure occurs, whichever occurs first. Our purpose is to determine an optimal extended replacement policy N^* such that the average cost rate of the system is minimized. The explicit expression of the average cost rate is derived, the corresponding optimal extended replacement policy N^* can be determined analytically or numerically. Further, we can show that the repair model for a system with multi-failure modes in this paper forms a general monotone process repair model, and the correlation between the general monotone process repair model and the geometric process repair model is also given. Finally, a numerical example is given to illustrate some theoretical results of the repair model in this paper.

For easy presentation, we first state the definitions of stochastic order and geometric process as follows.

Definition 1. Given two random variables ξ and η, ξ is said to be stochastically larger than η or η is stochastically smaller than ξ if

$$P(\xi > \alpha) \geqslant P(\eta > \alpha), \quad \text{for all real } \alpha$$

denoted by $\xi \geqslant_{st} \eta$ or $\eta \leqslant_{st} \xi$ (see e.g. Ross [17]). Furthermore, we say that a stochastic process $\{X_n, n = 1,2,\ldots\}$ is stochastically decreasing if $X_n \geqslant_{st} X_{n+1}$ or stochastically increasing if $X_n \leqslant_{st} X_{n+1}$ for all $n = 1, 2, \ldots$.

Definition 2. A stochastic process $\{\xi_n, n = 1,2,\ldots\}$ is a geometric process, if there exists a real $a > 0$ such that $\{a^{n-1}\xi_n, n = 1,2,\ldots\}$ forms a renewal process. The real a is called the ratio of the geometric process (see e.g., Lam [7] and Zhang [23] for more details).

Obviously, if $a > 1$, $\{\xi_n, n = 1,2,\ldots\}$ is stochastically decreasing, i.e.

$$\xi_n \geqslant_{st} \xi_{n+1}, \quad n = 1,2,\ldots$$

If $0 < a < 1$, $\{\xi_n, n = 1,2,\ldots\}$ is stochastically increasing, i.e.

$$\xi_n \leqslant_{st} \xi_{n+1}, \quad n = 1,2,\ldots.$$

If $a = 1$, the geometric process reduces to a renewal process.

If $E\xi_1 = \lambda$, we have $E\xi_n = \frac{\lambda}{a^{n-1}}, \; n = 1,2,\ldots.$

2. Model assumptions

We study a repair model for a deteriorating system with multi-failure modes by making the following assumptions.

Assumption 1. At the beginning, a new system is installed. Whenever the system fails, it will be repaired or replaced by a new and identical one.

Assumption 2. The system has $k + 1$ different failure modes. The $(k + 1)$th failure mode is the unrepairable catastrophic failure mode, and the rest modes are repairable non-catastrophic failure modes. Let S_n be the type of nth failure, clearly $S_n \in \{1,2,\ldots,k,k + 1\}$, $n = 1,2,\ldots$. The occurrences of these failure modes are stochastic and mutually exclusive.

$$
\begin{array}{l}
\quad X_1 \qquad Y_1 \qquad X_2 \qquad Y_2 \qquad\qquad X_n \qquad Y_n \qquad X_{n+1} \qquad Y_{n+1} \\
-----|\times\times|----|\times\times\times|\cdots|---|\times\times\times\times|--|\times\times\times\times\times|\cdots \\
\qquad\quad S_1 \qquad\qquad S_2 \qquad\qquad\quad S_n \qquad\qquad S_{n+1} \\
\qquad --\text{: working state;} \quad \times\times\text{: repair state}
\end{array}
$$

Fig. 1. A possible course of the system.

Assumption 3. Assume that the non-catastrophic failures occur with probability p and the catastrophic failure occurs with probability $q = 1 - p$. When one of the non-catastrophic failures occurs, the failed system cannot be repaired "as good as new". The system will be replaced by a new and identical one after the number of repairable failures reaches N or the catastrophic failure occurs. The time interval between the completion of the $(n-1)$ th repair and the completion of the nth repair of the system is defined as the nth cycle of the system, $n = 1, 2, \ldots$. Let X_n, Y_n and Z be respectively the working time after $(n-1)$ th repair, the repair time after nth repairable failure and the replacement time of the system, $n = 1, 2, \ldots$ (see Fig. 1), and let $EX_1 = \lambda > 0$, $EY_1 = \mu > 0$ and $EZ = \nu > 0$. Further assume that the system will be in ith type failure with probability p_i, i.e. $p_i = P(S_n = i)$, $i = 1, 2, \ldots, k, k+1$. Let M_{n-1} denote the number of the first $n-1$ failures that are respectively mode 1, mode 2, ..., mode k, $n = 1, 2, \ldots$. To determine the distribution functions of X_n and Y_n, we first introduce the following probabilities and conditional probabilities:

$$P(X_n \leqslant t, S_n = i | M_{n-1} = (\alpha_1, \alpha_2, \ldots, \alpha_k)) \equiv U_i\left(a_1^{\alpha_1} a_2^{\alpha_2} \ldots a_k^{\alpha_k} t\right), \tag{1}$$

where $i = 1, 2, \ldots, k, k+1$; $1 \leqslant a_1 \leqslant a_2 \leqslant \cdots \leqslant a_k$; while $\alpha_1, \alpha_2, \ldots, \alpha_k$ respectively indicate the occurrence number of the mode 1, mode 2, ..., mode k in the first $n-1$ failures, and $\alpha_1 + \alpha_2 + \cdots + \alpha_k = n-1$; $n = 1, 2, \ldots$.

Similarly,

$$P(Y_n \leqslant t, S_n = i | M_n = (\alpha_1, \alpha_2, \ldots, \alpha_k)) \equiv V_i\left(b_1^{\alpha_1} b_2^{\alpha_2} \ldots b_k^{\alpha_k} t\right), \tag{2}$$

where $i = 1, 2, \ldots, k$; $1 \geqslant b_1 \geqslant b_2 \geqslant \cdots \geqslant b_k > 0$; while $\alpha_1, \alpha_2, \ldots, \alpha_k$ are respectively indicate the occurrence number of the mode 1, mode 2, ..., mode k in the first n failures, and $\alpha_1 + \alpha_2 + \cdots + \alpha_k = n$; $n = 1, 2, \ldots$.

Assumption 4. X_n, Y_n, Z, $n = 1, 2, \ldots$ are independent.

Assumption 5. Assume that the repair cost rate of the system is c_r, the working reward rate of the system is c_w, and the replacement cost of the system comprises two parts: one part is the basic replacement cost C, the other part is the cost proportional to the length of replacement time Z at rate c_0.

Assumption 6. The extended replacement policy N is adopted by which the system will be replaced whenever the number of repairable failures reaches N or the unrepairable failure occurs, whichever occurs first.

Remarks:

Note that a_i indicates the impact on the lifetime of the system by each occurrence of ith type failure, and b_i denotes the impact on the repair time of the system by each occurrence of ith type failure. Thus, we can always arrange the k repairable failure modes in the light of failure ponderance. The conditions $1 \leqslant a_1 \leqslant a_2 \leqslant \cdots \leqslant a_k$ and $1 \geqslant b_1 \geqslant b_2 \geqslant \cdots \geqslant b_k > 0$ mean that the failure ponderance is increasing in i ($i = 1,2,\ldots,k$), i.e. $(i+1)$ th type failure is more serious than ith type failure for $i = 1, 2, \ldots, k-1$. Obviously, 1st type repairable failure mode is a failure mode with the lowest ponderance, so that for $i > 1$, we have

$$X_2 | S_1 = 1 \geqslant_{st} X_2 | S_1 = i, \quad Y_1 | S_1 = 1 \leqslant_{st} Y_1 | S_1 = i$$

while kth type repairable failure mode is a failure mode with the highest ponderance. Thus, it is logical that we define $(k+1)$ th type failure mode as an unrepairable failure (catastrophic failure) mode.

In the Eqs. (1) and (2), α_i represents the number of occurrence of ith type failure mode among the first $n-1$ failures. In other words, the conditional probability distributions of X_n and Y_n are both relational with failure mode i. We can also find that the distribution functions of X_n and Y_n are both relational with failure mode i in next section.

3. System analysis

To determine the distribution functions of X_n and Y_n, first of all, using the Eqs. (1) and (2), when $n = 1$, we have

$$P(X_1 \leqslant t, S_1 = i) = U_i(t), \quad (i = 1, 2, \ldots, k, k+1), \tag{3}$$

$$P(Y_1 \leqslant t, S_1 = i) = V_i(t), \quad (i = 1, 2, \ldots, k). \tag{4}$$

In general, we have

$$P(X_n \leqslant t|M_{n-1} = (\alpha_1,\alpha_2,\ldots,\alpha_k)) = \sum_{i=1}^{k+1} P(X_n \leqslant t, S_n = i|M_{n-1} = (\alpha_1,\alpha_2,\ldots,\alpha_k)) = \sum_{i=1}^{k+1} U_i(a_1^{\alpha_1}a_2^{\alpha_2}\ldots a_k^{\alpha_k}t), (n = 1,2,\ldots). \tag{5}$$

Similarly,

$$P(Y_n \leqslant t|M_n = (\alpha_1,\alpha_2,\ldots,\alpha_k)) = \sum_{i=1}^{k} V_i(b_1^{\alpha_1}b_2^{\alpha_2}\ldots b_k^{\alpha_k}t), \quad (n = 1,2,\ldots). \tag{6}$$

Now, according to the formula of total probability and a conclusion of the multinomial distribution, we can respectively obtain the distribution functions of X_n and Y_n.

$$\begin{aligned} F_n(t) = P(X_n \leqslant t) &= \sum_{\sum_{j=1}^k \alpha_j = n-1} P(X_n \leqslant t|M_{n-1} = (\alpha_1,\alpha_2,\ldots,\alpha_k))P(M_{n-1} = (\alpha_1,\alpha_2,\ldots,\alpha_k)) \\ &= \sum_{\sum_{j=1}^k \alpha_j = n-1} \sum_{i=1}^{k+1} U_i(a_1^{\alpha_1}a_2^{\alpha_2}\ldots a_k^{\alpha_k}t)P(M_{n-1} = (\alpha_1,\alpha_2,\ldots,\alpha_k)) \\ &= \sum_{i=1}^{k+1} \sum_{\sum_{j=1}^k \alpha_j = n-1} U_i(a_1^{\alpha_1}a_2^{\alpha_2}\ldots a_k^{\alpha_k}t)\frac{(n-1)!}{\alpha_1!\alpha_2!\ldots\alpha_k!}p_1^{\alpha_1}p_2^{\alpha_2}\ldots p_k^{\alpha_k}, \quad (n = 1,2,\ldots). \end{aligned} \tag{7}$$

Similarly,

$$G_n(t) = P(Y_n \leqslant t) = \sum_{i=1}^{k} \sum_{\sum_{j=1}^k \alpha_j = n} V_i(b_1^{\alpha_1}b_2^{\alpha_2}\ldots b_k^{\alpha_k}t)\frac{n!}{\alpha_1!\alpha_2!\ldots\alpha_k!}p_1^{\alpha_1}p_2^{\alpha_2}\ldots p_k^{\alpha_k}, (n = 1,2,\ldots). \tag{8}$$

Now, for such a deteriorating system with multi-failure modes in this paper, we can obtain the following two important conclusions, but their proofs are given in the Appendix.

Theorem 1.

$$EX_n = \lambda\left(\frac{p_1}{a_1} + \frac{p_2}{a_2} + \cdots + \frac{p_k}{a_k}\right)^{n-1}, \quad n = 1,2,\ldots, \tag{9}$$

where $\lambda = EX_1 = \sum_{i=1}^{k+1}\lambda_i$, *and* $\lambda_i \equiv \int_0^\infty t dU_i(t)$.

$$EY_n = \mu\left(\frac{p_1}{b_1} + \frac{p_2}{b_2} + \cdots + \frac{p_k}{b_k}\right)^{n}, \quad n = 1,2,\ldots, \tag{10}$$

where $\mu = EY_1 = \sum_{i=1}^{k}\mu_i$, *and* $\mu_i \equiv \int_0^\infty t dV_i(t)$.

Theorem 2. *For* $n = 1, 2, \ldots$ *and any real* $t > 0$, *then*

$$P(X_n > t) \geqslant P(X_{n+1} > t), \quad P(Y_n > t) \leqslant P(Y_{n+1} > t). \tag{11}$$

In other words, we have

$$X_n \geqslant_{st} X_{n+1}, \quad Y_n \leqslant_{st} Y_{n+1}.$$

Theorem 1 gives the expectations of X_n *and* Y_n *(*$n = 1,2,\ldots$*). It is very important for calculating the average cost rate of the system.*

Theorem 2 shows that $\{X_n, n = 1,2,\ldots\}$ *is stochastically decreasing while* $\{Y_n, n = 1,2,\ldots\}$ *is stochastically increasing. Thus, the repair model for the deteriorating system with multi-failure modes in this paper is a general monotone process repair model.*

For the following writing, we denote respectively

$$A_k = \left(\sum_{i=1}^{k}\frac{p_i}{a_i}\right)^{-1}, \quad B_k = \left(\sum_{i=1}^{k}\frac{p_i}{b_i}\right)^{-1}. \tag{12}$$

To understand the meaning of A_k and B_k in Eq. (12), we now introduce the concept of harmonic mean of a discrete random variable.

Definition 3. Given a random variable X with $E(\frac{1}{X}) \neq 0$, define $m_H = \frac{1}{E(\frac{1}{X})}$ as the harmonic mean of X.

Based on Definition 3, we note the following conclusions:

(1) If X is a discrete random variable such that $X = a_i$ with probability p_i for $i = 1, 2, \ldots, k$, then $m_H = \left(\sum_{i=1}^{k} \frac{p_i}{a_i}\right)^{-1}$ is the harmonic mean of $a_1, a_2, \ldots, a_k$.
(2) If $\alpha < X < \beta$, then $\alpha < m_H < \beta$.

Based on Eq. (12), we can interpret A_k and B_k as the harmonic means of $a_1, a_2, \ldots, a_k$ and $b_1, b_2, \ldots, b_k$, respectively. Based on conclusion (2) given above, we have

$$1 \leqslant a_1 \leqslant A_k \leqslant a_k, \quad 0 < b_k \leqslant B_k \leqslant b_1. \tag{13}$$

4. Average cost rate under extended policy N

Now, we use an extended replacement policy (or an extended policy) N by which the system will be replaced whenever the number of repairable failures of the system reaches N or the unrepairable failure first occurs before the repairable failure numbers N of the system, whichever occurs first. Our objective is to determine an optimal extended policy N^* such that the average cost rate is minimized. Let τ_1 be the first replacement time of the system under extended policy N. Let τ_n $(n \geqslant 2)$ be the time between the $(n-1)$ th replacement and the nth replacement of the system under extended policy N. Obviously, $\{\tau_1, \tau_2, \ldots\}$ forms a renewal process, while the interarrival time between two consecutive replacements is called a renewal cycle of the system.

Let $C(N)$ be the average cost rate of the system under extended policy N. According to renewal reward theorem (see, e.g. Ross [17]), we have

$$C(N) = \frac{\text{the expected cost incurred in a renewal cycle}}{\text{the expected length of the renewal cycle}}. \tag{14}$$

Now, let W be the length of a renewal cycle of the system under extended policy N. According to Assumptions 2 and 3, then

$$W = \left(\sum_{j=1}^{N} X_j + \sum_{j=1}^{N-1} Y_j + Z\right) I_A + \sum_{l=1}^{N} \left(\sum_{j=1}^{l} X_j + \sum_{j=1}^{l-1} Y_j + Z\right) I_{A_l}, \tag{15}$$

where the first term shows that the catastrophic failure does not occur before the number of repairable failures of the system reaching N; the second term shows that the catastrophic failure occurs before the number of repairable failures of the system reaching l while $l = 1, 2, \ldots, N$; so $A = \{S_1 \neq k+1, S_2 \neq k+1, \ldots, S_N \neq k+1\}$ while $A_l = \{S_1 \neq k+1, S_2 \neq k+1, \ldots, S_{l-1} \neq k+1, S_l = k+1\}$ and let $\overline{A} = \sum_{l=1}^{N} A_l$; I is an indicator function such that

$$I_D = \begin{cases} 1, & \text{if event } D \text{ occurs;} \\ 0, & \text{if event } D \text{ does not occur.} \end{cases}$$

Thus, the expectations of the indicator functions I_A and $I_{\overline{A}}$ are respectively given by

$$EI_A = P(A) = p^N, \quad EI_{\overline{A}} = P(\overline{A}) = P\left(\sum_{l=1}^{N} A_l\right) = \sum_{l=1}^{N} p^{l-1}(1-p), \quad 0 \leqslant p \leqslant 1. \tag{16}$$

Obviously, we have under extended policy N

$$p^N + \sum_{l=1}^{N} p^{l-1}(1-p) = 1.$$

According to Assumption 3, clearly $q = p_{k+1}$. Thus, whenever the system fails at a time we have

$$p + q = (p_1 + p_2 + \cdots + p_k) + p_{k+1} = 1.$$

According to assumptions, Theorem 1 and Eqs. (12)–(16), the average cost rate of the system under extended policy N is given by

$$
\begin{aligned}
C(N) &= \frac{\begin{bmatrix} E\Big[\Big(c_r\sum_{j=1}^{N-1}Y_j - c_w\sum_{j=1}^{N}X_j + C + c_0 Z\Big)I_A + \\ \Big(c_r\sum_{j=1}^{l-1}Y_j - c_w\sum_{j=1}^{l}X_j + C + c_0 Z\Big)I_{\bar{A}}\Big] \end{bmatrix}}{E\Big[\Big(\sum_{j=1}^{N}X_j + \sum_{j=1}^{N-1}Y_j + Z\Big)I_A + \Big(\sum_{j=1}^{l}X_j + \sum_{j=1}^{l-1}Y_j + Z\Big)I_{\bar{A}}\Big]} \\
&= \frac{\begin{bmatrix} \Big[c_r\sum_{j=1}^{N-1}\frac{\mu}{B_k^{j-1}} - c_w\sum_{j=1}^{N}\frac{\lambda}{A_k^{j-1}} + C + c_0 v\Big]p^N + \\ \sum_{l=1}^{N}\Big[c_r\sum_{j=1}^{l-1}\frac{\mu}{B_k^{j-1}} - c_w\sum_{j=1}^{l}\frac{\lambda}{A_k^{j-1}} + C + c_0 v\Big]p^{l-1}(1-p) \end{bmatrix}}{\Big[\sum_{j=1}^{N}\frac{\lambda}{A_k^{j-1}} + \sum_{j=1}^{N-1}\frac{\mu}{B_k^{j-1}} + v\Big]p^N + \sum_{l=1}^{N}\Big[\sum_{j=1}^{l}\frac{\lambda}{A_k^{j-1}} + \sum_{j=1}^{l-1}\frac{\mu}{B_k^{j-1}} + v\Big]p^{l-1}(1-p)} \\
&= \frac{\Big[c_r\sum_{j=1}^{N-1}\frac{\mu}{B_k^{j-1}} - c_w\sum_{j=1}^{N}\frac{\lambda}{A_k^{j-1}}\Big]p^N + \sum_{l=1}^{N}\Big[c_r\sum_{j=1}^{l-1}\frac{\mu}{B_k^{j-1}} - c_w\sum_{j=1}^{l}\frac{\lambda}{A_k^{j-1}}\Big]p^{l-1}q + C + c_0 v}{\Big[\sum_{j=1}^{N}\frac{\lambda}{A_k^{j-1}} + \sum_{j=1}^{N-1}\frac{\mu}{B_k^{j-1}}\Big]p^N + \sum_{l=1}^{N}\Big[\sum_{j=1}^{l}\frac{\lambda}{A_k^{j-1}} + \sum_{j=1}^{l-1}\frac{\mu}{B_k^{j-1}}\Big]p^{l-1}q + v}. \qquad (17)
\end{aligned}
$$

When $p = 1$, this system will become a deteriorating system with k distinct repairable failure modes. We denote the average cost rate of the system under the policy N by $C_1(N)$, then

$$
C_1(N) = \frac{c_r\sum_{j=1}^{N-1}\frac{\mu}{B_k^{j-1}} - c_w\sum_{j=1}^{N}\frac{\lambda}{A_k^{j-1}} + C + c_0 v}{\sum_{j=1}^{N}\frac{\lambda}{A_k^{j-1}} + \sum_{j=1}^{N-1}\frac{\mu}{B_k^{j-1}} + v}. \qquad (18)
$$

The optimal replacement policy N^* can be respectively determined by minimizing $C(N)$ and $C_1(N)$. Because the complicacy for the expression of the average cost rate $C(N)$, it is difficult to prove the existence of the optimal replacement policy N^* theoretically. However, we can show theoretically that the optimal replacement policy N^* is unique for minimizing $C_1(N)$. And in the following numerical example, we can see that the optimal replacement policy N^* can be numerically determined such that $C(N^*)$ is minimized.

5. Optimal replacement policy N^*

Now, our problem is to determine the optimal replacement policy N^* for minimizing $C_1(N)$ explicitly. For this purpose, first of all, we can rewrite the Eq. (18) as

$$
C_1(N) = A(N) - c_w,
$$

where

$$
A(N) = \frac{(c_r + c_w)\sum_{j=1}^{N-1}\frac{\mu}{B_k^{j-1}} + C + (c_0 + c_w)v}{\sum_{j=1}^{N}\frac{\lambda}{A_k^{j-1}} + \sum_{j=1}^{N-1}\frac{\mu}{B_k^{j-1}} + v}.
$$

Thus, to minimize $C_1(N)$ is equivalent to minimize $A(N)$. Now, we study the difference of $A(N+1)$ and $A(N)$:

$$
\begin{aligned}
A(N+1) - A(N) &= \frac{(c_r + c_w)\sum_{j=1}^{N}\frac{\mu}{B_k^{j-1}} + C + (c_0 + c_w)v}{\sum_{j=1}^{N+1}\frac{\lambda}{A_k^{j-1}} + \sum_{j=1}^{N}\frac{\mu}{B_k^{j-1}} + v} - \frac{(c_r + c_w)\sum_{j=1}^{N-1}\frac{\mu}{B_k^{j-1}} + C + (c_0 + c_w)v}{\sum_{j=1}^{N}\frac{\lambda}{A_k^{j-1}} + \sum_{j=1}^{N-1}\frac{\mu}{B_k^{j-1}} + v} \\
&= \frac{(c_r + c_w)\frac{\mu}{B_k^{N-1}}\sum_{j=1}^{N}B_k^{j-1} + C + (c_0 + c_w)v}{\frac{\lambda}{A_k^{N}}\sum_{j=1}^{N+1}A_k^{j-1} + \frac{\mu}{B_k^{N-1}}\sum_{j=1}^{N}B_k^{j-1} + v} - \frac{(c_r + c_w)\frac{\mu}{B_k^{N-2}}\sum_{j=1}^{N-1}B_k^{j-1} + C + (c_0 + c_w)v}{\frac{\lambda}{A_k^{N-1}}\sum_{j=1}^{N}A_k^{j-1} + \frac{\mu}{B_k^{N-2}}\sum_{j=1}^{N-1}B_k^{j-1} + v} \\
&= \frac{(c_r + c_w)\mu\Big[\lambda h(N) + vA_k^N\Big] - [C + (c_0 + c_w)v]\Big(\lambda B_k^{N-1} + \mu A_k^N\Big)}{A_k^N B_k^{N-1}\Big[\frac{\lambda}{A_k^N}\sum_{j=1}^{N+1}A_k^{j-1} + \frac{\mu}{B_k^{N-1}}\sum_{j=1}^{N}B_k^{j-1} + v\Big]\Big[\frac{\lambda}{A_k^{N-1}}\sum_{j=1}^{N}A_k^{j-1} + \frac{\mu}{B_k^{N-2}}\sum_{j=1}^{N-1}B_k^{j-1} + v\Big]},
\end{aligned}
$$

where $h(N) = \sum_{j=1}^{N}A_k^j - \sum_{j=1}^{N-1}B_k^j$.

According to the numerator of $A(N+1) - A(N)$, we structure an auxiliary function

$$
B(N) = \frac{(c_r + c_w)\mu\Big[\lambda h(N) + vA_k^N\Big]}{[C + (c_0 + c_w)v]\Big(\lambda B_k^{N-1} + \mu A_k^N\Big)}. \qquad (19)
$$

Because the denominator of $A(N+1) - A(N)$ is always positive, the sign of $A(N+1) - A(N)$ is the same as the sign of its numerator. Thus, the following lemma is straightforward.

Lemma 1.

$$A(N+1) \gtreqless A(N) \Longleftrightarrow B(N) \gtreqless 1$$

Lemma 1 shows that the monotonicity of $A(N)$ is determined by the value of $B(N)$. Now, we consider the difference of $B(N+1)$ and $B(N)$, and obtain the following result through calculation and simplification.

$$B(N+1) - B(N) = \frac{(c_r + c_w)\mu\left[\lambda h(N+1) + \nu A_k^{N+1}\right]}{[C + (c_0 + c_w)\nu]\left(\lambda B_k^N + \mu A_k^{N+1}\right)} - \frac{(c_r + c_w)\mu\left[\lambda h(N) + \nu A_k^N\right]}{[C + (c_0 + c_w)\nu]\left(\lambda B_k^{N-1} + \mu A_k^N\right)}$$
$$= \frac{\left[\begin{array}{l}(c_r + c_w)\mu\lambda\left\{\lambda B_k^{N-1}[h(N+1) - B_k h(N)] + \nu B_k^{N-1} A_k^N (A_k - B_k)\right. \\ \left. +\mu\left[A_k^N h(N+1) - A_k^{N+1} h(N)\right]\right\}\end{array}\right]}{\left(\lambda B_k^N + \mu A_k^{N+1}\right)\left(\lambda B_k^{N-1} + \mu A_k^N\right)}.$$

According to the Eq. (13), we have

$$h(N+1) - B_k h(N) = \left(\sum_{j=1}^{N+1} A_k^j - \sum_{j=1}^{N} B_k^j\right) - B_k\left(\sum_{j=1}^{N} A_k^j - \sum_{j=1}^{N-1} B_k^j\right) = (1 - B_k)\sum_{j=1}^{N} A_k^j + \left(A_k^{N+1} - B_k\right) \geqslant 0,$$

$$A_k - B_k \geqslant 0,$$

$$A_k^N h(N+1) - A_k^{N+1} h(N) = A_k^N\left(\sum_{j=1}^{N+1} A_k^j - \sum_{j=1}^{N} B_k^j\right) - A_k^{N+1}\left(\sum_{j=1}^{N} A_k^j - \sum_{j=1}^{N-1} B_k^j\right)$$
$$= \left(A_k^N \sum_{j=1}^{N+1} A_k^j - A_k^{N+1}\sum_{j=1}^{N} A_k^j\right) + \left(A_k^{N+1}\sum_{j=1}^{N-1} B_k^j - A_k^N \sum_{j=1}^{N} B_k^j\right) = A_k^N\left(A_k - B_k^N\right) + A_k^N (A_k - 1)\sum_{j=1}^{N-1} B_k^j \geqslant 0.$$

Thus, $B(N+1) - B(N) \geqslant 0$. This implies:

Lemma 2. *$B(N)$ is nondecreasing in N.*

According to Lemmas 1 and 2, an analytical expression for an optimal policy for minimizing $A(N)$ or $C_1(N)$ can be got through the study of $B(N)$. Therefore, we have the following theorem:

Theorem 3. *The optimal replacement policy N^* can be determined by*

$$N^* = \min\{N | B(N) \geqslant 1\}. \tag{20}$$

Furthermore, if $B(N^*) > 1$, then the optimal policy N^* is unique.

Because $B(N)$ is nondecreasing in N, there exists an integer N^* such that

$$B(N) \geqslant 1 \Longleftrightarrow N \geqslant N^*$$

and

$$B(N) < 1 \Longleftrightarrow N < N^*.$$

Note that N^* is the minimum satisfying (20), and the policy N^* is an optimal replacement policy. Furthermore, it is easy to see that if $B(N^*) > 1$, then the optimal policy is also uniquely existent.

6. A numerical example

In this section, a numerical example for a deteriorating repairable system with three failure modes, including two repairable failure (non-catastrophic failure) modes and one unrepairable failure (catastrophic failure) mode, is given to illustrate some theory results in which an optimal extended policy N^* is determined by minimizing $C(N)$. According to Theorem 1 and the Eq. (12), we have

$$EX_n = \lambda\left(\frac{p_1}{a_1} + \frac{p_2}{a_2}\right)^{n-1}, \quad \lambda = \sum_{i=1}^{3} \lambda_i; \quad n = 1, 2, \ldots,$$

$$EY_n = \mu\left(\frac{p_1}{b_1} + \frac{p_2}{b_2}\right)^{n}, \quad \mu = \sum_{i=1}^{2} \mu_i; \quad n = 1, 2, \ldots,$$

$$A_2 = \left(\frac{p_1}{a_1} + \frac{p_2}{a_2}\right)^{-1}, \quad B_2 = \left(\frac{p_1}{b_1} + \frac{p_2}{b_2}\right)^{-1}.$$

In fact, the Eq. (17) will become

$$C(N) = \frac{\left[c_r \sum_{j=1}^{N-1} \frac{\mu}{B_2^{j-1}} - c_w \sum_{j=1}^{N} \frac{\lambda}{A_2^{j-1}}\right] p^N + \sum_{l=1}^{N} \left[c_r \sum_{j=1}^{l-1} \frac{\mu}{B_2^{j-1}} - c_w \sum_{j=1}^{l} \frac{\lambda}{A_2^{j-1}}\right] p^{l-1} q + C + c_0 v}{\left[\sum_{j=1}^{N} \frac{\lambda}{A_2^{j-1}} + \sum_{j=1}^{N-1} \frac{\mu}{B_2^{j-1}}\right] p^N + \sum_{l=1}^{N} \left[\sum_{j=1}^{l} \frac{\lambda}{A_2^{j-1}} + \sum_{j=1}^{l-1} \frac{\mu}{B_2^{j-1}}\right] p^{l-1} q + v}, \tag{21}$$

where $\lambda = \sum_{i=1}^{3} \lambda_i;\ \mu = \sum_{i=1}^{2} \mu_i;\ p = p_1 + p_2;\ q = p_3$ and $p + q = p_1 + p_2 + p_3 = 1$.

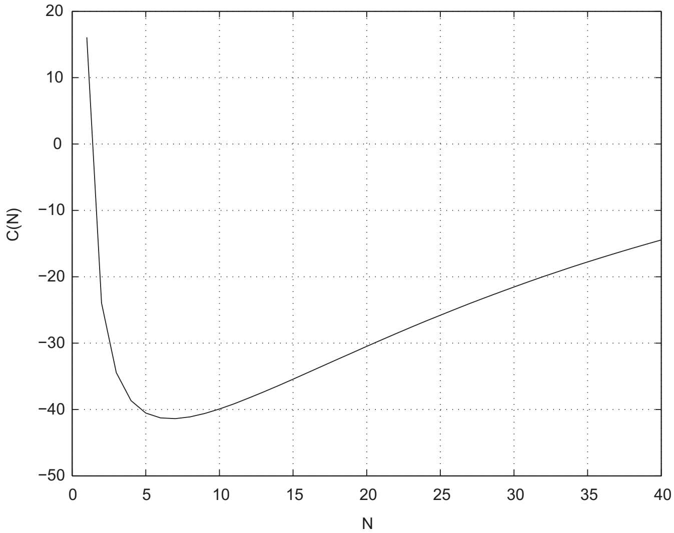

Fig. 2. The plot of average cost rate $C(N)$ against N.

Table 1
Some results obtained from the average cost rate $C(N)$, where N = 7 is the optimal replacement policy with minimal average cost rate -41.3786.

N	$C(N)$	N	$C(N)$	N	$C(N)$	N	$C(N)$
1	16.0870	11	−39.1379	21	−29.5205	31	−20.7413
2	−23.9646	12	−38.2750	22	−28.5640	32	−19.9683
3	−34.4452	13	−37.3592	23	−27.6225	33	−19.2146
4	−38.6624	14	−36.4078	24	−26.6973	34	−18.4802
5	−40.5301	15	−35.4335	25	−25.7896	35	−17.7647
6	−41.2674	16	−34.4460	26	−24.9001	36	−17.0679
7	−**41.3786**	17	−33.4528	27	−24.0295	37	−16.3893
8	−41.1105	18	−32.4595	28	−23.1781	38	−15.7287
9	−40.5987	19	−31.4708	29	−22.3462	39	−15.0856
10	−39.9240	20	−30.4902	30	−21.5339	40	−14.4596

Table 2
Optimal N^* and $C(N^*)$ obtained for different values of a_i and b_i.

a_1	a_2	b_1 = 0.96, b_2 = 0.94		b_1	b_2	a_1 = 1.08, a_2 = 1.05	
		N^*	$C(N^*)$			N^*	$C(N^*)$
1.02	1.01	9	−46.2339	0.99	0.97	8	−43.0953
1.03	1.02	8	−45.1335	0.98	0.96	8	−42.4548
1.05	1.04	7	−43.1261	0.97	0.95	7	−41.9241
1.08	1.05	7	−41.3786	0.96	0.94	7	−41.3786
1.09	1.07	6	−40.1379	0.95	0.93	6	−40.8436
1.12	1.09	6	−38.3400	0.92	0.90	6	−39.5002
1.14	1.12	6	−36.6022	0.88	0.88	5	−38.3215
1.18	1.16	5	−34.2973	0.85	0.84	5	−37.0583
1.20	1.20	5	−32.6943	0.80	0.79	4	−35.5000
1.30	1.25	4	−29.1721	0.75	0.73	4	−34.0620
1.35	1.30	4	−27.2885	0.72	0.70	4	−33.1905

We study the numerical example with the following parameter values: $a_1 = 1.08$, $a_2 = 1.05$, $b_1 = 0.96$, $b_2 = 0.94$, $p_1 = 0.49$, $p_2 = 0.49$, $q = p_3 = 0.02$, $c_w = 100$, $c_r = 15$, $c_0 = 5$, $v = 8$, $\lambda_1 = 16$, $\lambda_2 = 18$, $\lambda_3 = 4$, $\mu_1 = 12$, $\mu_2 = 8$ and $C = 4500$. Substituting the above values into the Eq. (21), we can obtain these results presented in Fig. 2 and Table 1. It is easy to find that $C(7) = -41.3786$ is the minimum of the average cost rate of the system. In other words, the optimal policy is $N^* = 7$, and we should replace the system when the extended policy reaches 7. We can see from Fig. 2 or Table 1 that the optimal extended policy N^* is unique.

To study the influence of the ratios of the geometric process on the optimal solution, we tabulate the optimal replacement policy N^* and the minimum $C(N^*)$ for different values of a_i, b_i $(i = 1,2)$ in Table 2, respectively.

In Table 2, when b_i $(i = 1,2)$ and other parameters are fixed, then N^* is nonincreasing in a_i $(i = 1,2)$, but $C(N^*)$ is increasing in a_i $(i = 1,2)$; when a_i $(i = 1,2)$ and other parameters are fixed, then N^* is nondecreasing in b_i $(i = 1,2)$, but $C(N^*)$ is decreasing in b_i $(i = 1,2)$. According to Table 2, it is easy to find that the optimal replacement policy N^* and the minimum $C(N^*)$ are sensitive to the tiny change of a_i or b_i $(i = 1,2)$, when the other parameters are fixed. At the same time, we can see that the optimal solution (N^* and $C(N^*)$) is uniquely existent for every group of parameters.

7. Discussion

For the repair model proposed in this paper, we can arrive at the following comparisons.

(1) According to Theorem 2, the repair model in this paper is a general monotone process repair model for a deteriorating system with multi-failure modes. Obviously, the general monotone process repair model will become a renewal process repair model for a repairable system with one failure mode when all $a_i = b_i = 1$.

(2) When $p = 1$ (i.e. $q = 0$), the repair model in this paper will reduce a general monotone process repair model for a deteriorating system with k distinct repairable failure modes. Thus, the Eq. (17) will become the Eq. (18).
If $C_1(N)$ denotes the long-run expected profit per unit time of the system, and let c_r, c_w, C, c_0, A_k and B_k in the Eq. (18) be $c_2, c_1, c_4, c_3, \frac{1}{A_k}$ and $\frac{1}{B_k}$, respectively, then the Eq. (18) will become the Eq. (20) in Zhang et al. [28].

(3) When $q = 1$ (i.e. $p = 0$), the system in this paper will be replaced as soon as it fails, namely the optimal policy is not considered. Thus, we can see that the different q will change the optimal solutions (N^* and $C(N^*)$) of the system. In practice application, how to select $q(0 < q < 1)$ is also very important for the deviser and manager of system engineering.

(4) When $p = 1$ and $a_1 = a_2 = \cdots = a_k \triangleq a$; $b_1 = b_2 = \cdots = b_k \triangleq b$; there is only one repairable failure mode, i.e. $i = 1$, then the general monotone process repair model for the deteriorating system in this paper will become a geometric process repair model. According to the Eq. (7), when $n = 1$ we have

$$F_1(t) = P(X_1 \leqslant t) = U_1(t) \equiv F(t).$$

When $n = 2$ we have

$$F_2(t) = P(X_2 \leqslant t) = \sum_{\alpha_1=0}^{1} U_1(at) p_1^{\alpha_1} (1 - p_1)^{1-\alpha_1} = U_1(at) \equiv F(at).$$

Generally, we have

$$F_n(t) = P(X_n \leqslant t) = \sum_{\alpha_1=0}^{n-1} U_1(a^{n-1}t) \frac{(n-1)}{\alpha_1![(n-1)-\alpha_1]!} p_1^{\alpha_1} (1 - p_1)^{(n-1)-\alpha_1} = U_1(a^{n-1}t) \equiv F(a^{n-1}t).$$

Similarly, using Eq. (8) we have

$$G_n(t) = P(Y_n \leqslant t) = V_1(b^{n-1}t) \equiv F(b^{n-1}t).$$

Thus, the geometric process repair model is a special case of the general monotone process repair model in this paper. Here

$$A_k = \left(\sum_{i=1}^{k} \frac{p_i}{a_i} \right)^{-1} = a, \quad B_k = \left(\sum_{i=1}^{k} \frac{p_i}{b_i} \right)^{-1} = b.$$

Then the Eq. (17) or (18) will become

$$C(N) = \frac{c_r \sum_{j=1}^{N-1} \frac{\mu}{b^{j-1}} - c_w \sum_{j=1}^{N} \frac{\lambda}{a^{j-1}} + C + c_0 v}{\sum_{j=1}^{N} \frac{\lambda}{a^{j-1}} + \sum_{j=1}^{N-1} \frac{\mu}{b^{j-1}} + v}. \tag{22}$$

If the replacement time is negligible, and let c_r, c_w and C in the Eq. (22) be C,1 and C_2 respectively, then the Eq. (22) will become the Eq. (5.1) in Lam [7].

(5) Further, we can obtain the following conclusion: when $p = 1$, the general monotone process repair model for a deteriorating system with multi-failure modes is equivalent to the geometric process repair model of a one-failure-mode system. The successive working times of the one-failure-mode system $\{X'_n,\ n = 1, 2, \ldots\}$ form a geometric process with rate $A_k \geqslant 1$, the harmonic mean of $a_1, a_2, \ldots, a_k$ while the consecutive repair times of one-failure-mode system $\{Y'_n,\ n = 1, 2, \ldots\}$ constitute a geometric process with rate $0 < B_k \leqslant 1$, the harmonic mean of $b_1, b_2, \ldots, b_k$. In other words, the general monotone process repair model for a system with multi-failure modes can be interpreted as an equivalent geometric process repair model for a one-failure-mode system in the sense that they have the same objective function for the average cost rate $C(N)$ and the same optimal policy N^*. The only difference would be the parameter values.

8. Concluding remarks

In this paper, a repair model for a deteriorating system with $k + 1$ failure modes including one unrepairable failure mode and k repairable failure modes, corresponding k distinct types is studied. Under the extended replacement policy N, the optimal policy N^* is theoretically obtained by minimizing the average cost rate $C_1(N)$. We also provide an example shown in Fig. 2, Tables 1 and 2 to illustrate not only the uniqueness of the optimal policy N^* but also the optimal policy N^* and the minimums $C(N^*)$ are sensitive to the tiny changes of a_i and $b_i (a_i = b_i \neq 1;\ i = 1,2)$, when the other parameters are fixed.

This paper shows that the repair model proposed is a general monotone process repair model, and it is a generalization of existing models, including the geometric process repair model and the general monotone process repair model without catastrophic failure. Therefore, the repair model in this paper should have not only interesting in reliability theory, but also valuable in reliability engineering.

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Appendix

Proof of Theorem 1. According to the Eqs. (7) and (8), when $n = 1$ we have

$$F_1(t) = P(X_1 \leqslant t) = \sum_{i=1}^{k+1} U_i(t),$$

$$G_1(t) = P(Y_1 \leqslant t) = \sum_{i=1}^{k} V_i(t).$$

To start with, let

$$EX_1 = \int_0^\infty t dF_1(t) = \lambda, \quad EY_1 = \int_0^\infty t dG_1(t) = \mu.$$

Then by using the Eq. (7), we have

$$\begin{aligned}
E(X_n) &= \int_0^\infty t dF_n(t) = \sum_{i=1}^{k+1} \sum_{\sum_{j=1}^{k} \alpha_j = n-1} \frac{(n-1)!}{\alpha_1! \alpha_2! \ldots \alpha_k!} \times p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k} \int_0^\infty t dU_i(a_1^{\alpha_1} a_2^{\alpha_2} \ldots a_k^{\alpha_k} t) \\
&= \sum_{i=1}^{k+1} \sum_{\sum_{j=1}^{k} \alpha_j = n-1} \frac{(n-1)!}{\alpha_1! \alpha_2! \ldots \alpha_k!} \times \frac{p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}}{a_1^{\alpha_1} a_2^{\alpha_2} \ldots a_k^{\alpha_k}} \int_0^\infty u dU_i(u) \\
&= \sum_{i=1}^{k+1} \lambda_i \sum_{\sum_{j=1}^{k} \alpha_j = n-1} \frac{(n-1)!}{\alpha_1! \alpha_2! \ldots \alpha_k!} \left(\frac{p_1}{a_1}\right)^{\alpha_1} \left(\frac{p_2}{a_2}\right)^{\alpha_2} \ldots \left(\frac{p_k}{a_k}\right)^{\alpha_k} \\
&= \lambda \left(\frac{p_1}{a_1} + \frac{p_2}{a_2} + \cdots + \frac{p_k}{a_k}\right)^{n-1}, \quad n = 1, 2, \ldots.
\end{aligned}$$

Therefore, the Eq. (9) is derived. Similarly, we can obtain the Eq. (10).

Proof of Theorem 2. Using the distribution function of X_n and the property of the multinomial distribution, we have

$$
\begin{aligned}
P(X_{n+1} \leqslant t) &= \sum_{i=1}^{k+1} \sum_{\sum_{j=1}^{k} \alpha_j = n} U_i\left(a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_k^{\alpha_k} t\right) \times \frac{n!}{\alpha_1! \alpha_2! \ldots \alpha_k!} \times p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k} \\
&= \sum_{i=1}^{k+1} \sum_{\sum_{j=1}^{k} \alpha_j = n} \left(\sum_{l=1}^{k} \frac{(n-1)! \alpha_l}{\alpha_1! \alpha_2! \cdots \alpha_k!} \times p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k} U_i\left(a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_k^{\alpha_k} t\right) \right) \\
&= \sum_{i=1}^{k+1} \sum_{\sum_{j=1}^{k} \alpha_j = n} \frac{(n-1)!}{(\alpha_1 - 1)! \alpha_2! \ldots \alpha_k!} \times p_1^{\alpha_1 - 1} p_2^{\alpha_2} \ldots p_k^{\alpha_k} p_1 U_i(a_1 a_1^{\alpha_1 - 1} a_2^{\alpha_2} \cdots a_k^{\alpha_k} t) \\
&\quad + \sum_{i=1}^{k+1} \sum_{\sum_{j=1}^{k} \alpha_j = n} \frac{(n-1)!}{\alpha_1! (\alpha_2 - 1)! \ldots \alpha_k!} \times p_1^{\alpha_1} p_2^{\alpha_2 - 1} \ldots p_k^{\alpha_k} p_2 U_i(a_2 a_1^{\alpha_1} a_2^{\alpha_2 - 1} \ldots a_k^{\alpha_k} t) + \cdots \\
&\quad + \sum_{i=1}^{k+1} \sum_{\sum_{j=1}^{k} \alpha_j = n} \frac{(n-1)!}{\alpha_1! \alpha_2! \ldots \alpha_{k-1} (\alpha_k - 1)!} \times p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k - 1} p_k U_i\left(a_k a_1^{\alpha_1} \ldots a_k^{\alpha_k - 1} t\right) \\
&= \sum_{i=1}^{k+1} \sum_{\sum_{j=1}^{k} \alpha_j = n-1} \frac{(n-1)!}{\alpha_1! \alpha_2! \ldots \alpha_k!} \times p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k} \left(\sum_{l=1}^{k} p_l U_i\left(a_l a_1^{\alpha_1} a_2^{\alpha_2} \ldots a_k^{\alpha_k} t\right) \right) \\
&\geqslant \sum_{i=1}^{k+1} \sum_{\sum_{j=1}^{k} \alpha_j = n-1} \frac{(n-1)!}{\alpha_1! \alpha_2! \ldots \alpha_k!} \times p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k} \left(\sum_{l=1}^{k} p_l U_i\left(a_1^{\alpha_1} a_2^{\alpha_2} \ldots a_k^{\alpha_k} t\right) \right) \\
&= \sum_{i=1}^{k+1} \sum_{\sum_{j=1}^{k} \alpha_j = n-1} \frac{(n-1)!}{\alpha_1! \alpha_2! \ldots \alpha_k!} \times p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k} U_i\left(a_1^{\alpha_1} \ldots a_k^{\alpha_k} t\right) = P(X_n \leqslant t).
\end{aligned}
$$

Therefore, we have P ($X_n > t$) $\geqslant$ P ($X_{n+1} > t$). Similarly, we can prove $P(Y_n > t) \leqslant P(Y_{n+1} > t)$.

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