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–Brief Paper–

DYNAMIC OUTPUT FEEDBACK CONSENSUALIZATION OF UNCERTAIN SWARM SYSTEMS WITH TIME DELAYS

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ABSTRACT

Consensus analysis and design problems of high-dimensional discrete-time swarm systems in directed networks with time delays and uncertainties are dealt with by using output information. Two subspaces are introduced, namely a consensus subspace and a complement consensus subspace. By projecting the state of a swarm system onto the two subspaces, a necessary and sufficient condition for consensus is presented, and based on different influences of time delays and uncertainties, an explicit expression of the consensus function is given which is very important in applications of swarm systems. A method to determine gain matrices of consensus protocols is proposed. Numerical simulations are presented to demonstrate theoretical results.

Key Words: Consensus, swarm system, uncertainty, time delay, dynamic output feedback.

I. INTRODUCTION

In the past few years, the research on swarm systems has attracted considerable attention from scientific communities. This is partly due to broad applications of swarm systems in various areas, such as physics (*e.g.*, collective motion of particles [1] and synchronization of networks [2,3]), engineering (*e.g.*, formation control [4,5]), and biology (*e.g.*, flocking [6]), *etc*.

In collective behaviors of swarm systems, consensus is one of the most interesting behaviors. In swarm systems, a number of agents can achieve an agreement over some variables of interest by local interactions. This problem is usually called a consensus problem. Actually, the motions of swarm systems consist of the relative motions among agents and the absolute motion as a whole. When the relative motions are asymptotically stable, then consensus is achieved. The key issue of consensus is to design a distributed control protocol for each agent based on information of its neighbors such that the whole swarm system can achieve consensus. Vicsek *et al*. [1] showed consensus behaviors of swarm systems by numerical simulations. A theoretical framework of consensus problems of swarm systems was presented in [7] and [8]. Ren [9] relaxed the conditions for consensus in [7] and [8], and pointed out that the communication topology having a spanning tree is critical for a swarm system to achieve consensus. In recent years, the study of consensus problems has developed fast, and many research results have been obtained. For example, consensus problems for swarm systems with time delays and/or uncertainties were discussed in [10–17], leader–follower consensus problems were dealt with in [18], fractional-order swarm systems were addressed in [19] and [20], stochastic consensus problems were investigated in [21] and [22], and high-dimensional consensus problems were considered in [23–35].

It is well-known that time delays and uncertainties may degrade the performance of control systems. In a swarm system, information delays appear naturally in the process of information transmission among agents and uncertainties exist due to the variations of the interaction strength. Based on linear matrix inequality (LMI) techniques, consensus problems for swarm systems with time delays and/or uncertainties are addressed in [10–13], where it is assumed that the dynamics of each agent is described by a first-order integrator. Wang *et al*. [23] give a sufficient condition for consensus of high-dimensional linear time-invariant (LTI) swarm systems with undirected communication topologies. A necessary and sufficient condition for consensus is given in [24], where it is supposed that the consensus function, which is the agreement state of each agent, is time-invariant. In [25–30], high-dimensional LTI swarm systems with time-varying consensus functions are dealt with.

In [23–30], all the states of neighboring agents are required when constructing the consensus protocols, but there exist the cases where each agent only obtains the outputs of its neighbors. Ma and Zhang [31] deal with consensus analysis and design problems via static output feedback consensus protocols. Dynamic output feedback consensus protocols are applied in [32–35], where time delays and uncertainties are not considered. Three important consensus problems are usually concerned in the literature: (i) What are the conditions to achieve consensus; (ii) How to determine consensus functions if consensus is achieved; (iii) How to design gain matrices of consensus protocols such that swarm systems achieve consensus. To the best of our knowledge, for high-dimensional discrete-time LTI swarm systems with dynamic output feedback consensus protocols including time delays and uncertainties, the above three consensus problems have not been investigated comprehensively.

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In the current paper, by using dynamic output feedback consensus protocols, both consensus analysis and design problems for high-dimensional discrete-time LTI swarm systems with time delays and uncertainties are studied and an approach is given to determine consensus functions. Firstly, a complex Euclidean space is decomposed into two subspaces with specific structures. By the state projection onto the two subspaces, consensus problems are transformed into asymptotic stability problems of reduced-order subsystems and a necessary and sufficient condition for consensus is given. Secondly, based on different influences of time delays and uncertainties, an explicit expression of the consensus function is shown. Finally, consensus design problems are addressed and a sufficient condition for consensualization is presented in terms of LMI techniques.

Compared with the existing works about consensus for high-dimensional LTI swarm systems, the current paper has the following three novel features. Firstly, in the current paper, both the consensus and disagreement parts are determined simultaneously, and a necessary and sufficient condition for consensus is presented. In [23], only the disagreement part was determined and a sufficient condition for consensus was given, where time delays and uncertainties were not dealt with. Secondly, the current paper shows an explicit expression of the consensus function and determines the impacts of the disagreement part on the consensus part. The approaches in [23] cannot do that when topologies are directed and time delays are considered. In [27], we proposed an initial state projection method to determine the consensus function, but this method cannot reveal the impacts of time delays and uncertainties on the consensus function. Thirdly, the current paper presents LMI criterions to determine the gain matrices of dynamic output feedback consensus protocols. The eigenvalue analysis approaches were used to determine the gain matrices of dynamic output feedback consensus protocols in [34] and [35], but these approaches were no longer valid when time delays and uncertainties were considered.

This paper is organized as follows. In Section II, some basic definitions and results in graph theory are presented, and the problem description is given. In Section III, a necessary and sufficient condition for consensus is shown. In Section IV, an approach to determine the consensus function is presented. A method to consensualize swarm systems is given in Section V. Numerical simulations are shown in Section VI. Finally, concluding remarks are stated in Section VII.

Notation.

In the current paper, for simplicity of notation, 0 is applied to denote zero matrices of any size with zero vectors and zero number as special cases and also to denote subspaces consisting of zero matrices. In symmetric block matrices, an asterisk (*) is used to represent a term which is induced by symmetry. The superscript H stands for the Hermitian transpose of a matrix. Let diag$\{\Gamma_1, \Gamma2, \ldots, \Gamma_n\}$ denote a diagonal block matrix with diagonal blocks $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ respectively.

II. PRELIMINARIES AND PROBLEM DESCRIPTION

In this section, first some basic concepts and results in graph theory are briefly summarized, then the problem description is presented.

2.1 Basic concepts and results in graph theory

A directed graph G consists of a node set $\mathcal{V}(G) = \{v_1, v_2, \cdots, v_N\}$, an edge set $\mathcal{E}(G) \subseteq \{(v_i, v_j) : v_i, v_j \in \mathcal{V}(G)\}$ and a weighted adjacency matrix $\tilde{W} = [w_{ij}] \in \mathbb{R}^{N\times N}$ with $w_{ij} \geq 0$. If (v_i, v_j) is an edge of G, v_i and v_j are defined as the parent and child nodes, respectively. If $w_{ji} > 0$, then $(v_i, v_j) \in \mathcal{E}(G)$. Moreover, it is assumed that $w_{ii} = 0$ for all $i \in \{1, 2, \ldots, N\}$. The set of neighbors of v_i is denoted by $\mathcal{N}_i = \{v_j \in \mathcal{V}(G) : (v_j, v_i) \in \mathcal{E}(G)\}$. The in-degree of v_i is defined as $\deg_{in}(v_i) = \sum_{j\in\mathcal{N}_i} w_{ij}$. Let $\tilde{D}$ be the degree matrix of G, which is defined as a diagonal matrix with the in-degree of each node along its diagonal. The Laplacian matrix of G is defined as $L = \tilde{D} - \tilde{W}$. A directed path from v_i to v_j is a sequence of ordered edges of the form $(v_i, v_{i_1}), (v_{i_1}, v_{i_2}), \cdots, (v_{i_l}, v_j)$, where $v_{i_k} \in \mathcal{V}(G)\,(k = 1, 2, \cdots, l)$. A directed graph having a spanning tree means that there exists at least one node having a directed path to all the other nodes. More details on graph theory can be found in [39]. The following lemmas show some basic properties of the Laplacian matrix L.

Lemma 1 [9]. Let L be the Laplacian matrix of a directed graph G and $\mathbf{1} = [1, 1, \ldots, 1]^{\mathrm{T}} \in \mathbb{R}^N$, then:

1. L at least has one zero eigenvalue, and $\mathbf{1}$ is the associated eigenvector; that is, $L\mathbf{1} = 0$;
2. If G has a spanning tree, then 0 is a simple eigenvalue of L, and all the other N–1 eigenvalues have positive real parts.

2.2 Problem description

Consider a swarm system with N agents which interact with each other via local information exchanges. A directed graph $G(k)$ can be used to describe the communication topology of the swarm system. For $i, j \in \{1, 2, \ldots, N\}$, the node v_i in a directed graph $G(k)$ represents agent i, the edge $(v_i, v_j) \in \mathcal{E}(G(k))$ corresponds to the information channel from agent i to agent j, and the nonnegative adjacency element w_{ji} denotes the transmitting strength of the channel (v_i, v_j).

Assume that all the agents share a common state space $\mathbb{R}^{d_1}$, and $x_i(k) \in \mathbb{R}^{d_1}$ $(i \in \{1, 2, \cdots, N\})$ denotes the state of agent i which needs to be coordinated, then the dynamics of agent i can be described by

$$\begin{cases} x_i(k+1) = Ax_i(k) + Bu_i(k), \\ y_i(k) = Cx_i(k), \end{cases} \tag{1}$$

where $A \in \mathbb{R}^{d_1 \times d_1}$, $B \in \mathbb{R}^{d_1 \times m}$, $C \in \mathbb{R}^{q \times d_1}$, $u_i(k)$ is the consensus protocol, and $y_i(k)$ is the measured output. Based on the theory of dynamic output feedback control, the following consensus protocol is proposed:

$$\begin{cases} z_i(k+1) = K_A z_i(k) + K_B \sum_{v_j \in \mathcal{N}_i} (w_{ij} + \Delta w_{ij}(k)) \\ \quad (y_j(k-\tau) - y_i(k-\tau)), \\ u_i(k) = K_C z_i(k) + K_D \sum_{v_j \in \mathcal{N}_i} (w_{ij} + \Delta w_{ij}(k)) \\ \quad (y_j(k-\tau) - y_i(k-\tau)), \\ z_i(k) = 0 \quad k \leqslant 0, \end{cases} \tag{2}$$

where $i, j \in \{1, 2, \dots, N\}$, $z_i(k) \in \mathbb{R}^{d_2}$ denotes the state variable of the protocol, K_A, K_B, K_C and K_D are constant matrices with appropriate dimensions, τ is a constant delay with $1 \leqslant \tau \leqslant \overline{\tau}$ where $\overline{\tau} \geqslant 1$ is a positive integer, and $\Delta w_{ij}(k)$ is the time-varying uncertainty of w_{ij} satisfying $|\Delta w_{ij}(k)| \le w_{ij} (i \neq j)$ and $\Delta w_{ii}(k) = -\sum_{m=1, m\neq i}^{N} \Delta w_{im}(k)$. One can see that the uncertainty matrix ΔL of L satisfies that $\Delta L \mathbf{1} = 0$. Let $v_i(k) = [x_i^{\mathrm{T}}(k), z_i^{\mathrm{T}}(k)]^{\mathrm{T}} \in \mathbb{R}^d$ $(i = 1, 2, \cdots, N)$ with $d = d_1 + d_2$ and $v(k) = [v_1^{\mathrm{T}}(k), v_2^{\mathrm{T}}(k), \cdots, v_N^{\mathrm{T}}(k)]^{\mathrm{T}}$, then the dynamics of a swarm system with protocol (2) can be described by

$$\begin{cases} v(k+1) = (I_N \otimes \overline{A}) v(k) - ((L + \Delta L) \otimes \overline{B}) v(k-\tau), k \in [0, \infty), \\ v(k) = \phi(k), k \in [-\tau, 0], \end{cases} \tag{3}$$

where $\varphi(k)$ is a vector-valued function on $k \in [-\tau, 0]$,

$$\overline{A} = \begin{bmatrix} A & BK_C \\ 0 & K_A \end{bmatrix} = \tilde{A} + \tilde{B}K\tilde{C}, \overline{B} = \begin{bmatrix} BK_D C & 0 \\ K_B C & 0 \end{bmatrix} = \tilde{B}K\tilde{C}_\tau,$$
$$\tilde{A} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \tilde{B} = \begin{bmatrix} B & 0 \\ 0 & I \end{bmatrix}, \tilde{C} = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}, \tilde{C}_\tau = \begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix},$$
$$K = \begin{bmatrix} K_D & K_C \\ K_B & K_A \end{bmatrix}.$$

Definition 1. Under a given protocol (2), system (3) is said to achieve consensus if for any given bounded initial condition, there exists a vector-valued function $c(k)$ dependent on the initial condition such that $\lim_{k\to\infty}(v(k) - \mathbf{1} \otimes c(k)) = 0$, where $c(k)$ is called a consensus function.

Definition 2. System (3) is said to be consensualizable by protocol (2) if there exists K such that it achieves consensus.

The following consensus problems of system (3) are investigated: (i) what are the conditions of consensus for a given K; (ii) how to determine the consensus function; and (iii) how to design K such that system (3) achieves consensus.

Remark 1. For a given swarm system, $\Delta w_{ij}(k) \equiv 0 (i, j = 1, 2, \dots, N)$ means that the communication topology is fixed. Otherwise, the communication topology is time-varying. Especially, if $\Delta w_{ij}(k) = -w_{ij}(k)$, then the communication topology is switching.

III. CONDITIONS FOR CONSENSUS

In this section, first system (3) is decomposed into two subsystems by a linear transformation. Then based on state projection, a necessary and sufficient condition for consensus is presented.

Let $U = [\overline{u}_1, \overline{U}] \in \mathbb{C}^{N\times N}$ with $\overline{u}_1 = \mathbf{1}$ and $\overline{U} = [\overline{u}_2, \cdots, \overline{u}_N]$ be nonsingular and $U^{-1} = [\upsilon^{\mathrm{H}}, \tilde{U}^{\mathrm{H}}]^{\mathrm{H}}$. Since $(L + \Delta L)\mathbf{1} = 0$, one can deduce

$$U^{-1}(L+\Delta L)U = \begin{bmatrix} \upsilon \\ \tilde{U} \end{bmatrix} (L+\Delta L)[\overline{u}_1, \overline{U}] = \begin{bmatrix} 0 & \upsilon(L+\Delta L)\overline{U} \\ 0 & \tilde{U}(L+\Delta L)\overline{U} \end{bmatrix}. \tag{4}$$

Let $\tilde{v}(k) = (U^{-1} \otimes I_d) v(k) = [\tilde{v}_1^{\mathrm{H}}(k), \tilde{v}_2^{\mathrm{H}}(k), \cdots, \tilde{v}_N^{\mathrm{H}}(k)]^{\mathrm{H}}$. By (4), system (3) can be transformed into

$$\tilde{v}(k+1) = (I_N \otimes \overline{A})\tilde{v}(k) - \left(\begin{bmatrix} 0 & \upsilon(L+\Delta L)\overline{U} \\ 0 & \tilde{U}(L+\Delta L)\overline{U} \end{bmatrix} \otimes \overline{B} \right) \tilde{v}(k-\tau). \tag{5}$$

Let $\varsigma(k) = [\tilde{v}_2^{\mathrm{H}}(k), \cdots, \tilde{v}_N^{\mathrm{H}}(k)]^{\mathrm{H}}$, then system (5) can be rewritten as follows

$$\begin{aligned} \begin{bmatrix} \tilde{v}_1(k+1) \\ \varsigma(k+1) \end{bmatrix} &= \begin{bmatrix} \overline{A} & 0 \\ 0 & I_{N-1} \otimes \overline{A} \end{bmatrix} \begin{bmatrix} \tilde{v}_1(k) \\ \varsigma(k) \end{bmatrix} \\ &\quad - \begin{bmatrix} 0 & \upsilon(L+\Delta L) \otimes \overline{B} \\ 0 & \tilde{U}(L+\Delta L) \otimes \overline{B} \end{bmatrix} \begin{bmatrix} \tilde{v}_1(k-\tau) \\ \varsigma(k-\tau) \end{bmatrix} \\ &= \begin{bmatrix} \overline{A}\tilde{v}_1(k) \\ (I_{N-1} \otimes \overline{A})\varsigma(k) \end{bmatrix} - \begin{bmatrix} (\upsilon(L+\Delta L) \otimes \overline{B})\varsigma(k-\tau) \\ (\tilde{U}(L+\Delta L) \otimes \overline{B})\varsigma(k-\tau) \end{bmatrix}; \end{aligned}$$

that is,

$$\tilde{v}_1(k+1) = \overline{A}\tilde{v}_1(k) - (\upsilon(L+\Delta L)\overline{U} \otimes \overline{B})\varsigma(k-\tau), \tag{6}$$

$$\varsigma(k+1) = (I_{N-1} \otimes \overline{A})\varsigma(k) - (\tilde{U}(L+\Delta L)\overline{U} \otimes \overline{B})\varsigma(k-\tau). \tag{7}$$

Consider d linear independent vectors $c_j \in \mathbb{R}^d$ $(j = 1, 2, \dots, d)$. The following two subspaces of $\mathbb{C}^{Nd}$ are introduced.

Definition 3. Let $p_j = \overline{u}_i \otimes c_m$ $(j = (i-1)d + m; i = 1, 2, \cdots, N; m = 1, 2, \cdots, d)$. A consensus subspace (CS) is defined as the subspace $\mathbb{C}(U)$ spanned by $p_1, p_2, \dots, p_d$ and a complement consensus subspace (CCS) as the subspace $\overline{\mathbb{C}}(U)$ spanned by $p_{d+1}, p_{d+2}, \dots, p_{Nd}$.

Note that any vector in $\mathbb{C}(U)$ has the form $\mathbf{1} \otimes \tilde{c}$, where $\tilde{c}$ is a d-dimensional vector. One sees that consensus is achieved if and only if $v(k)$ converges to a vector in $\mathbb{C}(U)$ as $k \to \infty$. This is the reason why $\mathbb{C}(U)$ is called a consensus subspace.

Since p_j ($j = 1, 2, \ldots, Nd$) are linear independent, the following lemma can be obtained.

Lemma 2. $\mathbb{C}(U) \oplus \bar{\mathbb{C}}(U) = \mathbb{C}^{Nd}$.

The following theorem presents a necessary and sufficient condition for system (3) to achieve consensus.

Theorem 1. For a given protocol (2), system (3) achieves consensus if and only if subsystem (7) is asymptotically stable.

Proof. By Lemma 2, the state $v(k)$ of system (3) can be uniquely projected onto $\mathbb{C}(U)$ and $\bar{\mathbb{C}}(U)$; that is,

$$v(k) = v_C(k) + v_{\bar{C}}(k), \tag{8}$$

where $v_C(k) = \sum_{j=1}^{d} \alpha_j(k) p_j$ and $v_{\bar{C}}(k) = \sum_{j=d+1}^{Nd} \alpha_j(k) p_j$. Due to $\tilde{v}(k) = (U^{-1} \otimes I_d) v(k)$, one has

$$\varsigma(k) = [0, I_{(N-1)d}](U^{-1} \otimes I_d) v(k). \tag{9}$$

Since $\bar{u}_i \otimes c_m = (U \otimes I_d)(e_i \otimes c_m)$, where $i \in \{1, 2, \ldots, N\}$, $m \in \{1, 2, \ldots, d\}$, and $e_i \in \mathbb{R}^N$ has a 1 as its ith component and 0 elsewhere, by (8), one can obtain that

$$\begin{aligned}(U^{-1} \otimes I_d) v(k) = &\sum_{m=1}^{d} \alpha_m(k)(e_1 \otimes c_m) \\ &+ \sum_{i=2}^{N} \sum_{m=1}^{d} \alpha_{(i-1)d+m}(k)(e_i \otimes c_m).\end{aligned} \tag{10}$$

It can be obtained by (9) and (10) that

$$\varsigma(k) = [0, I_{(N-1)d}] \sum_{i=2}^{N} \sum_{m=1}^{d} \alpha_{(i-1)d+m}(k)(e_i \otimes c_m). \tag{11}$$

Due to

$$v_{\bar{C}}(k) = (U \otimes I_d) \sum_{i=2}^{N} \sum_{m=1}^{d} \alpha_{(i-1)d+m}(k)(e_i \otimes c_m),$$

it can be shown by (11) that

$$v_{\bar{C}}(k) = (U \otimes I_d)[0, \varsigma^{\mathrm{H}}(k)]^{\mathrm{H}}. \tag{12}$$

Necessity. We prove the conclusion by contradiction. Assume that subsystem (7) with $\varsigma(s)$ being not identical to 0 for $s \in [-\tau, 0]$ is not asymptotically stable, then by (10), the limit of $v_{\bar{C}}(k)$ as $k \to \infty$ does not exist or is nonzero. Since system (3) achieves consensus, by the structure of p_j ($j = 1, 2, \ldots, d$) and (8), there exist $\beta_j(k) \in \mathbb{R}$($j = 1, 2, \ldots, d$) such that $v_{\bar{C}}(k) \to \sum_{j=1}^{d} \beta_j(k) p_j \in \mathbb{C}(U)$ as $k \to \infty$. Since $v_{\bar{C}}(k) \in \bar{\mathbb{C}}(U)$ and $\mathbb{C}(U) \cap \bar{\mathbb{C}}(U) = 0$, one has $\lim_{k\to\infty} v_{\bar{C}}(k) = 0$. A contradiction is obtained. Therefore, it is necessary that subsystem (7) is asymptotically stable.

Sufficiency. If subsystem (7) is asymptotically stable, then $\lim_{k\to\infty} v_{\bar{C}}(k) = 0$ by (12). According to the structure of p_j ($j = 1, 2, \ldots, d$) and (8), system (3) achieves consensus. The proof of Theorem 1 is completed.

Remark 2. From the proof of Theorem 1, $v_C(k)$ and $v_{\bar{C}}(k)$ are the state projection of $v(k)$ onto $\mathbb{C}(U)$ and $\bar{\mathbb{C}}(U)$. Hence, from Lemma 2, $v_C(k)$ and $v_{\bar{C}}(k)$ describe the agreement and disagreement parts of system (3) respectively.

IV. CONSENSUS FUNCTIONS

If system (3) achieves consensus, then states of all agents tend to be a common function; that is, the consensus function, which describes the absolute motion as a whole. In this case, a very interesting and challenging problem is how to determine the consensus function.

Let $G(k) = \bar{A}^k$ and $P_{\mathbb{C}(U),\bar{\mathbb{C}}(U)}$ be an oblique projector onto $\mathbb{C}(U)$ along $\bar{\mathbb{C}}(U)$. The following theorem presents a method to determine the consensus function.

Theorem 2. If system (3) achieves consensus, then subsystem (6) determines the consensus function $c(k)$, and

$$\lim_{k\to\infty}(c(k) - (c_0(k) + c_\tau(k) + c_\Delta(k))) = 0,$$

where

$$\begin{aligned}c_0(k) = &G(k)[I_d, 0, \cdots, 0] P_{\mathbb{C}(U),\bar{\mathbb{C}}(U)} v(0) \\ &- \sum_{i=0}^{k-1} G(k-i-1)(\upsilon L \bar{U} \otimes \bar{B}) \varsigma(i),\end{aligned}$$

$$c_\tau(k) = -\sum_{i=0}^{k-1} G(k-i-1)(\upsilon L \bar{U} \otimes \bar{B})(\varsigma(i-\tau) - \varsigma(i)),$$

$$c_\Delta(k) = -\sum_{i=0}^{k-1} G(k-i-1)(\upsilon \Delta L \bar{U} \otimes \bar{B}) \varsigma(i-\tau).$$

Proof. Due to $\tilde{v}(k) = (U^{-1} \otimes I_d) v(k)$, one obtains

$$\tilde{v}_1(k) = [I_d, 0](U^{-1} \otimes I_d) v(k). \tag{13}$$

Since $e_1 \otimes \tilde{v}_1(k) = [\tilde{v}_1^{\mathrm{H}}(k), 0]^{\mathrm{H}}$, from (8) and (13), one has

$$v_C(k) = (U \otimes I_d)[\tilde{v}_1^{\mathrm{H}}(k), 0]^{\mathrm{H}} = \mathbf{1} \otimes \tilde{v}_1(k). \tag{14}$$

By Theorem 1, if system (3) achieves consensus, then

$$\lim_{k\to\infty}(v_C(k) - \mathbf{1} \otimes c(k)) = 0. \tag{15}$$

It is clear by (14) and (15) that

$$\lim_{k\to\infty}(\tilde{v}_1(k) - c(k)) = 0; \tag{16}$$

that is, subsystem (6) determines the consensus function. By (6), one has

$$c(k)=G(k)\tilde{v}_1(0)-\sum_{i=0}^{k-1}G(k-i-1)(\upsilon(L+\Delta L)\bar{U}\otimes\bar{B})\varsigma(i-\tau). \quad (17)$$

By Lemma 2, one can obtain that $v_C(0)=P_{\mathbb{C}(U),\bar{\mathbb{C}}(U)}v(0)$. Hence, it can be shown by (12) that

$$\tilde{v}_1(0)=[I_d,0,\cdots,0]P_{\mathbb{C}(U),\bar{\mathbb{C}}(U)}v(0). \quad (18)$$

From (16) to (18), one can obtain that

$$\lim_{k\to\infty}(c(k)-(c_0(k)+c_\tau(k)+c_\Delta(k)))=0.$$

The proof of Theorem 2 is completed.

In Theorem 2, $c_0(k)$ is said to be a nominal consensus function, which describes the consensus function of a swarm system without the time delay and uncertainties. $c_\tau(k)$ and $c_\Delta(k)$ describe the impacts of time delays and uncertainties respectively.

Remark 3. It is well-known that the motions of swarm systems with N agents consist of the relative motions of N–1 order among agents and the absolute motion as a whole, which describe microscopic and macroscopic properties of these systems respectively. Consensus is achieved if and only if the relative motions of N–1 order are asymptotically stable, and the consensus function determines the absolute motion. Hence, under a given consensus protocol, the key of consensus is to transform the consensus problem into a stability problem. In [11], the state error method was used to convert consensus problems for swarm systems into stability problems of reduced-order subsystems, while the method cannot determine the absolute motion as a whole. The state projection method can describe both the relative motions and the absolute motion of swarm systems with time delays and uncertainties.

Remark 4. The consensus function describes the absolute motion of swarm systems, which is very important when analyzing and designing swarm systems. The χ-consensus problem was proposed to determine the consensus functions in [8], where the assumption that the communication topology is balanced and strongly connected was required. In [13], we presented an explicit expression of the consensus function based on the different impacts of delays and uncertainties. In [8] and [13], it was assumed that the dynamics of each agent is a first-order integrator, and their methods are no longer valid to deal with high-dimensional swarm systems. In [27], we presented an initial state decomposition method to determine consensus functions of high-dimensional swarm systems, but this method cannot be used to deal with the cases where delays and uncertainties are involved. Theorem 2 shows the influences of delays and uncertainties on the consensus function.

V. CONDITIONS FOR CONSENSUALIZATION

From the proof of Theorem 1, one can see that it is not related to the choice of $\bar{U}$ for system (3) to achieve consensus, which means that the choice of $\bar{U}$ is not unique. If U is a complex matrix, then the calculation complexity will be increased when solving LMIs. Therefore, it is assumed that $\bar{U}=[e_2,e_3,\cdots,e_N]$. The following lemmas are useful to obtain the conditions of consensualization.

Lemma 3 [12]. Let D be a 0–1 matrix with rows and columns indexed by the nodes and edges of G, and E be a 0–1 matrix with rows and columns indexed by the edges and nodes of G, defined as

$$D_{ve}=\begin{cases}1, & \text{if the node } v \text{ is the child node of the edge } e \text{ of } G,\\ 0, & \text{otherwise.}\end{cases}$$

$$E_{ev}=\begin{cases}1, & \text{if the node } v \text{ is the parent node of the edge } e \text{ of } G,\\ 0, & \text{otherwise.}\end{cases}$$

Let $\Lambda=\text{diag}\{\mu_1,\mu_2,\ldots,\mu_\kappa\}$ where μ_m ($m=1,2,\ldots,\kappa$) are the weight of the mth edge of G, and κ is the number of the edges of G. Then L can be denoted by $L=D\Lambda(D^{\mathrm{T}}–E)$.

By Lemma 3, the uncertainty matrix ΔL of L can be written as $\Delta L=DF(k)\bar{E}$, where $\bar{E}\in\mathbb{R}^{\kappa\times N}$ and $F(k)$ is a diagonal matrix whose diagonal elements are uncertainties of the edges. Since $|\Delta w_{ij}(k)|/w_{ij}\le 1(i,j\in\{1,2,\ldots,N\},\ i\neq j)$, it is assumed that $F^{\mathrm{T}}(k)F(k)\le I_\kappa$ ($\forall k$) without loss of generality.

Lemma 4 [36]. Given matrices $Q=Q^{\mathrm{T}}$, H and Z, for all $F(k)$ satisfying $F^{\mathrm{T}}(k)F(k)\le I$, $Q+HF(k)Z+Z^{\mathrm{T}}F^{\mathrm{T}}(k)H^T<0$ if and only if there exists a $\gamma>0$ such that $Q+\gamma HH^{\mathrm{T}}+\gamma^{-1}Z^{\mathrm{T}}Z<0$.

Lemma 5 [38]. Let R and Q be real symmetric matrices and assume that Q is invertible, then

$$\begin{bmatrix}R & Y\\ Y^{\mathrm{T}} & -Q\end{bmatrix}<0\Leftrightarrow\begin{cases}Q>0,\\ R+YQ^{-1}Y^{\mathrm{T}}<0.\end{cases}$$

The following theorem presents a criterion for system (3) to achieve consensus, which includes both LMI and matrix equality constraints, and an algorithm is given to check the criterion and determine the gain matrix K.

Theorem 3. For any $\tau\in[1,\bar{\tau}]$, system (3) can be consensualized by protocol (2) if there exist $R=R^{\mathrm{T}}>0$, $Q=Q^{\mathrm{T}}>0$, $S=S^{\mathrm{T}}>0$, $P_1=P_1^{\mathrm{T}}>0$, $P_2=P_2^{\mathrm{T}}>0$, $M=\begin{bmatrix}M_{11} & M_{12}\\ * & M_{22}\end{bmatrix}\geqslant 0$, X, Y and K satisfying

$$\Xi = \begin{bmatrix} \Xi_{11} & \Xi_{12} & \Xi_{13} & \Xi_{14} & 0 & 0 \\ * & \Xi_{22} & \Xi_{23} & \Xi_{24} & \Xi_{25} & 0 \\ * & * & -P_1 & 0 & 0 & \Xi_{36} \\ * & * & * & -\bar{\tau}^{-1}P_2 & 0 & \Xi_{46} \\ * & * & * & * & -I & 0 \\ * & * & * & * & * & -I \end{bmatrix} < 0, \tag{19}$$

$$\Psi = \begin{bmatrix} M_{11} & M_{12} & X \\ * & M_{22} & Y \\ * & * & S \end{bmatrix} \geqslant 0, \tag{20}$$

$$RP_1 = I, SP_2 = I, \tag{21}$$

where

$$\Xi_{11} = -R + \bar{\tau}Q + X + X^{\mathrm{T}} + \bar{\tau}M_{11},$$

$$\Xi_{12} = -X + Y^{\mathrm{T}} + \bar{\tau}M_{12},$$

$$\Xi_{13} = (I_{N-1} \otimes \bar{A})^{\mathrm{T}},$$

$$\Xi_{14} = (I_{N-1} \otimes \bar{A} - I)^{\mathrm{T}},$$

$$\Xi_{22} = -Q - Y - Y^{\mathrm{T}} + \bar{\tau}M_{22},$$

$$\Xi_{23} = \Xi_{24} = -(\tilde{U}L\bar{U} \otimes \bar{B})^{\mathrm{T}},$$

$$\Xi_{25} = -(\bar{E}\bar{U} \otimes \bar{B})^{\mathrm{T}},$$

$$\Xi_{36} = \Xi_{46} = \tilde{U}D \otimes I.$$

Proof. First discuss the stability of subsystem (7). Define

$$\varsigma(k+1) = \varsigma(k) + \eta(k), \tag{22}$$

then one can see that

$$\eta(k) = \varsigma(k+1) - \varsigma(k) = (I_{N-1} \otimes \bar{A} - I)\varsigma(k) - (\tilde{U}(L+\Delta L)\bar{U} \otimes \bar{B})\varsigma(k-\tau). \tag{23}$$

Consider the following Lyapunov function candidate

$$V(k) = V_1(k) + V_2(k) + V_3(k), \tag{24}$$

where

$$V_1(k) = \varsigma^{\mathrm{T}}(k)R\varsigma(k),$$

$$V_2(k) = \sum_{j=-\bar{\tau}+1}^{0} \sum_{i=k-1+j}^{k-1} \varsigma^{\mathrm{T}}(i)Q\varsigma(i),$$

$$V_3(k) = \sum_{j=-\bar{\tau}+1}^{0} \sum_{i=k-1+j}^{k-1} \eta^{\mathrm{T}}(i)S\eta(i).$$

In (24), R, Q and S are positive definite symmetric matrices to be determined. Define $\Delta V = V(k+1)-V(k)$, along the solution of subsystem (7), one has

$$\Delta V_1(k) = \xi_0^{\mathrm{T}}(k)\tilde{\Xi}_1\xi_0(k) - \varsigma^{\mathrm{T}}(k)R\varsigma(k), \tag{25}$$

$$\Delta V_2(k) = \bar{\tau}\varsigma^{\mathrm{T}}(k)Q\varsigma(k) - \sum_{i=k-\bar{\tau}}^{k-1} \varsigma^{\mathrm{T}}(i)Q\varsigma(i) \leqslant \bar{\tau}\varsigma^{\mathrm{T}}(k)Q\varsigma(k) - \varsigma^{\mathrm{T}}(k-\tau)Q\varsigma(k-\tau), \tag{26}$$

$$\Delta V_3(k) = \bar{\tau}\eta^{\mathrm{T}}(k)S\eta(k) - \sum_{i=k-\bar{\tau}}^{k-1} \eta^{\mathrm{T}}(i)S\eta(i) \leqslant \xi_0^{\mathrm{T}}(k)\tilde{\Xi}_2\xi_0(k) - \sum_{i=k-\tau}^{k-1} \eta^{\mathrm{T}}(i)S\eta(i), \tag{27}$$

where

$$\xi_0(k) = [\varsigma^{\mathrm{T}}(k), \varsigma^{\mathrm{T}}(k-\tau)]^{\mathrm{T}},$$

$$\tilde{\Xi}_1 = [I_{N-1} \otimes \bar{A}, -\tilde{U}(L+\Delta L)\bar{U} \otimes \bar{B}]^{\mathrm{T}} R[I_{N-1} \otimes \bar{A}, -\tilde{U}(L+\Delta L)\bar{U} \otimes \bar{B}],$$

$$\tilde{\Xi}_2 = \bar{\tau}[I_{N-1} \otimes \bar{A} - I, -\tilde{U}(L+\Delta L)\bar{U} \otimes \bar{B}]^{\mathrm{T}} S[I_{N-1} \otimes \bar{A} - I, -\tilde{U}(L+\Delta L)\bar{U} \otimes \bar{B}].$$

The following equation can be obtained by (22),

$$\Omega_1 = 2\xi_0^{\mathrm{T}}(k)\begin{bmatrix} X \\ Y \end{bmatrix}\left(\varsigma(k) - \varsigma(k-\tau) - \sum_{i=k-\tau}^{k-1} \eta(i)\right) = 0. \tag{28}$$

For any appropriately dimensioned matrix $M = \begin{bmatrix} M_{11} & M_{12} \\ * & M_{22} \end{bmatrix} \geqslant 0$, one has

$$\Omega_2 = \sum_{i=k-\bar{\tau}}^{k-1} \xi_0^{\mathrm{T}}(k)M\xi_0(k) - \sum_{i=k-\tau}^{k-1} \xi_0^{\mathrm{T}}(k)M\xi_0(k) = \bar{\tau}\xi_0^{\mathrm{T}}(k)M\xi_0(k) - \sum_{i=k-\tau}^{k-1} \xi_0^{\mathrm{T}}(k)M\xi_0(k) \geqslant 0. \tag{29}$$

From (24) to (29), one can obtain

$$\Delta V(k) \leqslant \Delta V_1(k) + \Delta V_2(k) + \Delta V_3(k) + \Omega_1 + \Omega_2 \leqslant \xi_0^{\mathrm{T}}(k)(\tilde{\Xi}_0 + \tilde{\Xi}_1 + \tilde{\Xi}_2)\xi_0(k) - \sum_{i=k-\tau}^{k-1} \xi_1^{\mathrm{T}}(k,i)\Psi\xi_1(k,i), \tag{30}$$

where $\tilde{\Xi}_0 = \begin{bmatrix} \Xi_{11} & \Xi_{12} \\ * & \Xi_{22} \end{bmatrix}$ and $\xi_1(k,i) = [\xi_0^{\mathrm{T}}(k), \eta^{\mathrm{T}}(i)]^{\mathrm{T}}$. By the Schur complement (Lemma 5), $\tilde{\Xi}_0 + \tilde{\Xi}_1 + \tilde{\Xi}_2 < 0$ is equivalent to

$$\tilde{\Xi} = \begin{bmatrix} \Xi_{11} & \Xi_{12} & \tilde{\Xi}_{13} & \tilde{\Xi}_{14} \\ * & \Xi_{22} & \tilde{\Xi}_{23} - (\tilde{U}\Delta L\bar{U} \otimes \bar{B})^{\mathrm{T}} R & \tilde{\Xi}_{24} - \bar{\tau}(\tilde{U}\Delta L\bar{U} \otimes \bar{B})^{\mathrm{T}} S \\ * & * & -R & 0 \\ * & * & * & -\bar{\tau}S \end{bmatrix} < 0, \tag{31}$$

where $\tilde{\Xi}_{13} = (I_{N-1} \otimes \bar{A})^{\mathrm{T}} R$, $\tilde{\Xi}_{23} = -(\tilde{U}L\bar{U} \otimes \bar{B})^{\mathrm{T}} R$, $\tilde{\Xi}_{14} = \bar{\tau}(I_{N-1} \otimes \bar{A} - I)^{\mathrm{T}} S$ and $\tilde{\Xi}_{24} = -\bar{\tau}(\tilde{U}L\bar{U} \otimes \bar{B})^{\mathrm{T}} S$. Since $\Delta L = DF(k)\bar{E}$, by the properties of Kronecker products, one obtains

$$\tilde{U}\Delta L\bar{U} \otimes \bar{B} = (\tilde{U}D \otimes I)(F(k) \otimes I)(\bar{E}\bar{U} \otimes \bar{B}). \tag{32}$$

Then it can be shown that (31) is equivalent to

$$\tilde{\Xi}_{\Delta} + H(F(k) \otimes I)Z + Z^{\mathrm{T}}(F(k) \otimes I)^{\mathrm{T}} H^{\mathrm{T}} < 0, \tag{33}$$

where

$$H = [0, 0, (\tilde{U}D \otimes I)^{\mathrm{T}} R, \bar{\tau}(\tilde{U}D \otimes I)^{\mathrm{T}} S]^{\mathrm{T}},$$

$$Z = [0, -\bar{E}\bar{U} \otimes \bar{B}, 0, 0],$$

$$\tilde{\Xi}_{\Delta} = \begin{bmatrix} \Xi_{11} & \Xi_{12} & \tilde{\Xi}_{13} & \tilde{\Xi}_{14} \\ * & \Xi_{22} & \tilde{\Xi}_{23} & \tilde{\Xi}_{24} \\ * & * & -R & 0 \\ * & * & * & -\bar{\tau}S \end{bmatrix}.$$

Due to $F^{\mathrm{T}}(k)F(k) \le I_{\kappa}$, one has $(F(k) \otimes I)^{\mathrm{T}}(F(k) \otimes I) \le I_{\kappa d}$. By Lemma 4, (33) holds if and only if there exits a $\gamma > 0$ such that

$$\tilde{\Xi}_{\Delta} + \gamma HH^{\mathrm{T}} + \gamma^{-1} Z^{\mathrm{T}} Z < 0. \tag{34}$$

Replacing γR, γQ, γS, γM, γX, and γY with R, Q, S, M, X, and Y respectively and using the Schur complement, (34) is equivalent to

$$\Phi = \begin{bmatrix} \Xi_{11} & \Xi_{12} & \tilde{\Xi}_{13} & \tilde{\Xi}_{14} & 0 & 0 \\ * & \Xi_{22} & \tilde{\Xi}_{23} & \tilde{\Xi}_{24} & \Xi_{25} & 0 \\ * & * & -R & 0 & 0 & \tilde{\Xi}_{36} \\ * & * & * & -\bar{\tau}S & 0 & \tilde{\Xi}_{46} \\ * & * & * & * & -I & 0 \\ * & * & * & * & * & -I \end{bmatrix} < 0, \tag{35}$$

where $\tilde{\Xi}_{36} = R(\tilde{U}D \otimes I)$ and $\tilde{\Xi}_{46} = \bar{\tau}S(\tilde{U}D \otimes I)$. From (30) to (35), one can see that if $\Phi < 0$ and $\Psi \ge 0$, then subsystem (7) is asymptotically stable.

Pre- and post-multiplying the left and right sides of Φ by $\mathrm{diag}\{I, I, R^{-1}, (\bar{\tau}S)^{-1}, I, I\}$, one obtains (19) and (21). By Theorem 1, if (19)–(21) are feasible, then system (3) can be consensualized by protocol (2). The proof of Theorem 3 is completed.

Due to (21), the conditions in Theorem 3 are not strict LMIs. However, the nonconvex feasibility problem can be solved by a cone complementarity linearization approach, whose basic principles and characteristics were addressed in detail in [37]. This method can convert the original problem into the following nonlinear minimization problem:

$$\begin{aligned} &\min && \mathrm{tr}(RP_1 + SP_2) \\ &\text{sbuject to} && (19), (20), \\ &&& R > 0, Q > 0, S > 0, P_1 > 0, P_2 > 0, M \geqslant 0, \end{aligned} \tag{36}$$

$$\begin{bmatrix} R & I \\ I & P_1 \end{bmatrix} \geqslant 0, \begin{bmatrix} S & I \\ I & P_2 \end{bmatrix} \geqslant 0. \tag{37}$$

The algorithm in [37] is applied to solve the above nonlinear problem.

Algorithm.

Step 1. Find a feasible set of R, Q, S, P_1, P_2, X, Y, M, K satisfying (19), (20), (36) and (37). Set $k = 0$.
Step 2. Solve the following LMI problem

$$\begin{aligned} &\min && \mathrm{tr}(RP_{1,k} + R_k P_1 + SP_{2,k} + S_k P_2) \\ &\text{subject to} && (19), (20), (36), \text{and } (37). \end{aligned}$$

Let $R_{k+1} = R$, $S_{k+1} = S$, $P_{1,k+1} = P_1$, and $P_{2,k+1} = P_2$.
Step 3. If LMIs (20) and (35) are feasible for the K obtained in Step 2, and $|tr(RP_1 + SP_2) - 2(N-1)d| < \delta$ for some sufficiently small scalar $\delta > 0$, then stop and output the feasible solutions R, Q, S, P_1, P_2, X, Y, M, K.
Step 4. If $k > N_{\max}$ where $N_{\max}$ is the maximum number of iterations allowed, then stop.
Step 5. Set $k = k + 1$ and go to Step 2.

Remark 5. Although the LMI tool cannot give the explicit expression of the controller gain, it is valid for dealing with stability and stabilization problems of time-delayed uncertain systems and has been extensively applied in the literature (*e.g.*, [36,40,41] for isolated systems and [10–13,33] for swarm systems). Lin and Jia [10] addressed average consensus problems for swarm systems with a single constant delay. Sun and Wang [11] dealt with swarm systems with multiple time delays, where uncertainties were not considered and the consensus function, which describes one of the important consensus properties, is difficult to determine by their methods. Swarm systems with time delays and uncertainties were considered in [12], where the consensus function is the average of states of all agents. In [13], we addressed the influences of both time delays and uncertainties on consensus properties. In [10–13], it was supposed that the dynamics of each agent is given by a first-order integrator. Liu *et al*. [33] discussed consensus problems for high-dimensional swarm systems with external disturbances in terms of LMIs, while their methods cannot be used to deal with the case where the consensus function is not the average of states of all agents.

The current paper deals with high-dimensional swarm systems with time delays and uncertainties, and an explicit expression of the consensus function is shown based on the impacts of time delays and uncertainties.

VI. NUMERICAL SIMULATIONS

In this section, a numerical example is given to illustrate the effectiveness of the theoretical results shown in previous sections.

Suppose that a swarm system consists of five agents with the dynamics of each agent described by (1), where

$$A=\begin{bmatrix}0.5 & 0.4\\ 0 & 1\end{bmatrix},\ B=\begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix},\ C=\begin{bmatrix}0.81 & 0\\ 0 & 1\end{bmatrix}.$$

A directed communication graph G of the system is shown in Fig. 1. The edges (v_4, v_2), (v_1, v_2), (v_1, v_3), (v_5, v_3), (v_3, v_5), (v_4, v_1), (v_1, v_4), (v_5, v_1), (v_1, v_5), (v_5, v_4) and (v_4, v_5) are labeled from 1 to 11 respectively. For simplicity, the adjacency matrix of G is set to be a 0-1 matrix. Uncertainties are described by $\Delta L = DF(k)\bar{E}$ where $\bar{E} = 0.5(D^{\mathrm{T}} - E)$, D and E can be obtained according to Lemma 3, and $F(k) = \mathrm{diag}\{0.2\sin(k), 0.3, 0.4\sin(k)\cos(k), 0.4, 0.35\sin(k), 0.6\cos(k), 0.02, 0.01, 0.15\sin(k), 0.35\cos(k)\}$. The time delay is chosen as $\tau = 5$. The initial condition is set as $\phi(k) = [8, -6, 0, 0, -2, 3\ 0, 0, 4, 6, 0, 0, -8, 5, 0, 0, 5, -7, 0, 0]^{\mathrm{T}}$ $(k \in [-\tau, 0]$

By using the iterative algorithm given in Section 5 to solve (19)–(21) in Theorem 3 with $\bar{\tau} = 10$, one can obtain

$$K=\begin{bmatrix}0.0045 & 0.0056 & 0 & 0\\ -0.0036 & -0.0030 & 0 & 0.0001\\ 0 & 0 & 0.5559 & 0\\ 0 & 0 & 0 & 0.5561\end{bmatrix}.$$

Fig. 2 shows the state trajectories of the swarm system, and one can see that the system achieves consensus. The nominal consensus function is denoted by circle markers in Fig. 2. The state trajectories deviate from the one formed by circle markers, which means that the time delay and uncertainties impact the consensus function as shown in Theorem 2.

VII. CONCLUSIONS

Based on the dynamic output feedback consensus protocol, consensus problems of high-dimensional discrete-time swarm systems with time delays and uncertainties were investigated. By projecting the state of a swarm system onto the consensus subspace and the complement consensus subspace, a necessary and sufficient condition for the system to achieve consensus was given, and an explicit expression of the consensus function was presented. An approach to consensualize the system was shown. Numerical simulations were given to illustrate the effectiveness of theoretical results.

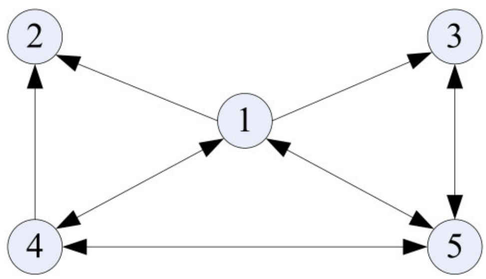

Fig. 1. Directed communication graph G.

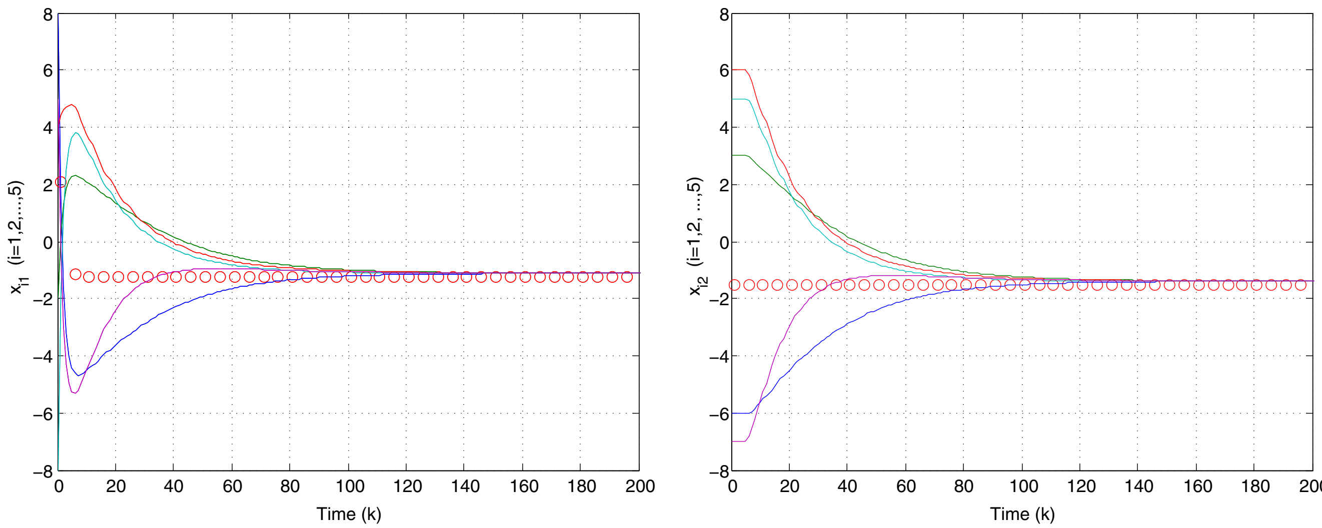

Fig. 2. State trajectories of the swarm system.

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