Chin. Phys. B Vol. 20, No. 1 (2011) 010301

A generalized Hadamard transformation from new entangled state*

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(Received 11 July 2010; revised manuscript received 16 August 2010)

A new entangled state $|\eta;\theta\rangle$ is proposed by the technique of integral within an ordered product. A generalized Hadamard transformation is derived by virtue of $|\eta;\theta\rangle$, which plays a role of Hadamard transformation for $(\hat{a}_1\sin\theta-\hat{a}_2\cos\theta)$ and $(\hat{a}_1\cos\theta+\hat{a}_2\sin\theta)$.

Keywords: generalized Hadamard transformation, entangled state, technique of integral within an ordered product

PACS: 03.65.–W, 42.50Dv **DOI:** 10.1088/1674-1056/20/1/010301

1. Introduction

In recent years quantum computer has attracted the attention of physicists because the principle of quantum computer obeys the laws of quantum mechanics. The representation theory is one of the foundations in the mathematics and physics of quantum mechanics. Fan[1] and Wünsche[2] have created the technique of integral within an ordered product (IWOP) of operators, which has developed the representation theory greatly. Since the publication of the paper of Einstein, Podolsky and Rosen (EPR) in 1935,[3] the conception of entanglement has become more and more fascinating and important as it plays a central role in quantum communication and quantum computation. The entangled state has been an important topic in quantum mechanics and quantum optics[4–6] since Glauber[7,8] and Klauder and Skagerstam[9] introduced the coherent state of the harmonic oscillator. In the theoretical study of quantum computer, of great importance is the Hadamard transform. The continuous Hadamard operator is defined as[10]

$$\hat{F}=\frac{1}{\sqrt{\pi}\sigma}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\mathrm{d}x\mathrm{d}y\exp\left(\frac{2\,\mathrm{i}\,xy}{\sigma^2}\right)|y\rangle\langle x|, \qquad (1)$$

where σ is the scale length.

From Eq. (1), Fan and Guo[11] found the following explicit form of Hadamard operator:

$$\begin{aligned}\hat{F}\ =\ &\frac{2\sigma}{\sqrt{\sigma^4+4}}\exp\left[\frac{\sigma^4-4}{2(\sigma^4+4)}\hat{a}^{\dagger 2}\right]\\ &\times\exp\left(\hat{a}^{\dagger}\hat{a}\ln\frac{4\,\mathrm{i}\,\sigma^2}{\sigma^4+4}\right)\exp\left[\frac{\sigma^4-4}{2(\sigma^4+4)}\hat{a}^2\right], \quad (2)\end{aligned}$$

and the following explicit form of two-mode Hadamard operator in entangled state representation:

$$\begin{aligned}\hat{F}=&\frac{4\sigma^2}{\sigma^4+4}\exp\left(\frac{4-\sigma^4}{\sigma^4+4}\hat{a}_1^{\dagger}\hat{a}_2^{\dagger}\right)\exp\Bigg[\hat{a}_1^{\dagger}\hat{a}_1\ln\left(\frac{4\sigma^2}{\sigma^4+4}\right)\\ &+\hat{a}_2^{\dagger}\hat{a}_2\ln\left(\frac{-4\sigma^2}{\sigma^4+4}\right)\Bigg]\exp\left(\frac{4-\sigma^4}{\sigma^4+4}\hat{a}_1\hat{a}_2\right). \qquad (3)\end{aligned}$$

In the present paper, employing a new type of entangled state representation $|\eta;\theta\rangle$ and IWOP technique, we construct a so-called generalized Hadmard operator which plays a role of two-mode Hadmard transform for $(\hat{a}_1\sin\theta-\hat{a}_2\cos\theta)$ and $(\hat{a}_1\cos\theta+\hat{a}_2\sin\theta)$.

2. Entangled state representation $|\eta;\theta\rangle$

Fan and Lu[12] constructed a kind of coherent-entangled state $|\alpha,x\rangle$. Xu *et al.*[13,14] constructed two

*Project supported by the National Natural Science Foundation of China (Grant No. 10574060) the Natural Science Foundation of Shandong Province of China (Grant No. Y2008A16), the University Experimental Technology Foundation of Shandong Province, China (Grant No. S04W138), and the Natural Science Foundation of Heze University of Shandong Province, China (Grant Nos. XY07WL01 and XY08WL03).
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kinds of states, i.e., coherent-entangled state $|\alpha, x; \lambda\rangle$ and intermediate entangled state $|\eta_1, \eta_2\rangle_{\lambda,\nu}$. By testing, we construct a new type of entangled state $|\eta;\theta\rangle$, which is a common eigenvector of $[(\hat{x}_1-\hat{x}_2)\sin\theta-(\hat{x}_1+\hat{x}_2)\cos\theta]$ and $[(\hat{p}_1+\hat{p}_2)\sin\theta+(\hat{p}_1-\hat{p}_2)\cos\theta]$, as follows:

$$\begin{aligned}|\eta;\theta\rangle = \exp\bigg[&-\frac{|\eta|^2}{2}+(\eta\sin\theta-\eta^*\cos\theta)\hat{a}_1^\dagger\\&-(\eta\cos\theta+\eta^*\sin\theta)\hat{a}_2^\dagger\\&+\hat{a}_1^{\dagger 2}\sin\theta\cos\theta-\hat{a}_2^{\dagger 2}\sin\theta\cos\theta\\&+\hat{a}_1^\dagger\hat{a}_2^\dagger(\sin^2\theta-\cos^2\theta)\bigg]|00\rangle,\end{aligned}\tag{4}$$

where

$$\begin{aligned}&\hat{x}_i=\frac{\hat{a}_i+\hat{a}_i^\dagger}{\sqrt{2}},\quad \hat{p}_i=\frac{\hat{a}_i-\hat{a}_i^\dagger}{\sqrt{2}\,\mathrm{i}},\\&(i=1,2),\quad \eta=\eta_1+\mathrm{i}\,\eta_2.\end{aligned}\tag{5}$$

In fact, using $[\hat{a}_i,\hat{a}_j^\dagger]=\delta_{ij}$, $[\hat{a}_i, :f(\hat{a}_i^\dagger,\hat{a}_i):]=:\frac{\partial}{\partial\hat{a}_i^\dagger}f(\hat{a}_i^\dagger,\hat{a}_i):$, and $\hat{a}_i|00\rangle=0$ $(i,j=1,2)$, we have

$$\begin{aligned}\hat{a}_1|\eta;\theta\rangle &= [(\eta\sin\theta-\eta^*\cos\theta)+2\hat{a}_1^\dagger\sin\theta\cos\theta\\&\quad+\hat{a}_2^\dagger(\sin^2\theta-\cos^2\theta)]|\eta;\theta\rangle,\\ \hat{a}_2|\eta;\theta\rangle &= [-(\eta\cos\theta+\eta^*\sin\theta)-2\hat{a}_2^\dagger\sin\theta\cos\theta\\&\quad+\hat{a}_1^\dagger(\sin^2\theta-\cos^2\theta)]|\eta;\theta\rangle.\end{aligned}\tag{6}$$

It follows that

$$\begin{aligned}&[(\hat{x}_1-\hat{x}_2)\sin\theta-(\hat{x}_1+\hat{x}_2)\cos\theta]|\eta;\theta\rangle=\sqrt{2}\eta_1|\eta;\theta\rangle,\\&[(\hat{p}_1+\hat{p}_2)\sin\theta+(\hat{p}_1-\hat{p}_2)\cos\theta]|\eta;\theta\rangle=\sqrt{2}\eta_2|\eta;\theta\rangle,\end{aligned}\tag{7}$$

where η is a complex number, whose real part η_1 and imaginary part η_2 multiplied by $\sqrt{2}$ are indeed the eigenvalues of $[(\hat{x}_1-\hat{x}_2)\sin\theta-(\hat{x}_1+\hat{x}_2)\cos\theta]$ and $[(\hat{p}_1+\hat{p}_2)\sin\theta+(\hat{p}_1-\hat{p}_2)\cos\theta]$, respectively.

Using the normal ordered product of the two-mode vacuum projector

$$|00\rangle\langle 00| =: \exp\{-\hat{a}_1^\dagger\hat{a}_1-\hat{a}_2^\dagger\hat{a}_2\}:\tag{8}$$

and IWOP technique, we can smoothly prove the completeness relation of $|\eta;\theta\rangle$

$$\begin{aligned}\int\frac{\mathrm{d}^2\eta}{\pi}|\eta;\theta\rangle\langle\eta;\theta| = \int\frac{\mathrm{d}^2\eta}{\pi}&:\exp[-|\eta|^2+(\eta\sin\theta-\eta^*\cos\theta)\hat{a}_1^\dagger-(\eta\cos\theta+\eta^*\sin\theta)\hat{a}_2^\dagger\\&+\hat{a}_1^{\dagger 2}\sin\theta\cos\theta-\hat{a}_2^{\dagger 2}\sin\theta\cos\theta+\hat{a}_1^\dagger\hat{a}_2^\dagger(\sin^2\theta-\cos^2\theta)-\hat{a}_1^\dagger\hat{a}_1-\hat{a}_2^\dagger\hat{a}_2\\&+(\eta^*\sin\theta-\eta\cos\theta)\hat{a}_1-(\eta^*\cos\theta+\eta\sin\theta)\hat{a}_2\\&+\hat{a}_1^2\sin\theta\cos\theta-\hat{a}_2^2\sin\theta\cos\theta+\hat{a}_1\hat{a}_2(\sin^2\theta-\cos^2\theta)]:=1.\end{aligned}\tag{9}$$

Employing the over-completeness relation of the two-mode coherent state

$$\int\frac{\mathrm{d}^2z_1\mathrm{d}^2z_2}{\pi^2}|z_1,z_2\rangle\langle z_1,z_2|=1\tag{10}$$

and noticing the overlap

$$\begin{aligned}\langle z_1,z_2|\eta;\theta\rangle = \exp\bigg[&-\frac{|z_1|^2}{2}-\frac{|z_2|^2}{2}-\frac{|\eta|^2}{2}+(\eta\sin\theta-\eta^*\cos\theta)z_1^*\\&-(\eta\cos\theta+\eta^*\sin\theta)z_2^*+z_1^{*2}\sin\theta\cos\theta-z_2^{*2}\sin\theta\cos\theta+z_1^*z_2^*(\sin^2\theta-\cos^2\theta)\bigg],\end{aligned}\tag{11}$$

we obtain

$$\begin{aligned}\langle\eta';\theta|\eta;\theta\rangle &= \int\frac{\mathrm{d}^2z_1\mathrm{d}^2z_2}{\pi^2}\langle\eta';\theta|z_1,z_2\rangle\langle z_1,z_2|\eta;\theta\rangle\\&=\int\frac{\mathrm{d}^2z_1\mathrm{d}^2z_2}{\pi^2}\exp\bigg\{-|z_1|^2+z_1[(\eta'^*\sin\theta-\eta'\cos\theta)+z_2(\sin^2\theta-\cos^2\theta)]\\&\quad+z_1^*[(\eta\sin\theta-\eta^*\cos\theta)+z_2^*(\sin^2\theta-\cos^2\theta)]+z_1^2\sin\theta\cos\theta\\&\quad+z_1^{*2}\sin\theta\cos\theta-|z_2|^2-z_2(\eta'\sin\theta+\eta'^*\cos\theta)-z_2^*(\eta\cos\theta+\eta^*\sin\theta)\\&\quad-z_2^2\sin\theta\cos\theta-z_2^{*2}\sin\theta\cos\theta-\frac{|\eta'|^2}{2}-\frac{|\eta|^2}{2}\bigg\}\\&=\pi\delta(\eta_1-\eta_1')\delta(\eta_2-\eta_2'),\end{aligned}\tag{12}$$

where we have used the mathematical formulas

$$\int \frac{\mathrm{d}^2 z}{\pi} \exp(\zeta|z|^2 + \xi z + \eta z^* + f z^2 + g z^{*2}) = \frac{1}{\sqrt{\zeta^2 - 4fg}} \exp\left(\frac{-\zeta\xi\eta + \xi^2 g + \eta^2 f}{\zeta^2 - 4fg}\right), \tag{13}$$

$$\delta(x) = \lim_{\varepsilon\to 0} \frac{1}{\sqrt{\pi\varepsilon}} \exp\left(\frac{-x^2}{\varepsilon}\right). \tag{14}$$

Thus the entangled state $|\eta;\theta\rangle$ is orthogonal.

Furthermore, by making the Fourier transformation and the inverse Fourier transform, we obtain

$$\begin{aligned}\left|\eta = \frac{\eta_1 + \mathrm{i}\eta_2}{\sqrt{2}};\theta\right\rangle = & \int_{-\infty}^{\infty} \mathrm{d}u \exp(\mathrm{i}u\eta_2)|x_1 = u(\sin\theta + \cos\theta) + \frac{\eta_1}{2}(\sin\theta - \cos\theta)\rangle \\ & \otimes\left|x_2 = u(\sin\theta - \cos\theta) - \frac{\eta_1}{2}(\sin\theta + \cos\theta)\right\rangle,\end{aligned} \tag{15}$$

which is the Schmidt decomposition of $|\eta;\theta\rangle$, and confirms that $|\eta;\theta\rangle$ itself is an entangled state, where

$$|x_1\rangle = \frac{1}{\pi^{1/4}} \exp\left(-\frac{x_1^2}{2} + \sqrt{2}x_1\hat{a}_1^\dagger - \frac{\hat{a}_1^{\dagger 2}}{2}\right)|0\rangle_1 \quad \text{and} \quad |x_2\rangle = \frac{1}{\pi^{1/4}} \exp\left(-\frac{x_2^2}{2} + \sqrt{2}x_2\hat{a}_2^\dagger - \frac{\hat{a}_2^{\dagger 2}}{2}\right)|0\rangle_2$$

are the first-mode and the second-mode coordinate eigenstates, respectively.

From Eqs. (9), (12) and (15), we conclude that $|\eta;\theta\rangle$ exhibits the properties of entangled states.

3. Properties of generalized Hadamard transformation operator

Note that

$$\begin{aligned}&[(\hat{a}_1\sin\theta - \hat{a}_2\cos\theta), (\hat{a}_1^\dagger\cos\theta + \hat{a}_2^\dagger\sin\theta)] = 0, \quad [(\hat{a}_1\cos\theta + \hat{a}_2\sin\theta), (\hat{a}_1^\dagger\sin\theta - \hat{a}_2^\dagger\cos\theta)] = 0, \\ &[(\hat{a}_1\sin\theta - \hat{a}_2\cos\theta), (\hat{a}_1^\dagger\sin\theta - \hat{a}_2^\dagger\cos\theta)] = 1, \quad [(\hat{a}_1\cos\theta + \hat{a}_2\sin\theta), (\hat{a}_1^\dagger\cos\theta + \hat{a}_2^\dagger\sin\theta)] = 1,\end{aligned} \tag{16}$$

$(\hat{a}_1\sin\theta - \hat{a}_2\cos\theta)$can be considered to be a mode independent of another mode $(\hat{a}_1\cos\theta + \hat{a}_2\sin\theta)$, then we will have the following operator identities:

$$\begin{aligned}&\exp[\lambda(\hat{a}_1^\dagger\sin\theta - \hat{a}_2^\dagger\cos\theta)(\hat{a}_1\sin\theta - \hat{a}_2\cos\theta)] =: \exp[(\mathrm{e}^\lambda - 1)(\hat{a}_1^\dagger\sin\theta - \hat{a}_2^\dagger\cos\theta)(\hat{a}_1\sin\theta - \hat{a}_2\cos\theta)] :, \\ &\exp[\lambda(\hat{a}_1^\dagger\cos\theta + \hat{a}_2^\dagger\sin\theta)(\hat{a}_1\cos\theta + \hat{a}_2\sin\theta)] =: \exp[(\mathrm{e}^\lambda - 1)(\hat{a}_1^\dagger\cos\theta + \hat{a}_2^\dagger\sin\theta)(\hat{a}_1\cos\theta + \hat{a}_2\sin\theta)] : .\end{aligned} \tag{17}$$

According to the entangled representation $|\eta;\theta\rangle$ and Eq. (2), we now construct the following ket-bra integration:

$$\hat{U} = \int \frac{\mathrm{d}^2\eta}{\sigma^2\pi} \int \frac{\mathrm{d}^2\eta'}{\pi} \exp\left(\frac{\eta^*\eta' - \eta\eta'^*}{\sigma^2}\right)|\eta;\theta\rangle\langle\eta';\theta|. \tag{18}$$

Substituting Eq. (4) into Eq. (18) and using the IWOP technique we obtain

$$\begin{aligned}\hat{U} = & \int \frac{\mathrm{d}^2\eta}{\sigma^2\pi} \int \frac{\mathrm{d}^2\eta'}{\pi} : \exp[-\frac{|\eta|^2 + |\eta'|^2}{2} + \frac{\eta^*\eta' - \eta\eta'^*}{\sigma^2} + (\eta\sin\theta - \eta^*\cos\theta)\hat{a}_1^\dagger \\ & - (\eta\cos\theta + \eta^*\sin\theta)\hat{a}_2^\dagger + \hat{a}_1^{\dagger 2}\sin\theta\cos\theta - \hat{a}_2^{\dagger 2}\sin\theta\cos\theta + \hat{a}_1^\dagger\hat{a}_2^\dagger(\sin^2\theta - \cos^2\theta) \\ & - \hat{a}_1^\dagger\hat{a}_1 - \hat{a}_2^\dagger\hat{a}_2 + (\eta'^*\sin\theta - \eta'\cos\theta)\hat{a}_1 - (\eta'^*\cos\theta + \eta'\sin\theta)\hat{a}_2 \\ & + \hat{a}_1^2\sin\theta\cos\theta - \hat{a}_2^2\sin\theta\cos\theta + \hat{a}_1\hat{a}_2(\sin^2\theta - \cos^2\theta)] : \\ = & \frac{4\sigma^2}{\sigma^4 + 4} : \exp\left[\frac{4 - \sigma^4}{\sigma^4 + 4}(\hat{a}_1^\dagger\sin\theta - \hat{a}_2^\dagger\cos\theta)(\hat{a}_1^\dagger\cos\theta + \hat{a}_2^\dagger\sin\theta)\right. \\ & + \left(\frac{4\sigma^2}{\sigma^4 + 4} - 1\right)(\hat{a}_1^\dagger\sin\theta - \hat{a}_2^\dagger\cos\theta)(\hat{a}_1\sin\theta - \hat{a}_2\cos\theta) \\ & + \left(\frac{-4\sigma^2}{\sigma^4 + 4} - 1\right)(\hat{a}_1^\dagger\cos\theta + \hat{a}_2^\dagger\sin\theta)(\hat{a}_1\cos\theta + \hat{a}_2\sin\theta)\end{aligned}$$

$$+\frac{4-\sigma^4}{\sigma^4+4}(\hat{a}_1\sin\theta-\hat{a}_2\cos\theta)(\hat{a}_1\cos\theta+\hat{a}_2\sin\theta)]:. \tag{19}$$

Using Eq. (17), we obtain

$$\begin{aligned}\hat{U} &= \frac{4\sigma^2}{\sigma^4+4}\exp\left[\frac{4-\sigma^4}{\sigma^4+4}(\hat{a}_1^\dagger\sin\theta-\hat{a}_2^\dagger\cos\theta)(\hat{a}_1^\dagger\cos\theta+\hat{a}_2^\dagger\sin\theta)\right]\\ &\quad\times\exp\left[(\hat{a}_1^\dagger\sin\theta-\hat{a}_2^\dagger\cos\theta)(\hat{a}_1\sin\theta-\hat{a}_2\cos\theta)\ln\left(\frac{4\sigma^2}{\sigma^4+4}\right)\right.\\ &\quad\left.+(\hat{a}_1^\dagger\cos\theta+\hat{a}_2^\dagger\sin\theta)(\hat{a}_1\cos\theta+\hat{a}_2\sin\theta)\ln\left(\frac{-4\sigma^2}{\sigma^4+4}\right)\right]\\ &\quad\times\exp\left[\frac{4-\sigma^4}{\sigma^4+4}(\hat{a}_1\sin\theta-\hat{a}_2\cos\theta)(\hat{a}_1\cos\theta+\hat{a}_2\sin\theta)\right],\end{aligned} \tag{20}$$

which is the normal ordered form of generalized Hadamard operator for $(\hat{a}_1\sin\theta-\hat{a}_2\cos\theta)$ and $(\hat{a}_1\cos\theta+\hat{a}_2\sin\theta)$ and is unitary $\hat{U}^+\hat{U}=\hat{U}\hat{U}^+=1$.

Using Eq. (20) we have

$$\begin{aligned}\hat{U}(\hat{a}_1\sin\theta-\hat{a}_2\cos\theta)\hat{U}^{-1} &= \frac{1}{4\sigma^2}[(\sigma^4+4)(\hat{a}_1\sin\theta-\hat{a}_2\cos\theta)+(\sigma^4-4)(\hat{a}_1^\dagger\cos\theta+\hat{a}_2^\dagger\sin\theta)],\\ \hat{U}(\hat{a}_1^\dagger\sin\theta-\hat{a}_2^\dagger\cos\theta)\hat{U}^{-1} &= \frac{1}{4\sigma^2}[(\sigma^4+4)(\hat{a}_1^\dagger\sin\theta-\hat{a}_2^\dagger\cos\theta)+(\sigma^4-4)(\hat{a}_1\cos\theta+\hat{a}_2\sin\theta)],\end{aligned} \tag{21}$$

and

$$\begin{aligned}\hat{U}(\hat{a}_1\cos\theta+\hat{a}_2\sin\theta)\hat{U}^{-1} &= \frac{-1}{4\sigma^2}[(\sigma^4+4)(\hat{a}_1\cos\theta+\hat{a}_2\sin\theta)+(\sigma^4-4)(\hat{a}_1^\dagger\sin\theta-\hat{a}_2^\dagger\cos\theta)],\\ \hat{U}(\hat{a}_1^\dagger\cos\theta+\hat{a}_2^\dagger\sin\theta)\hat{U}^{-1} &= \frac{-1}{4\sigma^2}[(\sigma^4+4)(\hat{a}_1^\dagger\cos\theta+\hat{a}_2^\dagger\sin\theta)+(\sigma^4-4)(\hat{a}_1\sin\theta-\hat{a}_2\cos\theta)],\end{aligned} \tag{22}$$

which lead to

$$\begin{aligned}\hat{U}[(\hat{x}_1-\hat{x}_2)\sin\theta-(\hat{x}_1+\hat{x}_2)\cos\theta)]\hat{U}^{-1} &= \frac{\sigma^2}{2}[(\hat{x}_1+\hat{x}_2)\sin\theta+(\hat{x}_1-\hat{x}_2)\cos\theta)],\\ \hat{U}[(\hat{p}_1+\hat{p}_2)\sin\theta+(\hat{p}_1-\hat{p}_2)\cos\theta)]\hat{U}^{-1} &= \frac{\sigma^2}{2}[(\hat{p}_1-\hat{p}_2)\sin\theta-(\hat{p}_1+\hat{p}_2)\cos\theta)],\end{aligned} \tag{23}$$

from which we see that the generalized Hadamard operator can also play a role in exchanging coordinates $[(\hat{x}_1-\hat{x}_2)\sin\theta-(\hat{x}_1+\hat{x}_2)\cos\theta)]$ and $[(\hat{x}_1+\hat{x}_2)\sin\theta+(\hat{x}_1-\hat{x}_2)\cos\theta)]$ or momenta $[(\hat{p}_1+\hat{p}_2)\sin\theta+(\hat{p}_1-\hat{p}_2)\cos\theta)]$ and $[(\hat{p}_1-\hat{p}_2)\sin\theta-(\hat{p}_1+\hat{p}_2)\cos\theta)]$, followed by a squeezing transform, with the squeezing parameter being $\frac{\sigma^2}{2}$.

In particular, when $\theta=\frac{\pi}{2}$, we have

$$\left|\eta;\frac{\pi}{2}\right\rangle=\exp\left[-\frac{|\eta|^2}{2}+\eta\hat{a}_1^\dagger-\eta^*\hat{a}_2^\dagger+\hat{a}_1^\dagger\hat{a}_2^\dagger\right]|00\rangle, \tag{24}$$

which is the common eigenvector of $(\hat{x}_1-\hat{x}_2)$ and $(\hat{p}_1+\hat{p}_2)$. The Hadamard operator in this entangled state representation is

$$\begin{aligned}\hat{U} &= \frac{4\sigma^2}{\sigma^4+4}\exp\left(\frac{4-\sigma^4}{\sigma^4+4}\hat{a}_1^\dagger\hat{a}_2^\dagger\right)\\ &\quad\times\exp\left[\hat{a}_1^\dagger\hat{a}_1\ln\left(\frac{4\sigma^2}{\sigma^4+4}\right)+\hat{a}_2^\dagger\hat{a}_2\ln\left(\frac{-4\sigma^2}{\sigma^4+4}\right)\right]\\ &\quad\times\exp\left(\frac{4-\sigma^4}{\sigma^4+4}\hat{a}_1\hat{a}_2\right).\end{aligned} \tag{25}$$

It then follows that

$$\begin{aligned}\hat{U}(\hat{x}_1-\hat{x}_2)]\hat{U}^{-1} &= \frac{\sigma^2}{2}(\hat{x}_1+\hat{x}_2),\\ \hat{U}[(\hat{p}_1+\hat{p}_2)\hat{U}^{-1} &= \frac{\sigma^2}{2}(\hat{p}_1-\hat{p}_2),\end{aligned} \tag{26}$$

which is in agreement with the result of Ref. [8].

In particular, when $\theta=\pi$, we have

$$\begin{aligned}|\eta=\xi^*;\pi\rangle &= \exp\left[-\frac{|\xi|^2}{2}+\xi\hat{a}_1^\dagger\right.\\ &\quad\left.+\xi^*\hat{a}_2^\dagger-\hat{a}_1^\dagger\hat{a}_2^\dagger\right]|00\rangle,\end{aligned} \tag{27}$$

which is the common eigenvector of $(\hat{x}_1+\hat{x}_2)$ and $(\hat{p}_1-\hat{p}_2)$. The Hadamard operator in this entangled state representation is

$$\hat{U} = \frac{4\sigma^2}{\sigma^4+4}\exp\left(\frac{\sigma^4-4}{\sigma^4+4}\hat{a}_1^\dagger\hat{a}_2^\dagger\right)$$

$$\times \exp\left[\left(\hat{a}_1^\dagger a_1\right)\ln\left(\frac{-4\sigma^2}{\sigma^4+4}\right) + (-\hat{a}_2^\dagger\hat{a}_2)\ln\left(\frac{4\sigma^2}{\sigma^4+4}\right)\right] \times \exp\left(\frac{\sigma^4-4}{\sigma^4+4}\hat{a}_1\hat{a}_2\right). \quad (28)$$

It then follows that

$$\hat{U}(\hat{x}_1+\hat{x}_2)]\hat{U}^{-1} = \frac{\sigma^2}{2}(\hat{x}_2-\hat{x}_1),$$
$$\hat{U}[(\hat{p}_2-\hat{p}_1)\hat{U}^{-1} = \frac{\sigma^2}{2}(\hat{p}_1+\hat{p}_2). \quad (29)$$

In particular, when $\theta = \frac{\pi}{4}$, we have

$$\left|\eta;\frac{\pi}{4}\right\rangle = \exp\left[-\frac{|\eta|^2}{2} + \eta\frac{\hat{a}_1^\dagger - \hat{a}_2^\dagger}{\sqrt{2}} - \eta^*\frac{\hat{a}_1^\dagger + \hat{a}_2^\dagger}{\sqrt{2}} + \frac{1}{2}(\hat{a}_1^{\dagger 2} - \hat{a}_2^{\dagger 2})\right]|00\rangle, \quad (30)$$

which is the common eigenvector of $(\hat{a}_1 - \hat{a}_1^\dagger)$ and $(\hat{a}_2 + \hat{a}_2^\dagger)$. The Hadamard operator in this entangled state representation is

$$\hat{U} = \frac{4\sigma^2}{\sigma^4+4}\exp\left[\frac{4-\sigma^4}{\sigma^4+4}\frac{(\hat{a}_1^\dagger - \hat{a}_2^\dagger)}{\sqrt{2}}\frac{(\hat{a}_1^\dagger + \hat{a}_2^\dagger)}{\sqrt{2}}\right] \times \exp\left[\frac{(\hat{a}_1^\dagger - \hat{a}_2^\dagger)}{\sqrt{2}}\frac{(\hat{a}_1 - \hat{a}_2)}{\sqrt{2}}\ln\left(\frac{4\sigma^2}{\sigma^4+4}\right) + \frac{(\hat{a}_1^\dagger + \hat{a}_2^\dagger)}{\sqrt{2}}\frac{(\hat{a}_1 + \hat{a}_2)}{\sqrt{2}}\ln\left(\frac{-4\sigma^2}{\sigma^4+4}\right)\right] \times \exp\left[\frac{4-\sigma^4}{\sigma^4+4}\frac{(\hat{a}_1 - \hat{a}_2)}{\sqrt{2}}\frac{(\hat{a}_1 + \hat{a}_2)}{\sqrt{2}}\right], \quad (31)$$

which is the unitary Hadamard operator for $(\frac{\hat{a}_1+\hat{a}_2}{\sqrt{2}})$ and $(\frac{\hat{a}_1-\hat{a}_2}{\sqrt{2}})$. It then follows that

$$\hat{U}(-\hat{x}_2)]\hat{U}^{-1} = \frac{\sigma^2}{2}\hat{x}_1, \quad \hat{U}\hat{p}_1\hat{U}^{-1} = \frac{\sigma^2}{2}(-\hat{p}_2). \quad (32)$$

4. Summary

With the aid of the IWOP technique, we have proposed a new entangled state representation $|\eta;\theta\rangle$. We have proved the completeness relation of $|\eta;\theta\rangle$ and shown that $|\eta;\theta\rangle$ is orthogonal and entangled. We have also derived a generalized Hadamard operator. This unitary operator plays a role of Hadamard transformation for $(\hat{a}_1\sin\theta - \hat{a}_2\cos\theta)$ and $(\hat{a}_1\cos\theta + \hat{a}_2\sin\theta)$. We have shown that this transformation is concisely expressed in the entangled state representation as a projective operator in integration form. In particular, when $\theta = \frac{\pi}{2}$ and $\theta = \pi$, this unitary operator plays a role of Hadamard transformation for $\hat{a}_1$ and $\hat{a}_2$, when $\theta = -\frac{\pi}{4}$, this unitary operator plays a role of Hadamard transformation for $(\frac{\hat{a}_1+\hat{a}_2}{\sqrt{2}})$ and $(\frac{\hat{a}_1-\hat{a}_2}{\sqrt{2}})$.

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