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A two-component μ-Hunter–Saxton equation

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Abstract
In this paper, we propose a two-component generalization of the generalized Hunter–Saxton equation obtained in Khesin *et al* (2008 *Math. Ann.* **342** 617–56). We will show that this equation is a bi-Hamiltonian Euler equation, and can also be viewed as a bi-variational equation.

1. Introduction

Arnold in [1] suggested a general framework for the Euler equations on an arbitrary (possibly infinite-dimensional) Lie algebra $\mathcal{G}$. In many cases, the Euler equations on $\mathcal{G}$ describe geodesic flows with respect to a suitable one-side invariant Riemannian metric on the corresponding group G. Now it is well known that Arnold's approach to the Euler equation works very well for the Virasoro algebra and its extensions, see [6, 10, 13–15, 19] and references therein.

Let $\mathcal{D}(\mathbb{S}^1)$ be a group of orientation preserving diffeomorphisms of the circle and $G = \mathcal{D}(\mathbb{S}^1) \oplus \mathbb{R}$ be the Bott–Virasoro group. In [6], Ovsienko and Khesin showed that the KdV equation is an Euler equation, describing a geodesic flow on G with respect to a right-invariant L^2 metric. Another interesting example is the Camassa–Holm equation, which was originally derived in [4] as an abstract equation with a bi-Hamiltonian structure, and independently in [9] as a shallow water approximation. In [10], Misiolek showed that the Camassa–Holm equation is also an Euler equation for a geodesic flow on G with respect to a right-invariant Sobolev H^1-metric.

In [13], Khesin and Misiolek extended Arnold's approach to homogeneous spaces and provided a beautiful geometric setting for the Hunter–Saxton equation, which firstly appeared in [8] as an asymptotic equation for rotators in liquid crystals, and its relatives. They showed that the Hunter–Saxton equation is an Euler equation describing the geodesic flow on the homogeneous spaces of the Bott–Virasoro group G modulo rotations with respect to a right-invariant homogeneous $\dot{H}^1$-metric.

Furthermore, by using extended Bott–Virasoro groups, Guha and others [11, 16, 21] generalized the above results to two-component integrable systems, including several coupled KdV-type systems, and two-component peak-type systems, especially two-component

Camassa–Holm equation which was introduced by Chen *et al* [17] and independently by Falqui [18]. Another interesting topic is to discuss the super or supersymmetric analogue, see [6, 12, 16, 20, 23, 24] and references therein.

Recently Khesin *et al* in [22] introduced a generalized Hunter–Saxton (μ-HS in brief) equation lying midway between the periodic Hunter–Saxton and Camassa–Holm equations:

$$-f_{txx} = -2\mu(f)f_x + 2f_xf_{xx} + ff_{xxx}, \tag{1.1}$$

where $f = f(t,x)$ is a time-dependent function on the unit circle $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ and $\mu(f) = \int_{\mathbb{S}^1} f\,\mathrm{d}x$ denotes its mean. This equation describes evolution of rotators in liquid crystals with an external magnetic field and self-interaction.

Let $\mathcal{D}^s(\mathbb{S}^1)$ be a group of orientation preserving Sobolev H^s diffeomorphisms of the circle. They proved that the μ-HS equation (1.1) describes a geodesic flow on $\mathcal{D}^s(\mathbb{S}^1)$ with a right-invariant metric given at the identity by the inner product

$$\langle f, g\rangle_\mu = \mu(f)\mu(g) + \int_{\mathbb{S}^1} f'(x)g'(x)\,\mathrm{d}x. \tag{1.2}$$

They also showed that (1.2) is bi-Hamiltonian and admits both cusped and smooth travelling wave solutions which are natural candidates for solitons. In this paper, we want to generalize these to a two-component μ-HS (2-μHS in brief) equation. Our main object is the Lie algebra $\mathcal{G} = \mathrm{Vect}^s(\mathbb{S}^1) \ltimes C^\infty(\mathbb{S}^1)$ and its three-dimensional central extension $\widehat{\mathcal{G}}$. Firstly, we introduce an inner product on $\widehat{\mathcal{G}}$ given by

$$\langle \hat{f}, \hat{g}\rangle_\mu = \mu(f)\mu(g) + \int_{\mathbb{S}^1} (f'(x)g'(x) + a(x)b(x))\,\mathrm{d}x + \overrightarrow{\alpha}\cdot\overrightarrow{\beta}, \tag{1.3}$$

where $\hat{f} = \left(f(x)\frac{\mathrm{d}}{\mathrm{d}x}, a(x), \overrightarrow{\alpha}\right), \hat{g} = \left(g(x)\frac{\mathrm{d}}{\mathrm{d}x}, b(x), \overrightarrow{\beta}\right)$ and $\overrightarrow{\alpha}, \overrightarrow{\beta} \in \mathbb{R}^3$. Afterwards, we have

Theorem 1.1 (=Theorem 2.2)**.** *The Euler equation on $\widehat{\mathcal{G}}^*_{\mathrm{reg}}$ with respect to (1.3) is a 2-μHS equation*

$$\begin{cases} -f_{xxt} = 2\mu(f)f_x - 2f_xf_{xx} - ff_{xxx} + v_xv - \gamma_1 f_{xxx} + \gamma_2 v_{xx}, \\ v_t = (vf)_x - \gamma_2 f_{xx} + 2\gamma_3 v_x, \end{cases} \tag{1.4}$$

where $\gamma_j \in \mathbb{R}$, $j = 1, 2, 3$.

Actually from the geometric view, if we extend the inner product (1.3) to a left-invariant metric on $\widehat{G} = \mathcal{D}^s(\mathbb{S}^1) \ltimes C^\infty(\mathbb{S}^1) \oplus \mathbb{R}^3$, we can view the 2-$\mu$HS equation (1.4) as a geodesic flow on $\widehat{G}$ with respect to this left-invariant metric. Obviously, if we choose $v = 0$ and $\gamma_j = 0$, $j = 1, 2, 3$, and replace t by $-t$, (1.4) reduces to (1.1). Furthermore, we show that

Theorem 1.2 (=Theorems 3.1 and 4.1)**.** *The 2-μHS equation (1.4) can be viewed as a bi-Hamiltonian and bi-variational equation.*

This paper is organized as follows. In section 2, we calculate the Euler equation on $\widehat{\mathcal{G}}^*_{\mathrm{reg}}$. In section 3, we study the Hamiltonian nature and the Lax pair of the 2-μHS equation (1.4). Section 4 is devoted to discuss the variational nature of (1.4). In the last section we describe the interrelation between bi-Hamiltonian natures and bi-variational natures.

2. Eulerian nature of the 2-μHS equation

Let $\mathcal{D}^s(\mathbb{S}^1)$ be a group of orientation preserving Sobolev H^s diffeomorphisms of the circle and let $T_{id}\mathcal{D}^s(\mathbb{S}^1)$ be the corresponding Lie algebra of vector fields denoted by $\mathrm{Vect}^s(\mathbb{S}^1) = \left\{ f(x)\frac{\mathrm{d}}{\mathrm{d}x} \middle| f(x) \in H^s(\mathbb{S}^1)\right\}$.

The main objects in our paper will be the group $\mathcal{D}^s(\mathbb{S}^1) \ltimes \mathrm{C}^\infty(\mathbb{S}^1)$, its Lie algebra $\mathcal{G} = \mathrm{Vect}^s(\mathbb{S}^1) \ltimes \mathrm{C}^\infty(\mathbb{S}^1)$ with the Lie bracket given by

$$\left[\left(f(x)\frac{\mathrm{d}}{\mathrm{d}x}, a(x)\right), \left(g(x)\frac{\mathrm{d}}{\mathrm{d}x}, b(x)\right)\right] = \left((f(x)g'(x) - f'(x)g(x))\frac{\mathrm{d}}{\mathrm{d}x}, f(x)b'(x) - a'(x)g(x)\right),$$

and their central extensions. It is well known in [3, 7] that the algebra $\mathcal{G}$ has a three-dimensional central extension given by the following nontrivial cocycles:

$$\begin{aligned}
\omega_1\left(\left(f(x)\frac{\mathrm{d}}{\mathrm{d}x}, a(x)\right), \left(g(x)\frac{\mathrm{d}}{\mathrm{d}x}, b(x)\right)\right) &= \int_{\mathbb{S}^1} f'(x)g''(x)\,\mathrm{d}x,\\
\omega_2\left(\left(f(x)\frac{\mathrm{d}}{\mathrm{d}x}, a(x)\right), \left(g(x)\frac{\mathrm{d}}{\mathrm{d}x}, b(x)\right)\right) &= \int_{\mathbb{S}^1} [f''(x)b(x) - g''(x)a(x)]\,\mathrm{d}x,\\
\omega_3\left(\left(f(x)\frac{\mathrm{d}}{\mathrm{d}x}, a(x)\right), \left(g(x)\frac{\mathrm{d}}{\mathrm{d}x}, b(x)\right)\right) &= 2\int_{\mathbb{S}^1} a(x)b''(x)\,\mathrm{d}x,
\end{aligned} \tag{2.1}$$

where $f(x), g(x) \in H^s(\mathbb{S}^1)$ and $a(x), b(x) \in \mathrm{C}^\infty(\mathbb{S}^1)$. Note that the first cocycle ω_1 is the well-known Gelfand–Fuchs cocycle [2, 5]. The Virasoro algebra $\mathrm{Vir} = \mathrm{Vect}^s(\mathbb{S}^1) \oplus \mathbb{R}$ is the unique non-trivial central extension of $\mathrm{Vect}^s(\mathbb{S}^1)$ via the Gelfand–Fuchs cocycle ω_1. Sometimes we would like to use the modified Gelfand–Fuchs cocycle

$$\tilde{\omega}_1\left(\left(f(x)\frac{\mathrm{d}}{\mathrm{d}x}, a(x)\right), \left(g(x)\frac{\mathrm{d}}{\mathrm{d}x}, b(x)\right)\right) = \int_{\mathbb{S}^1} (c_1 f'(x)g''(x) + c_2 f'(x)g(x))\,\mathrm{d}x, \tag{2.2}$$

which is cohomologeous to the Gelfand–Fuchs cocycle ω_1, where $c_1, c_2 \in \mathbb{R}$.

Definition 2.1. *The algebra $\widehat{\mathcal{G}}$ is an extension of $\mathcal{G}$ defined by*

$$\widehat{\mathcal{G}} = \mathrm{Vect}^s(\mathbb{S}^1) \ltimes C^\infty(\mathbb{S}^1) \oplus \mathbb{R}^3 \tag{2.3}$$

with the commutation relation

$$[\hat{f}, \hat{g}] = \left((fg' - f'g)\frac{\mathrm{d}}{\mathrm{d}x}, fb' - a'g, \overrightarrow{\omega}\right), \tag{2.4}$$

where $\hat{f} = \left(f(x)\frac{\mathrm{d}}{\mathrm{d}x}, a(x), \overrightarrow{\alpha}\right)$, $\hat{g} = \left(g(x)\frac{\mathrm{d}}{\mathrm{d}x}, b(x), \overrightarrow{\beta}\right)$ *and* $\overrightarrow{\alpha}, \overrightarrow{\beta} \in \mathbb{R}^3$ *and* $\overrightarrow{\omega} = (\omega_1, \omega_2, \omega_3) \in \mathbb{R}^3$.

Let

$$\widehat{\mathcal{G}}^*_{\mathrm{reg}} = \mathrm{C}^\infty(\mathbb{S}^1) \oplus \mathrm{C}^\infty(\mathbb{S}^1) \oplus \mathbb{R}^3 \tag{2.5}$$

denote the regular part of the dual space $\widehat{\mathcal{G}}^*$ to the Lie algebra $\widehat{\mathcal{G}}$ under the pairing

$$\langle \hat{u}, \hat{f}\rangle^* = \int_{\mathbb{S}^1} (u(x)f(x) + a(x)v(x))\,\mathrm{d}x + \overrightarrow{\alpha} \cdot \overrightarrow{\gamma}, \tag{2.6}$$

where $\hat{u} = (u(x)(\mathrm{d}x)^2, v(x), \overrightarrow{\gamma}) \in \widehat{\mathcal{G}}^*$. Of particular interest are the coadjoint orbits in $\widehat{\mathcal{G}}^*_{\mathrm{reg}}$.

On $\widehat{\mathcal{G}}$, let us introduce an inner product

$$\langle \hat{f}, \hat{g}\rangle_\mu = \mu(f)\mu(g) + \int_{\mathbb{S}^1} (f'(x)g'(x) + a(x)b(x))\,\mathrm{d}x + \overrightarrow{\alpha} \cdot \overrightarrow{\beta}. \tag{2.7}$$

A direct computation gives

$$\langle \hat{f}, \hat{g}\rangle_\mu = \langle \hat{f}, (\Lambda(g)(\mathrm{d}x)^2, b(x), \overrightarrow{\beta})\rangle^*, \qquad \Lambda(g) = \mu(g) - g''(x),$$

which induces an inertia operator $\mathcal{A} : \widehat{\mathcal{G}} \longrightarrow \widehat{\mathcal{G}}^*_{\mathrm{reg}}$ given by

$$\mathcal{A}(\hat{g}) = (\Lambda(g)(\mathrm{d}x)^2, b(x), \overrightarrow{\beta}). \tag{2.8}$$

Theorem 2.2. *The 2-μHS equation (1.4) is an Euler equation on $\widehat{\mathcal{G}}^*_{\mathrm{reg}}$ with respect to the inner product (2.7).*

Proof. By definition,

$$\begin{aligned}\langle ad^*_{\hat{f}}(\hat{u}), \hat{g}\rangle^* &= -\langle \hat{u}, [\hat{f}, \hat{g}]\rangle^* \qquad \text{(by using integration by parts)}\\ &= \langle ((2uf_x + u_x f + a_x v - \alpha_1 f_{xxx} + \alpha_2 a_{xx})(\mathrm{d}x)^2, (vf)_x - \alpha_2 f_{xx} + 2\alpha_3 a_x, 0), \hat{g}\rangle^*.\end{aligned}$$

This gives

$$ad^*_{\hat{f}}(\hat{u}) = ((2uf_x + u_x f + a_x v - \alpha_1 f_{xxx} + \alpha_2 a_{xx})(\mathrm{d}x)^2, (vf)_x - \alpha_2 f_{xx} + 2\alpha_3 a_x, 0).$$

By definition in [13], the Euler equation on $\widehat{\mathcal{G}}^*_{\mathrm{reg}}$ is given by

$$\frac{\mathrm{d}\hat{u}}{\mathrm{d}t} = -ad^*_{\mathcal{A}^{-1}\hat{u}}\hat{u} \tag{2.9}$$

as an evolution of a point $\hat{u} \in \widehat{\mathcal{G}}^*_{\mathrm{reg}}$. That is to say, the Euler equation on $\widehat{\mathcal{G}}^*_{\mathrm{reg}}$ is

$$\begin{aligned} u_t &= 2uf_x + u_x f + v_x v - \gamma_1 f_{xxx} + \gamma_2 v_{xx},\\ v_t &= (vf)_x - \gamma_2 f_{xx} + 2\gamma_3 v_x,\end{aligned}$$

where $u(x,t) = \Lambda(f(x,t)) = \mu(f) - f_{xx}$. By integrating both sides of this equation over the circle and using periodicity, we obtain

$$\mu(f_t) = \mu(f)_t = 0.$$

This yields that

$$\begin{aligned} -f_{xxt} &= 2\mu(f)f_x - 2f_x f_{xx} - f f_{xxx} + v_x v - \gamma_1 f_{xxx} + \gamma_2 v_{xx},\\ v_t &= (vf)_x - \gamma_2 f_{xx} + 2\gamma_3 v_x,\end{aligned}$$

which is the 2-μHS equation (1.4). □

Remark 2.3. If we replace the Gelfand–Fuchs cocycle ω_1 by the modified cocycle $\tilde{\omega}_1$, the Euler equation $\widehat{\mathcal{G}}^*_{\mathrm{reg}}$ is of the form

$$\begin{aligned} -f_{xxt} &= 2\mu(f)f_x - 2f_x f_{xx} - f f_{xxx} + v_x v - \gamma_1 c_1 f_{xxx} + \gamma_2 v_{xx} + \gamma_1 c_2 f_x,\\ v_t &= (vf)_x - \gamma_2 f_{xx} + 2\gamma_3 v_x.\end{aligned}$$

3. Hamiltonian nature of the 2-μHS equation

In this section, we will study the Hamiltonian nature of the 2-μHS equation (1.4) and its geometric meaning. We will show that

Theorem 3.1. *The 2-μHS equation (1.4) is bi-Hamiltonian.*

Proof. Let us define $u(x,t) = \Lambda(f) = \mu(f) - f_{xx}$ and

$$H_1 = \frac{1}{2}\int_{\mathbb{S}^1} (uf + v^2)\,\mathrm{d}x \tag{3.1}$$

and

$$H_2 = \int_{\mathbb{S}^1} \left(\mu(f) f^2 + \frac{1}{2} f f_x^2 + \frac{1}{2} f v^2 - \gamma_2 v f_x + \gamma_3 v^2 - \frac{\gamma_1}{2} f f_{xx} \right) \mathrm{d}x. \tag{3.2}$$

It is easy to check that the 2-μHS equation can be written as

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \mathcal{J}_1 \begin{pmatrix} \frac{\delta H_2}{\delta u} \\ \frac{\delta H_2}{\delta v} \end{pmatrix} = \mathcal{J}_2 \begin{pmatrix} \frac{\delta H_1}{\delta u} \\ \frac{\delta H_1}{\delta v} \end{pmatrix}, \tag{3.3}$$

where the Hamiltonian operators are

$$\mathcal{J}_1 = \begin{pmatrix} \partial_x \Lambda & 0 \\ 0 & \partial_x \end{pmatrix}, \qquad \mathcal{J}_2 = \begin{pmatrix} u\partial_x + \partial_x u - \gamma_1 \partial_x^3 & v\partial_x + \gamma_2 \partial_x^2 \\ \partial_x v - \gamma_2 \partial_x^2 & 2\gamma_3 \partial_x \end{pmatrix}. \tag{3.4}$$

By a direct and lengthy calculation we could show that the Hamiltonian operators $\mathcal{J}_1$ and $\mathcal{J}_2$ are compatible. □

Next we will explain the geometric meaning of the bi-Hamiltonian structures of the 2-μHS equation (1.4). Let $F_i : \widehat{\mathcal{G}}^*_{\mathrm{reg}} \to \mathbb{R}, i = 1, 2$, be the two arbitrary smooth functionals. It is well known that the dual space $\widehat{\mathcal{G}}^*_{\mathrm{reg}}$ carries the canonical Lie–Poisson bracket

$$\{F_1, F_2\}_2(\hat{u}) = \left\langle \hat{u}, \left[\frac{\delta F_1}{\delta \hat{u}}, \frac{\delta F_2}{\delta \hat{u}} \right] \right\rangle^*, \tag{3.5}$$

where $\hat{u} = (u(x,t)(\mathrm{d}x)^2, v(x,t), \vec{\gamma}) \in \widehat{\mathcal{G}}^*_{\mathrm{reg}}$ and $\frac{\delta F_i}{\delta \hat{u}} = \left(\frac{\delta F_i}{\delta u}, \frac{\delta F_i}{\delta v}, \frac{\delta F_i}{\delta \vec{\gamma}} \right) \in \widehat{\mathcal{G}}, i = 1, 2$. By the definition of the Euler equation (2.9), we know that the Lie–Poisson structure (3.5) is exactly the second Poisson bracket, induced by $\mathcal{J}_2$, of the 2-μHS equation (1.4).

To explain the first Hamiltonian structure, in the following we will use the 'frozen Lie–Poisson' method introduced in [13]. Let us define a frozen (or constant) Poisson bracket

$$\{F_1, F_2\}_1(\hat{u}) = \left\langle \hat{u}_0, \left[\frac{\delta F_1}{\delta \hat{u}}, \frac{\delta F_2}{\delta \hat{u}} \right] \right\rangle^*, \tag{3.6}$$

where $\hat{u}_0 = (u_0(\mathrm{d}x)^2, v_0, \vec{\gamma}_0) \in \widehat{\mathcal{G}}^*_{\mathrm{reg}}$. The corresponding Hamiltonian equation for any functional $F : \widehat{\mathcal{G}}^*_{\mathrm{reg}} \to \mathbb{R}$ reads

$$\frac{\mathrm{d}\hat{u}}{\mathrm{d}t} = ad^*_{\frac{\delta F}{\delta \hat{u}}} \hat{u}_0 \tag{3.7}$$

which gives

$$\begin{aligned} u_t &= 2u_0 \left(\frac{\delta F}{\delta u} \right)_x + \left(\frac{\delta F}{\delta v} \right)_x v_0 - \gamma_1^0 \left(\frac{\delta F}{\delta u} \right)_{xxx} + \gamma_2^0 \left(\frac{\delta F}{\delta v} \right)_{xx}, \\ v_t &= \left(v_0 \frac{\delta F}{\delta u} \right)_x - \gamma_2^0 \left(\frac{\delta F}{\delta u} \right)_{xx} + 2\gamma_3^0 \left(\frac{\delta F}{\delta v} \right)_x, \\ \vec{\gamma}_{0,t} &= 0. \end{aligned} \tag{3.8}$$

Let us take the Hamiltonian functional F to be

$$H_2 = \int_{\mathbb{S}^1} \left(\mu(f) f^2 + \frac{1}{2} f f_x^2 + \frac{1}{2} f v^2 - \gamma_2 v f_x + \gamma_3 v^2 - \frac{\gamma_1}{2} f f_{xx} \right) \mathrm{d}x \tag{3.9}$$

and set $u(x,t) = \Lambda(f(x,t)) = \mu(f) - f_{xx}$. Then we have

$$\begin{aligned} \frac{\delta F}{\delta u} &= \Lambda^{-1} \left(\mu(f^2) + 2f\mu(f) - \frac{1}{2} f_x^2 - f f_{xx} - \gamma_1 f_{xx} + \gamma_2 v_x \right), \\ \frac{\delta F}{\delta v} &= vf - \gamma_2 f_x + 2\gamma_3 v. \end{aligned} \tag{3.10}$$

Let us choose a fixed point

$$\hat{u}_0 = (u_0, v_0, \vec{\gamma}_0) = \left(0, 0, \left(1, 0, \tfrac{1}{2}\right)\right).$$

Observe that $\partial_x^3 \Lambda^{-1} = -\partial_x$. By substituting (3.10) into (3.8), we obtain the 2-μHS equation (1.4). According to proposition 5.3 in [13], $\{,\}_1$ and $\{,\}_2$ are compatible for every freezing point $\hat{u}_0$. Consequently we have

Theorem 3.2. *The 2-μHS equation (1.4) is Hamiltonian with respect to two compatible Poisson structures (3.5) and (3.6) on $\widehat{\mathcal{G}}^*_{\text{reg}}$, where the first bracket is frozen at the point $\hat{u}_0 = (u_0, v_0, \vec{\gamma}_0) = \left(0, 0, \left(1, 0, \tfrac{1}{2}\right)\right)$.*

Let us point out that the constant bracket depends on the choice of the freezing point $\hat{u}_0$, while the Lie–Poisson bracket is only determined by the Lie algebra structure.

To this end we want to derive a Lax pair of 2-μHS equation (1.4) with $\overrightarrow{\gamma} = 0$, i.e.

$$-f_{xxt} = 2\mu(f) f_x - 2f_x f_{xx} - f f_{xxx} + v_x v, \qquad v_t = (vf)_x. \tag{3.11}$$

Motivated by the Lax pair of the two-component Camassa–Holm equation in [17], we could assume that the Lax pair of (3.11) has the following form:

$$\Psi_x = U\Psi, \qquad \Psi_t = V\Psi \tag{3.12}$$

with

$$U = \begin{pmatrix} 0 & 1 \\ \lambda\Lambda(f) - \lambda^2 v^2 & 0 \end{pmatrix} \qquad \text{and} \qquad V = \begin{pmatrix} p & r \\ q & -p \end{pmatrix},$$

where λ is a spectral parameter. The compatibility condition

$$U_t - V_x + UV - VU = 0$$

in componentwise form reads

$$p = -\frac{r_x}{2}, \qquad q = p_x + r(\lambda\Lambda(f) - \lambda^2 v^2),$$
$$2\lambda^2 v v_t + \lambda f_{xxt} + q_x - 2p(\lambda\Lambda(f) - \lambda^2 v^2) = 0.$$

By choosing $r = f - \frac{1}{2\lambda}$, we have

$$p = -\frac{f_x}{2}, \qquad q = -\frac{f_{xx}}{2} + \left(f - \frac{1}{2\lambda}\right)(\lambda\Lambda(f) - \lambda^2 v^2)$$

and

$$f_{xxt} + 2\mu(f) f_x - 2f_x f_{xx} - f f_{xxx} + v_x v + 2\lambda v(v_t - (vf)_x) = 0$$

which yields the system (3.11). Let us write $\Psi = \binom{\psi}{\psi_x}$. We have

Proposition 3.3. *The system (3.11) has a Lax pair given by*

$$\psi_{xx} = (\lambda\Lambda(f) - \lambda^2 v^2)\psi, \qquad \psi_t = \left(f - \frac{1}{2\lambda}\right)\psi_x - \frac{1}{2} f_x \psi,$$

where $\lambda \in \mathbb{C} - \{0\}$ is a spectral parameter.

4. Variational nature of the 2-μHS equation

In [22], they have shown that the μ-HS equation (1.1) can be obtained from two distinct variational principles. In this section we will show that the 2-μHS equation (1.4) also arises as the equation

$$\delta\mathcal{S} = 0$$

for the action functional

$$\mathcal{S} = \int\left(\int \mathcal{L}\,\mathrm{d}x\right)\mathrm{d}t$$

with two different densities $\mathcal{L}$. That is to say,

Theorem 4.1. *The 2-μHS equation (1.4) satisfies two different variational principles.*

Proof. Motivated by the Lagrangian densities for the μ-HS equation (1.1) in [22], by some conjectural computations we find two generalized Lagrangian densities for the 2-μHS equation (1.4). More precisely,

Case I. Let us consider the first Lagrangian density

$$\mathcal{L}_1 = \tfrac{1}{2}f_x^2 + \tfrac{1}{2}\mu(f)f + \tfrac{1}{2}v^2 - vz_x + w(fz_x - z_t + \tilde{\gamma}_3 v) + \gamma_2 w_x f - 2\gamma_1 f, \tag{4.1}$$

where $\tilde{\gamma}_3 = \gamma_3 - \frac{1}{2}\gamma_1$. Varying the corresponding action with respect to f, v, w and z, respectively, we get

$$\begin{aligned} f_{xx} &= \mu(f) + wz_x + \gamma_2 w_x - 2\gamma_1, \\ z_x &= v + \tilde{\gamma}_3 w, \\ z_t &= fz_x + \tilde{\gamma}_3 v - \gamma_2 f_x, \\ w_t &= (wf)_x - v_x. \end{aligned} \tag{4.2}$$

By using (4.2), we have

$$\begin{aligned} v_t = z_{xt} - \gamma_3 w_t &= [f(v + \tilde{\gamma}_3 w) + \tilde{\gamma}_3 v - \gamma_2 f_x]_x - \tilde{\gamma}_3((wf)_x - v_x), \\ &= (vf)_x - \gamma_2 f_{xx} + (2\gamma_3 - \gamma_1)v_x, \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} -f_{xxt} + f_x f_{xx} + ff_{xxx} &= -(\mu(f) + wz_x + \gamma_2 w_x)_t + f_x(\mu(f) \\ &\quad + wz_x + \gamma_2 w_x - 2\gamma_1) + f(\mu(f) + wz_x + \gamma_2 w_x)_x \\ &= -w_t z_x - wz_{xt} + \gamma w_{xt} + f_x wz_x + fw_x z_x + fwz_{xx} + \gamma_2 fw_{xx} + 2\gamma_1 f_x \\ &= vv_x + 2\mu(f)f_x + \gamma_2 v_{xx} - 2\gamma_1 f_x. \end{aligned} \tag{4.4}$$

Note that if we replace f by $f + \gamma_1$ in the system (4.3) and (4.4), this gives the 2-μHS equation (1.4).

Case II. The second variational representation can be obtained from the Lagrangian density

$$\mathcal{L}_2 = -f_x f_t + 2\mu(f)f^2 + ff_x^2 + f\phi_x^2 - \gamma_1 ff_{xx} - 2\gamma_2\phi_x f_x + 2\gamma_3\phi_x^2 - \phi_x\phi_t. \tag{4.5}$$

The variational principle $\delta\mathcal{S} = 0$ gives the Euler–Lagrange equation

$$\begin{aligned} -f_{xt} &= 2\mu(f)f + \mu(f^2) - \tfrac{1}{2}f_x^2 - ff_{xx} + \tfrac{1}{2}\phi_x^2 - \gamma_1 f_x + \gamma_2\phi_{xx}, \\ \phi_{xt} &= (f\phi_x)_x - \gamma_2 f_{xx} + 2\gamma_3\phi_{xx}. \end{aligned} \tag{4.6}$$

If we set $\phi_x = v$ and take the x-derivative of the first term in (4.6), this yields the 2-μHS equation (1.4). □

5. Relation between Hamiltonian nature and variational nature

Recall that we have shown that the 2-μHS equation (1.4) is bi-Hamiltonian and has two different variational principles. In the last section we will study the relation between Hamiltonian natures and bi-variational principles and prove that

Theorem 5.1. *The two variational formulations for the 2-μHS equation (1.4) formally correspond to the two Hamiltonian formulations of this equation with the Hamiltonian functionals H_1 and H_2.*

Proof. The action is related to the Lagrangian by $\mathcal{S} = \int \left(\int \mathcal{L} \mathrm{d}x \right) \mathrm{d}t$. The first variational principle has the Lagrangian density

$$\mathcal{L}_1 = \tfrac{1}{2} f_x^2 + \tfrac{1}{2}\mu(f) f + \tfrac{1}{2} v^2 - v z_x + w(f z_x - z_t + \tilde{\gamma}_3 v) + \gamma_2 w_x f - 2\gamma_1 f.$$

The momenta conjugate to the velocities f_t, v_t, z_t and w_t, respectively, are

$$\frac{\partial \mathcal{L}_1}{\partial f_t} = 0, \qquad \frac{\partial \mathcal{L}_1}{\partial w_t} = 0, \qquad \frac{\partial \mathcal{L}_1}{\partial z_t} = -w, \qquad \frac{\partial \mathcal{L}_1}{\partial w_t} = 0.$$

Consequently, the Hamiltonian density is

$$\begin{aligned}
\mathcal{H} &= -z_t w - \mathcal{L}_1 \\
&= -\tfrac{1}{2} f_x^2 - \tfrac{1}{2}\mu(f) f - \tfrac{1}{2} v^2 + v z_x - w(f z_x + \tilde{\gamma}_3 v) - \gamma_2 w_x f + 2\gamma_1 \\
&= \tfrac{1}{2}\mu(f) f - \tfrac{1}{2} f_x^2 + \tfrac{1}{2} v^2 - f f_{xx}, \quad \text{by using (4.2).}
\end{aligned}$$

Therefore, the Hamiltonian is

$$\begin{aligned}
H &= \int \mathcal{H}\, \mathrm{d}x = \int \left(\frac{1}{2}\mu(f) f - \frac{1}{2} f_x^2 + \frac{1}{2} v^2 - f f_{xx} \right) \mathrm{d}x \\
&= \frac{1}{2} \int (\mu(f) f - f f_{xx} + v^2)\, \mathrm{d}x,
\end{aligned}$$

which is exactly H_1 defined in (3.1).

In the second principle the Lagrangian density is

$$\mathcal{L}_2 = -f_x f_t + 2\mu(f) f^2 + f f_x^2 + f \phi_x^2 - \gamma_1 f f_{xx} - 2\gamma_2 \phi_x f_x + 2\gamma_3 \phi_x^2 - \phi_x \phi_t.$$

The momenta conjugate to the velocities f_t and ϕ_t, respectively, are

$$\frac{\partial \mathcal{L}_2}{\partial f_t} = -f_x, \qquad \frac{\partial \mathcal{L}_2}{\partial \phi_t} = -\phi_x.$$

Consequently, the Hamiltonian density is

$$\begin{aligned}
\mathcal{H} &= -f_x f_t - \phi_x \phi_t - \mathcal{L}_2 \\
&= -2\mu(f) f^2 - f f_x^2 - f \phi_x^2 + \gamma_1 f f_{xx} + 2\gamma_2 \phi_x f_x - 2\gamma_3 \phi_x^2.
\end{aligned}$$

Now let us set $\phi_x = v$ and so

$$H = \int (-2\mu(f) f^2 - f f_x^2 - f v^2 + -\gamma_1 f f_{xx} + 2\gamma_2 v f_x - 2\gamma_3 v^2)\, \mathrm{d}x = -\frac{H_2}{2},$$

where H_2 is defined in (3.2). □

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