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Nonlinear Analysis: Real World Applications 8 (2007) 1390 – 1397

Nonlinear Analysis Real World Applications

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Asymptotic expansion method for some nonlinear two point boundary value problems with rapidly oscillating coefficients

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Received 9 May 2006; accepted 19 July 2006

Abstract

In this paper, using asymptotic expansion method, we obtain accurate solutions for some nonlinear two point boundary value problems with rapidly oscillating coefficients.

Keywords: Nonlinear two point boundary value problems; Rapidly oscillating coefficients; Asymptotic expansion method

1. Introduction

For elliptic problems with rapidly oscillating coefficients, the method of multi-scale asymptotic expansions which is thoroughly described in numerous sources (see e.g. [1–6]) is very effective, for it couples the macroscopic and microscopic scales together; it not only reflects the global mechanical and physical properties of the structure, but also the effect of micro-configuration of the composite materials.

For the following one-dimensional Dirichlet problems with rapidly oscillating coefficient

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}x}\left(a\left(\dfrac{x}{\varepsilon}\right)\dfrac{\mathrm{d}u^{\varepsilon}}{\mathrm{d}x}\right)=f(x), \quad x\in(c,d),\\ u^{\varepsilon}(c)=u_0,\\ u^{\varepsilon}(d)=u_1, \end{cases} \tag{1.1}$$

where $a(x/\varepsilon)$ is a 1-periodic function.

In [1], homogenization method is introduced to solve problem (1.1). The procedures are presented as follows:

(1) $\quad a^0=\dfrac{1}{\int_0^1 (1/a(\xi))\,\mathrm{d}\xi}.$

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1468-1218/$ - see front matter
doi:10.1016/j.nonrwa.2006.07.006

(2) Solve the following problem:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}x}\left(a^0\dfrac{\mathrm{d}u}{\mathrm{d}x}\right)=f(x), \quad x\in(c,d), \\ u(c)=u_0, \\ u(d)=u_1. \end{cases} \tag{1.2}$$

Then u is the homogenization solution for problem (1.1).

It is proved that $u^\varepsilon \rightharpoonup u$ in $H^1(c,d)$, but the error estimate is not given.

On basis of [1], using asymptotic expansion method, [3] obtain an accurate expression of u^ε for problem (1.1).

In this paper, we will consider how to propose an asymptotic expansion method for the following nonlinear two point boundary value problems with rapidly oscillating coefficients:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}x}\left(a\left(\dfrac{x}{\varepsilon}\right)\dfrac{\mathrm{d}u^\varepsilon}{\mathrm{d}x}\right)+b_1\left(\dfrac{x}{\varepsilon}\right)\psi_1\left(\dfrac{\mathrm{d}u^\varepsilon}{\mathrm{d}x}\right)+b_2\left(\dfrac{x}{\varepsilon}\right)\psi_2(u^\varepsilon)=f(x), \quad x\in(0,l), \\ u^\varepsilon(0)=u_0, \quad u^\varepsilon(l)=u_1, \end{cases} \tag{1.3}$$

where $f\in C^\infty[0,l]$ and $a(x/\varepsilon), b_1(x/\varepsilon), b_2(x/\varepsilon)$ are three 1-periodic functions, $\psi_1(\mathrm{d}u^\varepsilon/\mathrm{d}x)$ is a function about $\mathrm{d}u^\varepsilon/\mathrm{d}x$ and $\psi_2(u^\varepsilon)$ is a function about u^ε. For example, $\psi_1(\mathrm{d}u^\varepsilon/\mathrm{d}x)=\mathrm{d}u^\varepsilon/\mathrm{d}x$, $\psi_2(u^\varepsilon)=(u^\varepsilon)^2$.

2. Asymptotic expansion method for problem (1.3)

In this section, let us consider how to solve problem

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}x}\left(a\left(\dfrac{x}{\varepsilon}\right)\dfrac{\mathrm{d}u^\varepsilon}{\mathrm{d}x}\right)+b_1\left(\dfrac{x}{\varepsilon}\right)\psi_1\left(\dfrac{\mathrm{d}u^\varepsilon}{\mathrm{d}x}\right)+b_2\left(\dfrac{x}{\varepsilon}\right)\psi_2(u^\varepsilon)=f(x), \quad x\in(0,l), \\ u^\varepsilon(0)=u_0, \quad u^\varepsilon(l)=u_1, \end{cases} \tag{2.1}$$

where $f\in C^\infty[0,l]$, $b_1(x/\varepsilon)$ and $b_2(x/\varepsilon)$ are two 1-periodic functions, $\psi_1(\mathrm{d}u^\varepsilon/\mathrm{d}x)$ is a function about $\mathrm{d}u^\varepsilon/\mathrm{d}x$ and $\psi_2(u^\varepsilon)$ is a function about u^ε. For example, $\psi_1(\mathrm{d}u^\varepsilon/\mathrm{d}x)=\mathrm{d}u^\varepsilon/\mathrm{d}x$, $\psi_2(u^\varepsilon)=(u^\varepsilon)^2$.

For simplicity, we assume that $l=m\varepsilon$ $(m\in N)$.

At first,we will take following steps to obtain an asymptotic expansion expression of $u^\varepsilon(x)$ for $x=i\varepsilon$ $(i\in N, 1\leqslant i\leqslant m-1)$.

(a) Obtain constants β and β_k $(k\in N\cup\{0\})$ and let $u^\varepsilon(i\varepsilon)$ be expressed as follows:

$$u^\varepsilon(i\varepsilon)=\beta[u^\varepsilon((i-1)\varepsilon)+u^\varepsilon((i+1)\varepsilon)]+\sum_{k=0}^{\infty}\beta_k\varepsilon^{k+2}f^{(k)}(i\varepsilon). \tag{2.2}$$

(b) Have constants a_0, b_0 and function $F(x)$ and then introduce function $U(x)$ by following equation:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}x}\left(a_0\dfrac{\mathrm{d}U(x)}{\mathrm{d}x}\right)+b_0\psi_1\left(\dfrac{\mathrm{d}U(x)}{\mathrm{d}x}\right)+\psi_2(U(x))=F(x), \quad x\in(0,l), \\ U(0)=u_0, \quad U(l)=u_1, \end{cases} \tag{2.3}$$

which such that

$$u^\varepsilon(i\varepsilon)=U(i\varepsilon), \tag{2.4}$$

where $i=0,1,\ldots,m$.

Now let us consider (a) firstly.

Set

$$E=[(i-1)\varepsilon,(i+1)\varepsilon], \quad \xi=\frac{x-i\varepsilon}{\varepsilon} \tag{2.5}$$

and denote function $\omega(\xi)$ by

$$\omega(\xi) = u^{\varepsilon}(x). \tag{2.6}$$

Considering

$$\frac{\mathrm{d}^k u^{\varepsilon}(x)}{\mathrm{d}x^k} = \varepsilon^{-k}\frac{\mathrm{d}^k \omega(\xi)}{\mathrm{d}\xi^k}, \quad \frac{\mathrm{d}a(x/\varepsilon)}{\mathrm{d}x} = \varepsilon^{-1}\frac{\mathrm{d}a(\xi)}{\mathrm{d}\xi}, \qquad x \in E = [(i-1)\varepsilon, (i+1)\varepsilon], \tag{2.7}$$

where $k = 1, 2$.

We see from (2.1) and (2.7) that

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}\xi}\left(a(\xi)\dfrac{\mathrm{d}\omega(\xi)}{\mathrm{d}\xi}\right) + \varepsilon b_1(\xi)\psi_1\left(\dfrac{\mathrm{d}\omega(\xi)}{\mathrm{d}\xi}\right) + \varepsilon^2 b_2(\xi)\psi_2(\omega(\xi)) = \varepsilon^2 f(x), \\ \omega(-1) = u^{\varepsilon}((i-1)\varepsilon), \quad \omega(1) = u^{\varepsilon}((i+1)\varepsilon), \end{cases} \tag{2.8}$$

where $x \in E$ and $\xi = (x - i\varepsilon)/\varepsilon$.

It is seen from (2.8) that $u^{\varepsilon}(i\varepsilon) = \omega(0)$. In the following, we will consider how to obtain $\omega(0)$.

Using Taylor expansion, we have

$$f(x) = \sum_{k=0}^{\infty}\frac{(x-i\varepsilon)^k}{k!}f^{(k)}(i\varepsilon) = \sum_{k=0}^{\infty}\frac{\xi^k\varepsilon^k}{k!}f^{(k)}(i\varepsilon), \quad x \in E, \tag{2.9}$$

and then obtain

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}\xi}\left(a(\xi)\dfrac{\mathrm{d}\omega(\xi)}{\mathrm{d}\xi}\right) + \varepsilon b_1(\xi)\psi_1\left(\dfrac{\mathrm{d}\omega(\xi)}{\mathrm{d}\xi}\right) + \varepsilon^2 b_2(\xi)\psi_2(\omega(\xi)) = \sum\limits_{k=0}^{\infty}\dfrac{\xi^k\varepsilon^{k+2}}{k!}f^{(k)}(i\varepsilon), \\ \omega(-1) = u^{\varepsilon}((i-1)\varepsilon), \quad \omega(1) = u^{\varepsilon}((i+1)\varepsilon), \end{cases} \tag{2.10}$$

where $x \in E$ and $\xi = (x - i\varepsilon)/\varepsilon$.

Introduce functions $\alpha(\xi)$ and $\hat{\alpha}_k(\xi)(k \in N \cup \{0\})$ by

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}\xi}\left(a(\xi)\dfrac{\mathrm{d}\alpha(\xi)}{\mathrm{d}\xi}\right) + \varepsilon b_1(\xi)\psi_1\left(\dfrac{\mathrm{d}\alpha(\xi)}{\mathrm{d}\xi}\right) + \varepsilon^2 b_2(\xi)\psi_2(\alpha(\xi)) = 0, \\ \alpha(-1) = u^{\varepsilon}((i-1)\varepsilon), \quad \alpha(1) = u^{\varepsilon}((i+1)\varepsilon), \end{cases} \tag{2.11}$$

and

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}x}\left(a(\xi)\dfrac{\mathrm{d}\hat{\alpha}_k(\xi)}{\mathrm{d}\xi}\right) + \varepsilon b_1(\xi)\psi_1\left(\dfrac{\mathrm{d}\hat{\alpha}_k(\xi)}{\mathrm{d}\xi}\right) + \varepsilon^2 b_2(\xi)\psi_2(\hat{\alpha}_k(\xi)) = \dfrac{\xi^k}{k!}, \\ \hat{\alpha}_k(-1) = 0, \quad \hat{\alpha}_k(1) = 0. \end{cases} \tag{2.12}$$

Further, we see from (2.11) and (2.12) that

$$\omega(\xi) = \alpha(\xi) + \sum_{k=0}^{\infty}\varepsilon^{k+2}\hat{\alpha}_k(\xi)f^{(k)}(i\varepsilon) \tag{2.13}$$

and then obtain

$$\omega(0) = \alpha(0) + \sum_{k=0}^{\infty}\varepsilon^{k+2}\hat{\alpha}_k(0)f^{(k)}(i\varepsilon). \tag{2.14}$$

Set

$$\beta_k = \hat{\alpha}_k(0). \tag{2.15}$$

Now we know the key to have (a) is to solve (2.11) for $\xi = 0$.

Assume that functions $\hat{\beta}_1(\xi)$ and $\hat{\beta}_2(\xi)$ are defined by following (2.16) and (2.17):

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}\xi}\left(a(\xi)\dfrac{\mathrm{d}\hat{\beta}_1(\xi)}{\mathrm{d}\xi}\right) + \varepsilon b_1(\xi)\psi_1\left(\dfrac{\mathrm{d}\hat{\beta}_1(\xi)}{\mathrm{d}\xi}\right) + \varepsilon^2 b_2(\xi)\psi_2(\hat{\beta}_1(\xi)) = 0, \\ \hat{\beta}_1(-1) = 1, \quad \hat{\beta}_1(1) = 0 \end{cases} \tag{2.16}$$

and

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}\xi}\left(a(\xi)\dfrac{\mathrm{d}\hat{\beta}_2(\xi)}{\mathrm{d}\xi}\right) + \varepsilon b_1(\xi)\psi_1\left(\dfrac{\mathrm{d}\hat{\beta}_2(\xi)}{\mathrm{d}\xi}\right) + \varepsilon^2 b_2(\xi)\psi_2(\hat{\beta}_2(\xi)) = 0, \\ \hat{\beta}_2(-1) = 0, \quad \hat{\beta}_2(1) = 1. \end{cases} \tag{2.17}$$

We get from (2.11), (2.16) and (2.17) that

$$\alpha(\xi) = \hat{\beta}_1(\xi)u^\varepsilon((i-1)\varepsilon) + \hat{\beta}_2(\xi)u^\varepsilon((i+1)\varepsilon)$$

and then obtain

$$\alpha(0) = \hat{\beta}_1(0)u^\varepsilon((i-1)\varepsilon) + \hat{\beta}_2(0)u^\varepsilon((i+1)\varepsilon).$$

It is obvious that $\hat{\beta}_1(0) = \hat{\beta}_2(0)$, set

$$\beta = \hat{\beta}_1(0) = \hat{\beta}_2(0) \tag{2.18}$$

and we have

$$\alpha(0) = \beta[u^\varepsilon((i-1)\varepsilon) + u^\varepsilon((i+1)\varepsilon)]. \tag{2.19}$$

Combining (2.13) and (2.19)–(2.15), we see

$$\omega(0) = \beta[u^\varepsilon((i-1)\varepsilon) + u^\varepsilon((i+1)\varepsilon)] + \sum_{k=0}^{\infty} \beta_k \varepsilon^{k+2} f^{(k)}(i\varepsilon). \tag{2.20}$$

That is to say, we obtain

$$u^\varepsilon(i\varepsilon) = \beta[u^\varepsilon((i-1)\varepsilon) + u^\varepsilon((i+1)\varepsilon)] + \sum_{k=0}^{\infty} \beta_k \varepsilon^{k+2} f^{(k)}(i\varepsilon). \tag{2.21}$$

We finish (a).

Now, let us consider (b).

Assume that a_0 and b_0 are two constants and the following equation:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(a_0\frac{\mathrm{d}\overline{\alpha}(x)}{\mathrm{d}x}\right) + b_0\psi_1\left(\frac{\mathrm{d}\overline{\alpha}(x)}{\mathrm{d}x}\right) + \psi_2(\overline{\alpha}(x)) = 0, \quad x \in E, \tag{2.22}$$

such that

$$\overline{\alpha}(i\varepsilon) = \beta[\overline{\alpha}((i-1)\varepsilon) + \overline{\alpha}((i+1)\varepsilon)]. \tag{2.23}$$

For $k = 0, 1, 2, \ldots$, denote functions $\overline{\alpha}_k(\xi)$ by

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}\xi}\left(a_0\dfrac{\mathrm{d}\overline{\alpha}_k(\xi)}{\mathrm{d}\xi}\right) + \varepsilon b_0\psi_1\left(\dfrac{\mathrm{d}\overline{\alpha}_k(\xi)}{\mathrm{d}\xi}\right) + \varepsilon^2\psi_2(\overline{\alpha}_k(\xi)) = \dfrac{\xi^k}{k!}, \quad \xi \in [-1, 1], \\ \overline{\alpha}_k(-1) = \overline{\alpha}_k(1) = 0. \end{cases} \tag{2.24}$$

Solve problem (2.24) for $k \in N \cup \{0\}$ and set

$$\overline{\beta}_k = \overline{\alpha}_k(0). \tag{2.25}$$

Combining (2.21) and (2.25), we have

$$u^\varepsilon(i\varepsilon) = \beta[u^\varepsilon((i-1)\varepsilon) + u^\varepsilon((i+1)\varepsilon)] + \sum_{k=0}^{\infty} \frac{\beta_k}{\overline{\beta}_k} \overline{\alpha}_k(0)\varepsilon^{k+2} f^{(k)}(i\varepsilon). \tag{2.26}$$

Introduce function $F(x) \in C^\infty(0,l)$ by

$$F^{(k)}(0) = \frac{\beta_k}{\overline{\beta}_k} f^{(k)}(0) \tag{2.27}$$

and we obtain that

$$F^{(k)}(i\varepsilon) = \frac{\beta_k}{\overline{\beta}_k} f^{(k)}(i\varepsilon), \tag{2.28}$$

where $i = 0, 1, \ldots, m$ and $k \in N \cup \{0\}$.

Finally, introduce function $U(x)$ by

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}x}\left(a_0 \dfrac{\mathrm{d}U(x)}{\mathrm{d}x}\right) + b_0\psi_1\left(\dfrac{\mathrm{d}U(x)}{\mathrm{d}x}\right) + \psi_2(U(x)) = F(x), \quad x \in (0,l), \\ U(0) = u_0, \quad U(l) = u_1. \end{cases} \tag{2.29}$$

In the following, we will prove

$$u^\varepsilon(x) = U(x), \tag{2.30}$$

where $x = i\varepsilon$ $(i = 0, 1, \ldots, m)$.

For $x \in [(i-1)\varepsilon, (i+1)\varepsilon]$, in view of

$$F(x) = \sum_{k=0}^{\infty} \frac{(x-i\varepsilon)^k}{k!} F^{(k)}(i\varepsilon) = \sum_{k=0}^{\infty} \frac{\xi^k}{k!} \varepsilon^k F^{(k)}(i\varepsilon). \tag{2.31}$$

We have from (2.22) and (2.24) that

$$\begin{aligned} U(i\varepsilon) &= \beta[U((i-1)\varepsilon) + U((i+1)\varepsilon)] + \sum_{k=0}^{\infty} \frac{\beta_k}{\overline{\beta}_k} \varepsilon^k f^{(k)}(i\varepsilon)\overline{\alpha}_k \\ &= \beta[U((i-1)\varepsilon) + U((i+1)\varepsilon)] + \sum_{k=0}^{\infty} \beta_k \varepsilon^k f^{(k)}(i\varepsilon). \end{aligned} \tag{2.32}$$

Note that

$$U(0) = u^\varepsilon(0), \quad U(l) = u^\varepsilon(l).$$

For $i = 1, 2, \ldots, m-1$, we have from (2.21) and (2.32) that

$$\begin{cases} u^\varepsilon(i\varepsilon) - U(i\varepsilon) = \beta[(u^\varepsilon((i-1)\varepsilon) - U((i-1)\varepsilon)) + (u^\varepsilon((i+1)\varepsilon) - U((i+1)\varepsilon))], \\ u^\varepsilon(0) - U(0) = u^\varepsilon(l) - U(l) = 0 \end{cases}$$

and then obtain

$$u^{\varepsilon}(i\varepsilon) - U(i\varepsilon) = 0.$$

Thus, we have (2.30).

We finish (b).

According to above analysis, we have the following result.

Theorem 2.1. *Assume that u^{ε} and U are defined as* (2.1) *and* (2.22), *respectively. Then, for $i = 0, 1, \ldots, m$, there exists*

$$u^{\varepsilon}(i\varepsilon) = U(i\varepsilon). \tag{2.33}$$

Above, we solve problem (2.1) for $x = i\varepsilon$ $(i = 0, 1, \ldots, m)$.

In the following, we will consider how to solve problem (2.1) for $x \in ((i-1)\varepsilon, i\varepsilon)$.

Set

$$\xi = \frac{x - (i-1)\varepsilon}{\varepsilon}, \quad x \in ((i-1)\varepsilon, i\varepsilon)$$

and introduce function $\omega(\xi)$ by

$$\omega(\xi) = u^{\varepsilon}(x).$$

Considering

$$\frac{d^k u^{\varepsilon}(x)}{dx^k} = \varepsilon^{-k}\frac{d^k \omega(\xi)}{d\xi^k}, \quad \frac{da(x/\varepsilon)}{dx} = \varepsilon^{-1}\frac{da(\xi)}{d\xi}, \quad x \in E = [(i-1)\varepsilon, (i+1)\varepsilon], \tag{2.34}$$

where $k = 1, 2$.

It follows from (2.1) that

$$\begin{cases} \dfrac{d}{d\xi}\left(a(\xi)\dfrac{d\omega(\xi)}{d\xi}\right) + \varepsilon b_1(\xi)\psi_1\left(\dfrac{d\omega(\xi)}{d\xi}\right) + \varepsilon^2 b_2(\xi)\psi_2(\omega(\xi)) = \varepsilon^2 f(x), \quad x \in [(i-1)\varepsilon, i\varepsilon], \\ \omega(0) = U((i-1)\varepsilon), \quad \omega(1) = U(i\varepsilon). \end{cases} \tag{2.35}$$

Denote functions $\tilde{\omega}(\xi)$ and $\overline{\omega}(\xi)$ by

$$\begin{cases} \dfrac{d}{d\xi}\left(a(\xi)\dfrac{d\tilde{\omega}}{d\xi}\right) + \varepsilon b_1(\xi)\psi_1\left(\dfrac{d\tilde{\omega}(\xi)}{d\xi}\right) + \varepsilon^2 b_2(\xi)\psi_2(\tilde{\omega}(\xi)) = 0, \quad \xi \in [0, 1], \\ \tilde{\omega}(0) = U((i-1)\varepsilon), \quad \tilde{\omega}(1) = U(i\varepsilon) \end{cases} \tag{2.36}$$

and

$$\begin{cases} \dfrac{d}{d\xi}\left(a(\xi)\dfrac{d\overline{\omega}(\xi)}{d\xi}\right) + \varepsilon b_1(\xi)\psi_1\left(\dfrac{d\overline{\omega}(\xi)}{d\xi}\right) + \varepsilon^2 b_2(\xi)\psi_2(\overline{\omega}(\xi)) = \varepsilon^2 f(x), \quad x \in [(i-1)\varepsilon, i\varepsilon], \\ \overline{\omega}(0) = 0, \quad \overline{\omega}(1) = 0. \end{cases} \tag{2.37}$$

Combining (2.35) with (2.36)–(2.37), we obtain

$$\omega(\xi) = \tilde{\omega}(\xi) + \overline{\omega}(\xi). \tag{2.38}$$

We know from (2.38) that the key to get $\omega(\xi)$ is to solve problem (2.36) and (2.37) together.

Firstly, let us consider (2.36).

Introduce functions $\beta_1(\xi)$ and $\beta_2(\xi)$ by

$$\begin{cases} \dfrac{d}{d\xi}\left(a(\xi)\dfrac{d\beta_1(\xi)}{d\xi}\right) + \varepsilon b_1(\xi)\psi_1\left(\dfrac{d\beta_1(\xi)}{d\xi}\right) + \varepsilon^2 b_2(\xi)\psi_2(\beta_1(\xi)) = 0, \quad \xi \in [0, 1], \\ \beta_1(0) = 1, \quad \beta_1(1) = 0 \end{cases} \tag{2.39}$$

and

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}\xi}\left(a(\xi)\dfrac{\mathrm{d}\beta_2(\xi)}{\mathrm{d}\xi}\right) + \varepsilon b_1(\xi)\psi_1\left(\dfrac{\mathrm{d}\beta_2(\xi)}{\mathrm{d}\xi}\right) + \varepsilon^2 b_2(\xi)\psi_2(\beta_2(\xi)) = 0, \quad \xi \in [0, 1], \\ \beta_2(0) = 0, \quad \beta_2(1) = 1. \end{cases} \tag{2.40}$$

We conclude from (2.36), (2.39) and (2.40) that

$$\tilde{\omega}(\xi) = \beta_1(\xi)U((i-1)\varepsilon) + \beta_2(\xi)U(i\varepsilon). \tag{2.41}$$

Note that, for $x \in [(i-1)\varepsilon, i\varepsilon]$, Taylor expansion leads to

$$U((i-1)\varepsilon) = \sum_{k=0}^{\infty} \frac{[(i-1)\varepsilon - x]^k}{k!}\frac{\mathrm{d}^k U(x)}{\mathrm{d}x^k} = \sum_{k=0}^{\infty} \frac{\xi^k}{k!}(-1)^k\varepsilon^k\frac{\mathrm{d}^k U(x)}{\mathrm{d}x^k}$$

and

$$U(i\varepsilon) = \sum_{k=0}^{\infty} \frac{[i\varepsilon - x]^k}{k!}\frac{\mathrm{d}^k U(x)}{\mathrm{d}x^k} = \sum_{k=0}^{\infty} \frac{\xi^k}{k!}(1-\varepsilon)^k\frac{\mathrm{d}^k U(x)}{\mathrm{d}x^k}.$$

We have from (2.41) that

$$\begin{aligned} \tilde{\omega}(\xi) &= \beta_1(\xi)\sum_{k=0}^{\infty} \frac{\xi^k}{k!}(-1)^k\varepsilon^k\frac{\mathrm{d}^k U(x)}{\mathrm{d}x^k} + \beta_2(\xi)\sum_{k=0}^{\infty} \frac{\xi^k}{k!}(1-\varepsilon)^k\frac{\mathrm{d}^k U(x)}{\mathrm{d}x^k} \\ &= \sum_{k=0}^{\infty}\left[\beta_1(\xi)(-1)^k\varepsilon^k + \beta_2(\xi)(1-\varepsilon)^k\right]\frac{\xi^k}{k!}\frac{\mathrm{d}^k U(x)}{\mathrm{d}x^k}. \end{aligned} \tag{2.42}$$

Now let us consider how to solve problem (2.37).

Using Taylor expansion, we have

$$f(x) = \sum_{k=0}^{\infty} \frac{[x-(i-1)\varepsilon]^k}{k!} f^{(k)}((i-1)\varepsilon) = \frac{\xi^k}{k!}\varepsilon^k f^{(k)}((i-1)\varepsilon). \tag{2.43}$$

For $k = 0, 1, 2, \ldots,$ introduce problem

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}\xi}\left(a(\xi)\dfrac{\mathrm{d}\overline{\omega}_k(\xi)}{\mathrm{d}\xi}\right) + \varepsilon b_1(\xi)\psi_1\left(\dfrac{\mathrm{d}\overline{\omega}_k(\xi)}{\mathrm{d}\xi}\right) + \varepsilon^2 b_2(\xi)\psi_2(\overline{\omega}_k(\xi)) = \dfrac{\xi^k}{k!}, \quad x \in [(i-1)\varepsilon, i\varepsilon], \\ \overline{\omega}_k(0) = 0, \quad \overline{\omega}_k(1) = 0. \end{cases} \tag{2.44}$$

We have from (2.37) and (2.43)–(2.44) that

$$\overline{\omega}(\xi) = \sum_{k=0}^{\infty} \overline{\omega}_k(\xi)\varepsilon^{k+2} f^{(k)}((i-1)\varepsilon). \tag{2.45}$$

Considering

$$f^{(k)}((i-1)\varepsilon) = \sum_{j=0}^{\infty} \frac{[(i-1)\varepsilon - x]^j}{j!} f^{(k+j)}(x) = \sum_{j=0}^{\infty}(-1)^j\varepsilon^j\frac{\xi^j}{j!}f^{(k+j)}(x). \tag{2.46}$$

Combining (2.45) with (2.46), we see

$$\begin{aligned} \overline{\omega}(\xi) &= \sum_{k=0}^{\infty} \overline{\omega}_k(\xi)\varepsilon^{k+2}\sum_{j=0}^{\infty}(-1)^j\varepsilon^j\frac{\xi^j}{j!}f^{(k+j)}(x) \\ &= \sum_{k=0}^{\infty} \overline{\omega}_k(\xi)\left(\sum_{j=0}^{\infty}(-1)^j\varepsilon^{k+j+2}\frac{\xi^j}{j!}f^{(k+j)}(x)\right). \end{aligned} \tag{2.47}$$

Finally, from (2.38), (2.42) and (2.47), we have

$$\omega(\xi) = \tilde{\omega}(\xi) + \overline{\omega}(\xi) = \sum_{k=0}^{\infty}\left[\beta_1(\xi)(-1)^k\varepsilon^k + \beta_2(\xi)(1-\varepsilon)^k\right]\frac{\xi^k}{k!}\frac{\mathrm{d}^k U(x)}{\mathrm{d}x^k} + \sum_{k=0}^{\infty}\overline{\omega}_k(\xi)\left(\sum_{j=0}^{\infty}(-1)^j\varepsilon^{k+j+2}\frac{\xi^j}{j!}f^{(k+j)}(x)\right). \tag{2.48}$$

Concluding from above analysis, we have an asymptotic expansion expression of $u^\varepsilon(x)$ as follows:

$$u^\varepsilon(x) = \sum_{k=0}^{\infty}\left[\beta_1(\xi)(-1)^k\varepsilon^k + \beta_2(\xi)(1-\varepsilon)^k\right]\frac{\xi^k}{k!}\frac{\mathrm{d}^k U(x)}{\mathrm{d}x^k} + \sum_{k=0}^{\infty}\overline{\omega}_k(\xi)\left(\sum_{j=0}^{\infty}(-1)^j\varepsilon^{k+j+2}\frac{\xi^j}{j!}f^{(k+j)}(x)\right). \tag{2.49}$$

Thus, we have the following result.

Theorem 2.2. *Assume that u^ε is the solution for problem* (2.1). *Then u^ε can be expressed as follows*:

$$u^\varepsilon(x) = \sum_{k=0}^{\infty}\left[\beta_1(\xi)(-1)^k\varepsilon^k + \beta_2(\xi)(1-\varepsilon)^k\right]\frac{\xi^k}{k!}\frac{\mathrm{d}^k U(x)}{\mathrm{d}x^k} + \sum_{k=0}^{\infty}\overline{\omega}_k(\xi)\left(\sum_{j=0}^{\infty}(-1)^j\varepsilon^{k+j+2}\frac{\xi^j}{j!}f^{(k+j)}(x)\right). \tag{2.50}$$

Notation: Theorems 2.1 and 2.2 propose an asymptotic expansion method to solve problem (1.3). Then we want to know whether we can obtain solution for following common nonlinear problem by using asymptotic expansion method

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}x}\left(a\left(\dfrac{x}{\varepsilon}\right)\dfrac{\mathrm{d}u^\varepsilon}{\mathrm{d}x}\right) + \psi\left(u^\varepsilon, \dfrac{\mathrm{d}u^\varepsilon}{\mathrm{d}x}, \xi, x\right) = f(x), \quad x \in (0, l), \\ u^\varepsilon(0) = u_0, \quad u^\varepsilon(l) = u_1, \end{cases} \tag{2.51}$$

where $\psi(u^\varepsilon, \mathrm{d}u^\varepsilon/\mathrm{d}x, \xi, x)$ is a function about u^ε, $\mathrm{d}u^\varepsilon/\mathrm{d}x$, ξ and x.

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