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A rigidity theorem for complete noncompact Bach-flat manifolds ☆

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ABSTRACT

Let (M^4, g) be a four-dimensional complete noncompact Bach-flat Riemannian manifold with positive Yamabe constant. In this paper, we show that (M^4, g) has a constant curvature if it has a nonnegative constant scalar curvature and sufficiently small L_2-norm of trace-free Riemannian curvature tensor. Moreover, we get a gap theorem for (M^4, g) with positive scalar curvature.

1. Introduction

Let (M, g) be a four-dimensional Riemannian manifold with Weyl curvature tensor W, Ricci tensor Rc and scalar curvature R. In local coordinates, we denote by $g = (g_{ij})$, the Riemannian metric on M with coefficients g_{ij}, and denote the inverse matrix by $(g^{ij}) = (g_{ij})^{-1}$. Throughout this paper we adopt the Einstein summation convention. Recall (see [1]) that the Bach tensor is defined as

$$B_{ij} \equiv \nabla^k \nabla^l W_{kijl} + \frac{1}{2} R^{kl} W_{kijl}, \tag{1.1}$$

where W_{kijl} and $R^{kl} = g^{ki} g^{lj} R_{ij}$ are the components of the Weyl and Ricci tensors respectively. In [2], Korzyński and Lewandowski proved that the Bach tensor can be identified with the Yang–Mills current of the Cartan normal conformal connection.

A metric is called Bach-flat if it is a critical metric of the functional

$$\mathcal{W} : g \mapsto \int_M |W_g|^2 \mathrm{d}V_g. \tag{1.2}$$

The Bach-flat condition is equivalent to the vanishing of Bach tensor (see [3]). Note that, in particular, metrics which are locally conformal to an Einstein metric are Bach-flat, and half conformally flat metrics are Bach-flat (see [4]). However, Hill and Nurowski [5] and Leistner and Nurowski [6] obtained a large class of Bach-flat examples which are not conformally Einstein.

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We now introduce the definition of Yamabe constant. Given an n-dimensional complete noncompact Riemannian manifold (M, g) with scalar curvature R and $n \geq 3$, the Yamabe constant $Q(M, g)$ is defined as

$$Q(M,g) \equiv \inf_{0 \neq u \in C_0^\infty(M)} \frac{\int_M \left(|\nabla u|^2 + \frac{n-2}{4(n-1)} R u^2 \right) \mathrm{d}V_g}{\left(\int_M |u|^{2n/(n-2)} \mathrm{d}V_g \right)^{(n-2)/n}}. \tag{1.3}$$

The important work of Yamabe [7], Schoen [8] and Trudinger [9] showed that the infimum in (1.3) is always achieved. Furthermore, for compact manifolds, $Q(M, g)$ is determined by the sign of scalar curvature R (see [10,11]), for noncompact case, $Q(M, g)$ is always positive if R vanishes (see [12]).

Bach-flat manifold (M^4, g) with positive Yamabe constant has been studied by many mathematicians. When M^4 is compact, Chang et al. [13] proved that (M^4, g) is conformally equivalent to the standard four-sphere if it has a sufficiently small L_2-norm of Weyl tensor. They also showed that there is only finite diffeomorphism class with an L_2-norm bound of Weyl tensor. When M^4 is a complete noncompact manifold with vanished scalar curvature, Tian and Viaclovsky [14] showed that it is almost locally Euclidean of order 0 under the assumption of bounded L_2-norm of curvature tensor, bounded first Betti number and the uniform volume growth for any geodesic ball. In 2010, Streets [15] extends the removal of singularities result for special classes of Bach-flat metrics obtained in [14].

Recently, Kim [16] studied the rigidity phenomena for complete noncompact Bach-flat manifolds and proved the following results (Theorems A and B).

Theorem A. *Assume that (M, g) is a complete noncompact Bach-flat Riemannian 4-manifold with zero scalar curvature and $Q(M, g) > 0$. Then there exists a small number c_0 such that if $\int_M |Rm|^2 \mathrm{d}V_g \leq c_0$, then M is flat, where Rm is the Riemannian curvature tensor.*

Theorem B. *Let (M, g) be a complete noncompact Riemannian 4-manifold with a nonnegative constant scalar curvature R and traceless Ricci curvature $\overset{\circ}{Rc}$. Assume that (M, g) is Bach-flat and $Q(M, g) > 0$. Then there exists a small number c_0 such that if $\int_M (|W|^2 + |\overset{\circ}{Rc}|^2) \mathrm{d}V_g \leq c_0$, then (M, g) is an Einstein manifold.*

Inspired by the historical development on the study of the Bach-flat manifolds with positive Yamabe constant, we continue to study rigidity phenomena for these manifolds, in this paper. We first give the Laplacian of the norm square of trace-free curvature tensor in Lemma 2.1. By employing an elliptic estimation on the Laplacian, we prove the following result which is more stronger than Theorems A and B.

Theorem 1.1. *Let (M, g) be a complete noncompact Bach-flat Riemannian 4-manifold with a nonnegative constant scalar curvature and $Q(M, g) > 0$. Then there exists a small number c_1 such that if $\int_M |\overset{\circ}{R}m|^2 \mathrm{d}V_g \leq c_1$, then M has a constant curvature, where $\overset{\circ}{R}m = \{\overset{\circ}{R}_{ijkl}\} = \left\{R_{ijkl} - \frac{R}{12}(g_{ik}g_{jl} - g_{il}g_{jk})\right\}$ is a trace-free curvature tensor.*

Remark 1.2. As the referee pointed out, the nonnegativity of the scalar curvature in Theorem 1.1 is necessary (consider the hyperbolic space for example).

Since there is no complete noncompact constant curvature manifold with positive scalar curvature, Theorem 1.1 implies the following gap theorem.

Theorem 1.3. *Let (M, g) be a complete noncompact Bach-flat Riemannian 4-manifold with a positive constant scalar curvature and $Q(M, g) > 0$. Then there exists a positive constant c_2 such that $\int_M |\overset{\circ}{R}m|^2 \mathrm{d}V_g > c_2$.*

Remark 1.4. The constants c_1 and c_2 in Theorems 1.1 and 1.3 depend on the Yamabe constant.

Remark 1.5. When the scalar curvature of (M, g) vanishes, a consequence of Theorem 1.1 is Theorem A in [16].

2. Proof of Theorem 1.1

Suppose that (M, g) is a Riemannian manifold of dimension $n \geq 3$, it is well known that the curvature tensor $Rm = \{R_{ijkl}\}$ of M^n can be decomposed into three orthogonal components which have the same symmetries as Rm:

$$Rm = U + V + W, \tag{2.1}$$

where U, V and W denote the scalar curvature part, the traceless Ricci part and the Weyl curvature tensor, respectively (see [17]). In local coordinates, the norm of a tensor T is defined as

$$|T|^2 = |T_{ijkl}|^2 = g^{im}g^{jn}g^{ks}g^{lt}T_{ijkl}T_{mnst} \equiv T^{ijkl}T_{ijkl}.$$

We denote by $\mathring{R}m = \{\mathring{R}_{ijkl}\} = \left\{R_{ijkl} - \frac{R}{n(n-1)}(g_{ik}g_{jl} - g_{il}g_{jk})\right\}$ and $\mathring{R}c = \{\mathring{R}_{ij}\} = \left\{R_{ij} - \frac{1}{n}Rg_{ij}\right\}$ the trace-free Riemannian curvature tensor and the traceless Ricci tensor, then the following equalities are easily obtained from the properties of Riemannian curvature tensor:

$$g^{ik}\mathring{R}_{ijkl} = \mathring{R}_{jl}, \tag{2.2}$$

$$\mathring{R}_{ijkl} + \mathring{R}_{iljk} + \mathring{R}_{iklj} = 0, \tag{2.3}$$

$$\mathring{R}_{ijkl} = \mathring{R}_{klij} = -\mathring{R}_{jikl} = -\mathring{R}_{ijlk}, \tag{2.4}$$

$$|\mathring{R}m|^2 = |W|^2 + |V|^2 = |W|^2 + \frac{4}{n-2}|\mathring{R}c|^2. \tag{2.5}$$

Moreover, the assumption of constant scalar curvature yields

$$\nabla_m\mathring{R}_{ijkl} + \nabla_l\mathring{R}_{ijmk} + \nabla_k\mathring{R}_{ijlm} = 0. \tag{2.6}$$

Note that, for a four-dimensional manifold, (2.5) implies that

$$|\mathring{R}c|^2 \le |\mathring{R}m|^2. \tag{2.7}$$

Now, we establish a general formula for the Laplacian of $|\mathring{R}m|^2$.

Lemma 2.1. *Let (M^n, g) $(n \ge 3)$ be an n-dimensional Riemannian manifold, then*

$$\begin{aligned} \frac{1}{2}\Delta|\mathring{R}_{ijkl}|^2 &= |\nabla\mathring{R}m|^2 + 2\mathring{R}^{ijkl}g^{pq}g^{rs}(\mathring{R}_{pilr}\mathring{R}_{qjks} - \mathring{R}_{pjlr}\mathring{R}_{qiks} + \mathring{R}_{plkr}\mathring{R}_{ijqs}) \\ &\quad + 2\mathring{R}^{ijkl}\mathring{R}^{h}_{ijk}\mathring{R}_{hl} + 2\mathring{R}^{ijkl}\nabla_l\nabla^m\mathring{R}_{mkji} + \frac{2}{n}R|\mathring{R}m|^2 - \frac{4R|\mathring{R}c|^2}{n(n-1)}, \end{aligned} \tag{2.8}$$

where Δf denotes the Laplacian of f given by the trace of Hess$f$.

Proof. To simplify the notations, we will compute at an arbitrarily chosen point $p \in M$ in normal coordinates centered at p so that $g_{ij} = \delta_{ij}$. By the definition of $|\mathring{R}m|^2$, we obtain from (2.4) and (2.6) that

$$\begin{aligned} \frac{1}{2}\Delta|\mathring{R}_{ijkl}|^2 &= |\nabla\mathring{R}m|^2 + \mathring{R}^{ijkl}\nabla^m\nabla_m\mathring{R}_{ijkl} \\ &= |\nabla\mathring{R}m|^2 + \mathring{R}^{ijkl}\nabla^m(\nabla_l\mathring{R}_{ijkm} + \nabla_k\mathring{R}_{ijml}) \\ &= |\nabla\mathring{R}m|^2 + 2\mathring{R}^{ijkl}\nabla^m\nabla_l\mathring{R}_{ijkm} \\ &= |\nabla\mathring{R}m|^2 + 2\mathring{R}^{ijkl}(\nabla_l\nabla^m\mathring{R}_{ijkm} + R^m_{lih}\mathring{R}^h_{jkm} + R^m_{ljh}\mathring{R}^h_{ikm} + R^m_{lkh}\mathring{R}^h_{ijm} + R^m_{lmh}\mathring{R}^h_{ijk}), \end{aligned} \tag{2.9}$$

where the Ricci identities are used in the last equality of (2.9). By the definition of trace-free Riemannian curvature tensor and (2.2), we get

$$\begin{aligned} \frac{1}{2}\Delta|\mathring{R}_{ijkl}|^2 &= |\nabla\mathring{R}m|^2 + 2\mathring{R}^{ijkl}(\nabla_l\nabla^m\mathring{R}_{mkji} + \mathring{R}^m_{hil}\mathring{R}^h_{jkm} - \mathring{R}^m_{hjl}\mathring{R}^h_{ikm} + \mathring{R}^m_{lkh}\mathring{R}^h_{ijm}) \\ &\quad + 2\mathring{R}^{ijkl}\mathring{R}^h_{ijk}\left(\mathring{R}_{lh} + \frac{R}{n}g_{lh}\right) + \frac{2R}{n(n-1)}\mathring{R}^{ijkl}(\mathring{R}^h_{jkm}(g^m_i g_{lh} - g^m_h g_{li}) \\ &\quad + \mathring{R}^h_{ikm}(g^m_j g_{lh} - g^m_h g_{lj}) + \mathring{R}^h_{ijm}(g^m_k g_{lh} - g^m_h g_{lk})) \\ &= |\nabla\mathring{R}m|^2 + 2\mathring{R}^{ijkl}(\nabla_l\nabla^m\mathring{R}_{mkji} + \mathring{R}^m_{hil}\mathring{R}^h_{jkm} - \mathring{R}^m_{hjl}\mathring{R}^h_{ikm} + \mathring{R}^m_{lkh}\mathring{R}^h_{ijm}) + 2\mathring{R}^{ijkl}\mathring{R}^h_{ijk}\mathring{R}_{lh} + \frac{2R}{n}|\mathring{R}m|^2 \\ &\quad + \frac{2R}{n(n-1)}\mathring{R}^{ijkl}(\mathring{R}_{ljki} + \mathring{R}_{ilkj} + \mathring{R}_{ijlk} + \mathring{R}_{jk}g_{li} - \mathring{R}_{ik}g_{lj}). \end{aligned} \tag{2.10}$$

Combining (2.2), (2.3) with (2.10), we have

$$\begin{aligned} \frac{1}{2}\Delta|\mathring{R}_{ijkl}|^2 &= |\nabla\mathring{R}m|^2 + 2\mathring{R}^{ijkl}(\mathring{R}^h_{ijk}\mathring{R}_{lh} + \mathring{R}^m_{hil}\mathring{R}^h_{jkm} - \mathring{R}^m_{hjl}\mathring{R}^h_{ikm} + \mathring{R}^m_{lkh}\mathring{R}^h_{ijm}) \\ &\quad + 2\mathring{R}^{ijkl}\nabla_l\nabla^m\mathring{R}_{mkji} + \frac{2R}{n}|\mathring{R}m|^2 - \frac{4R}{n(n-1)}|\mathring{R}c|^2. \end{aligned} \tag{2.11}$$

This proves Lemma 2.1. □

Making use of the $*$-notation (see [18]), we rewrite (2.8) as

$$\overset{\circ}{R}{}^{ijkl}\Delta\overset{\circ}{R}_{ijkl}=\overset{\circ}{R}m*\overset{\circ}{R}m*\overset{\circ}{R}m+2\overset{\circ}{R}{}^{ijkl}\nabla_l\nabla^m\overset{\circ}{R}_{mkji}+\frac{2}{n}R|\overset{\circ}{R}m|^2-\frac{4R|\overset{\circ}{R}c|^2}{n(n-1)},\tag{2.12}$$

where $\overset{\circ}{R}m*\overset{\circ}{R}m*\overset{\circ}{R}m$ denotes cubic terms of $\overset{\circ}{R}m$ (see Eq. (2.5)), which is bounded by $c|\overset{\circ}{R}m|^3$ for a positive constant c.

In dimension 4, we conclude from (2.12), (2.7) and $R\geq 0$ that

$$\begin{aligned}-\overset{\circ}{R}{}^{ijkl}\Delta\overset{\circ}{R}_{ijkl}&\leq c|\overset{\circ}{R}m|^3-2\overset{\circ}{R}{}^{ijkl}\nabla_l\nabla^m\overset{\circ}{R}_{mkji}-\frac{R}{2}|\overset{\circ}{R}m|^2+\frac{R|\overset{\circ}{R}c|^2}{3}\\&\leq c|\overset{\circ}{R}m|^3-2\overset{\circ}{R}{}^{ijkl}\nabla_l\nabla^m\overset{\circ}{R}_{mkji}-\frac{R}{6}|\overset{\circ}{R}m|^2.\end{aligned}\tag{2.13}$$

Assume that, ϕ is a smooth compact supported function on M and $u=|\overset{\circ}{R}m|$. From Lemma 2.1 and (2.13), we derive the following lemma.

Lemma 2.2. *Let (M,g) be a four-dimensional Bach-flat Riemannian manifold with a nonnegative constant scalar curvature, then*

$$\int_M(\phi^2|\nabla u|^2+2\phi\nabla u\cdot\nabla\phi)\mathrm{d}V_g\leq\int_M\left(c_3\phi^2u^3+c_3|\nabla\phi|^2u^2-\frac{R}{6}\phi^2u^2\right)\mathrm{d}V_g,$$

where c_3 is a positive constant.

Proof. By the Kato inequality $|\nabla\overset{\circ}{R}m|^2\geq|\nabla|\overset{\circ}{R}m||^2$, we get

$$\begin{aligned}\overset{\circ}{R}{}^{ijkl}\Delta\overset{\circ}{R}_{ijkl}&\leq\overset{\circ}{R}{}^{ijkl}\Delta\overset{\circ}{R}_{ijkl}+|\nabla\overset{\circ}{R}m|^2-|\nabla|\overset{\circ}{R}m||^2\\&=\frac{1}{2}\Delta|\overset{\circ}{R}m|^2-|\nabla|\overset{\circ}{R}m||^2\\&=|\overset{\circ}{R}m|\Delta|\overset{\circ}{R}m|=u\Delta u.\end{aligned}\tag{2.14}$$

This together with (2.13) yields

$$-\int_M\phi^2u\Delta u\mathrm{d}V_g\leq\int_M\phi^2\left(cu^3-2\overset{\circ}{R}{}^{ijkl}\nabla_l\nabla^m\overset{\circ}{R}_{mkji}-\frac{R}{6}u^2\right)\mathrm{d}V_g,\tag{2.15}$$

where

$$\begin{aligned}-2\int_M\phi^2\overset{\circ}{R}{}^{ijkl}\nabla_l\nabla^m\overset{\circ}{R}_{mkji}\mathrm{d}V_g&=2\int_M\left(\nabla_l(\phi^2)\overset{\circ}{R}_{ijkl}\nabla^m\overset{\circ}{R}_{mkji}+\phi^2\nabla_l\overset{\circ}{R}{}^{ijkl}\cdot\nabla^m\overset{\circ}{R}_{mkji}\right)\mathrm{d}V_g\\&\leq 2\int_M\left((1+\epsilon)\phi^2|\nabla^i\overset{\circ}{R}_{ijkl}|^2+\frac{1}{\varepsilon}|\nabla\phi|^2|\overset{\circ}{R}_{ijkl}|^2\right)\mathrm{d}V_g,\end{aligned}\tag{2.16}$$

here ϵ is a positive constant.

Recall that R is a constant, and then $\nabla^iR_{ijkl}=\nabla^i\overset{\circ}{R}_{ijkl}$. Under the assumption of Bach-flat, we can get the following relationship from (13) and (17) in [16].

$$\begin{aligned}\int_M\phi^2|\nabla^i\overset{\circ}{R}_{ijkl}|^2\mathrm{d}V_g&=4\int_M\phi^2|\nabla^iW_{ijkl}|^2\mathrm{d}V_g\\&\leq\frac{4}{1-\eta}\int_M\left(\frac{1}{2}\phi^2W_{ikjl}\overset{\circ}{R}{}^{kl}\overset{\circ}{R}{}^{ij}+\frac{1}{\eta}|\nabla\phi|^2|\overset{\circ}{R}_{ij}|^2\right)\mathrm{d}V_g,\end{aligned}\tag{2.17}$$

where η is a positive constant. Combining (2.15), (2.16) with (2.17), we have

$$\begin{aligned}-\int_M\phi^2u\Delta u\mathrm{d}V_g&\leq\int_M\left(c\phi^2u^3+2(1+\epsilon)\phi^2|\nabla^i\overset{\circ}{R}_{ijkl}|^2+\frac{2}{\epsilon}|\nabla\phi|^2u^2-\frac{R}{6}\phi^2u^2\right)\mathrm{d}V_g\\&\leq\int_M\left(c\phi^2u^3+\frac{2}{\epsilon}|\nabla\phi|^2u^2+\frac{4(1+\epsilon)}{1-\eta}\phi^2W_{ikjl}\overset{\circ}{R}{}^{kl}\overset{\circ}{R}{}^{ij}+\frac{8(1+\epsilon)}{(1-\eta)\eta}|\nabla\phi|^2|\overset{\circ}{R}_{ij}|^2-\frac{R}{6}\phi^2u^2\right)\mathrm{d}V_g.\end{aligned}$$

Note that $W_{ikjl}\overset{\circ}{R}{}^{kl}\overset{\circ}{R}{}^{ij}$ and $|\overset{\circ}{R}_{ij}|^2$ are cubic and quadratic terms of $\overset{\circ}{R}m$ (see Eq. (2.5)), respectively. Therefore, there exists a constant $c_3 > 0$, such that

$$\begin{aligned}\int_M (\phi^2|\nabla u|^2 + 2\phi\nabla u\cdot\nabla\phi)\mathrm{d}V_g &= -\int_M \phi^2 u\Delta u \mathrm{d}V_g\\ &\le \int_M \left(c_3\phi^2u^3 + c_3|\nabla\phi|^2u^2 - \frac{R}{6}\phi^2u^2\right)\mathrm{d}V_g. \end{aligned} \tag{2.18}$$

This completes the proof of Lemma 2.2. □

Proof of Theorem 1.1. It is sufficient to prove $u = 0$. From the definition of Yamabe constant $\Lambda_0 \equiv Q(M,g)$ and Lemma 2.2, we have

$$\begin{aligned}\Lambda_0\left(\int_M(\phi u)^4\mathrm{d}V_g\right)^{\frac{1}{2}} &\le \int_M\left(|\nabla(\phi u)|^2 + \frac{R}{6}\phi^2u^2\right)\mathrm{d}V_g\\ &\le \int_M\left(u^2|\nabla\phi|^2 + \phi^2|\nabla u|^2 + 2u\phi\nabla u\cdot\nabla\phi + \frac{R}{6}\phi^2u^2\right)\mathrm{d}V_g\\ &\le \int_M((c_3+1)u^2|\nabla\phi|^2 + c_3\phi^2u^3)\mathrm{d}V_g. \end{aligned}\tag{2.19}$$

Note that

$$\int_M c_3\phi^2u^3\mathrm{d}V_g \le c'\left(\int_M(\phi u)^4\mathrm{d}V_g\right)^{\frac{1}{2}}\left(\int_M u^2\mathrm{d}V_g\right)^{\frac{1}{2}}, \tag{2.20}$$

where $c' > 0$ is a constant. Since $\int_M u^2\mathrm{d}V_g$ is sufficiently small, $\int_M u^2\mathrm{d}V_g$ can be absorbed into the left-hand side of (2.19). Therefore, there exists a constant C, such that

$$C\left(\int_M\phi^4u^4\mathrm{d}V_g\right)^{\frac{1}{2}} \le \int_M|\nabla\phi|^2u^2\mathrm{d}V_g. \tag{2.21}$$

Let $B_r = \{x\in M : d(x,x_0)\le r\}$ for some fixed point $x_0\in M$, choose ϕ as

$$\phi = \begin{cases} 1 & \text{on } B_r,\\ 0 & \text{on } M - B_{2r},\\ |\nabla\phi| \le \dfrac{2}{r} & \text{on } B_{2r} - B_r,\end{cases}$$

and $0\le\phi\le 1$. From (2.21), we get

$$\begin{aligned}C\left(\int_M\phi^4u^4\mathrm{d}V_g\right)^{\frac{1}{2}} &\le \frac{4}{r^2}\int_{B_{2r}-B_r}u^2\mathrm{d}V_g\\ &\le \frac{4}{r^2}c_1.\end{aligned}$$

By taking $r\to\infty$, we have $u = 0$, this completes the proof of Theorem 1.1. □

Remark 2.3. For a Riemannian 4-manifold, (2.5) implies the equivalence between the assumption of $\int_M|\overset{\circ}{R}m|^2\mathrm{d}V_g \le c_1$ in Theorem 1.1 and $\int_M|\overset{\circ}{Rc}|^2 + |W|^2\mathrm{d}V_g \le c_0$ in Theorem B. Therefore, Theorem 1.1 expands Theorem B in [16].

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