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# Multifractal dimension inequalities in a probability space

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**Abstract**

Let $v$ be a probability measure on $\Omega$. We define the upper and lower multifractal box dimension (the measure $v$ with respect to $\mu$) on a probability space and investigate the relation between the multifractal box dimension and the multifractal Hausdorff dimension, the multifractal pre-packing dimension . Then, we generalize the dimension inequalities of multifractal Hausdorff measures and multifractal packing measures in a probability space.


## 1. Introduction

Multifractal theory has been discussed by numerous authors and it is developing rapidly. Very recently there has been an enormous interest in verifying the multifractal formalism and computing the multifractal spectrum of measures in the mathematical literature and within the 5 or 6 years the multifractal spectra of various classes of measures in Euclidean space $\mathbf{R}^d$ exhibiting some degree of self-similarity have been computed rigorously. But the nature many things have the smoothness and the complexity often are not stochastic. Therefore, the research multifractal theory needs in the general probability space.

In 1995 Olsen established a multifractal formalism (see [1]). In 2000, Olsen developed the dimension inequalities of multifractal Hausdorff measures and multifractal packing measures (see [2]). Li and Dai established a multifractal formalism in a probability space in 2006 (see [3]). Applying the above idea, we may develop the analogue of the dimension inequalities of multifractal Hausdorff measures and multifractal packing measures.

We now give a brief description of the organization the paper. Section 2 contains preliminaries . In Section 3, we define the lower and upper multifractal box dimension (the measure $v$ with respect to the measure $\mu$) in a probability space and investigate the relation between the multifractal box dimension and the multifractal Hausdorff dimension, the multifractal pre-packing dimension. In Section 4, we generalize the dimension inequalities of multifractal Hausdorff measures and multifractal packing measure in a probability space.

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## 2. Preliminaries

In this paper, we want to yield a generalization multifractal box dimension in a probability space.

Let we start with a fixed stochastic process $\{X_n, n \in \mathbf{N}\}$ on a probability space $(\Omega, \mathscr{F}, \mu)$ taking values in a finite or countable state space $E$. A cylinder set $C$ of rank $n$ is of the form

$$C = \{\omega : X_i(\omega) = k_i,\ i = 1, 2, \ldots, n\}$$

with $k_i \in E$. For each $\omega_0 \in \Omega$ there is a unique cylinder set of rank $n$, denoted by $u_n(\omega_0)$, which contains $\omega_0$. Thus

$$u_n(\omega_0) = \{\omega : X_i(\omega) = X_i(\omega_0),\ i = 1, 2, \ldots, n\}.$$

We assume the process is $\mathscr{F}$-measure, that is that $\mathscr{C} \subset \mathscr{F}$,where $\mathscr{C}$ is the class of all cylinder sets. We use sets in $\mathscr{C}$ for both covering and packing. Many details of classical proofs are greatly simplified because $\mathscr{C}$ is nested; that is, give $C_1, C_2 \in \mathscr{C}$, either $C_1 \cap C_2 = \emptyset$ or $C_1 \subset C_2$ or $C_2 \subset C_1$. We use sets in $\mathscr{C}$ for both covering and packing.

In this paper, we will assume that $\mu$ is $\mathscr{H} = \sigma(\mathscr{C})$-continuous, that is

$$\lim_{n\to\infty} \mu(u_n(\omega)) = 0 \quad \text{for all } \omega \in \Omega.$$

Let $A \subseteq \Omega$ and $\delta > 0$. A countable family $\mathscr{B} = \{C_i = u_n(\omega_i)\}_i$ is called $\delta$-covering of $A$ if $A \subset \cup_i C_i$, $\omega_i \in A$ and $\mu(u_n(\omega_i)) < \delta$ for all $i$. The family $\mathscr{B} = \{C_i = u_n(\omega_i)\}_i$ is called centered $\delta$-packing of $A$ if $\omega_i \in A$, $\mu(u_n(\omega_i)) < \delta$ and $C_i \cap C_j = \emptyset$ for all $i \neq j$. Let $v$ be a probability measure on $\Omega$.

For $q \in \mathbf{R}$, define $\varphi_q : [0, \infty) \to \overline{\mathbf{R}}_+ = [0, \infty]$ by

$$\varphi_q(x) = \begin{cases} +\infty & \text{for } x = 0, \\ x^q & \text{for } x > 0, \end{cases} \quad \text{for } q < 0;$$

$$\varphi_q(x) = 1 \quad \text{for } q = 0;$$

$$\varphi_q(x) = \begin{cases} 0, & \text{for } x = 0, \\ x^q & \text{for } x > 0, \end{cases} \quad \text{for } q > 0.$$

Let $\emptyset \neq A \subset \Omega$ and $\delta > 0$, Suppose $v$ is a probability measure on $(\Omega, \mathscr{F})$. For $q, t \in \mathbf{R}$, write

$$\begin{aligned} \widetilde{\mathscr{H}}^{q,t}_{\mu,v,\delta}(A) &= \inf\left\{\sum_i \varphi_q(v(C_i))\varphi_t(\mu(C_i)) : A \subset \cup_i C_i,\ \mu(C_i) < \delta \text{ and } C_i = u_n(\omega) \text{ with } \omega \in A\right\}, \\ \widetilde{\mathscr{H}}^{q,t}_{\mu,v}(A) &= \lim_{\delta\to 0} \widetilde{\mathscr{H}}^{q,t}_{\mu,v,\delta}(A) = \sup_{\delta>0} \widetilde{\mathscr{H}}^{q,t}_{\mu,v,\delta}(A), \\ \mathscr{H}^{q,t}_{\mu,v}(A) &= \sup_{A_i \subset A} \widetilde{\mathscr{H}}^{q,t}_{\mu,v}(A_i) \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} \widetilde{\mathscr{P}}^{q,t}_{\mu,v,\delta}(A) &= \sup\left\{\sum_i \varphi_q(v(C_i))\varphi_t(\mu(C_i)) : C_i \cap C_j = \emptyset,\ i \neq j,\ \mu(C_i) < \delta,\ \text{and } C_i = u_n(\omega) \text{ with } \omega \in A\right\}, \\ \widetilde{\mathscr{P}}^{q,t}_{\mu,v}(A) &= \lim_{\delta\to 0} \widetilde{\mathscr{P}}^{q,t}_{\mu,v,\delta}(A) = \inf_{\delta>0} \widetilde{\mathscr{P}}^{q,t}_{\mu,v,\delta}(A), \\ \mathscr{P}^{q,t}_{\mu,v}(A) &= \inf\left\{\sum_i \widetilde{\mathscr{P}}^{q,t}_{\mu,v}(A_i) : A \subset \cup_i A_i\right\}. \end{aligned} \tag{2.2}$$

In fact, it is easily seen that the following holds for $t \geqslant 0$:

$$\begin{aligned} L^t_\mu(A) &\leqslant \mathscr{H}^{0,t}_{\mu,v}(A), \\ \mathscr{P}^t_\mu(A) &= \mathscr{P}^{0,t}_{\mu,v}(A), \\ \widetilde{\mathscr{P}}^t_\mu(A) &= \widetilde{\mathscr{P}}^{0,t}_{\mu,v}(A), \end{aligned} \tag{2.3}$$

where $L^t_\mu(A)$ denotes the $t$-dimensional Hausdorff measure with respect to $\mu$, $\mathscr{P}^t_\mu$ denotes the $t$-dimensional packing measure and $\widetilde{\mathscr{P}}^t_\mu$ denotes the $t$-dimensional pre-packing measure with respect to $\mu$. It is easily seen that the usual assign way a dimension to each subset $A$ of $\Omega$: there exist unique numbers $\dim^q_{\mu,v}(A)$, $\Delta^q_{\mu,v}(A)$, $\mathrm{Dim}^q_{\mu,v}(A) \in [-\infty, +\infty]$ such that

$$\begin{aligned}
&\dim^q_{\mu,\nu}(A) = \sup\left\{t : \mathscr{H}^{q,t}_{\mu,\nu}(A) = +\infty\right\} = \inf\left\{t : \mathscr{H}^{q,t}_{\mu,\nu}(A) = 0\right\},\\
&\Delta^q_{\mu,\nu}(A) = \sup\left\{t : \widetilde{\mathscr{P}}^{q,t}_{\mu,\nu}(A) = +\infty\right\} = \inf\left\{t : \widetilde{\mathscr{P}}^{q,t}_{\mu,\nu}(A) = 0\right\},\\
&\mathsf{Dim}^q_{\mu,\nu}(A) = \sup\left\{t : \mathscr{P}^{q,t}_{\mu,\nu}(A) = +\infty\right\} = \inf\left\{t : \mathscr{P}^{q,t}_{\mu,\nu}(A) = 0\right\},
\end{aligned} \tag{2.4}$$

which are respectively called multifractal Hausdorff dimension, pre-packing dimension, packing dimension with respect to $\mu$ in a probability space. For convenience, we write

$$\begin{aligned}
&b_{\mu,\nu,A}(q) := \dim^q_{\mu,\nu}(A),\\
&\Lambda_{\mu,\nu,A}(q) := \Delta^q_{\mu,\nu}(A),\\
&B_{\mu,\nu,A}(q) := \mathsf{Dim}^q_{\mu,\nu}(A),\\
&b(q) := b_{\mu,\nu}(q) = \dim^q_{\mu,\nu}(\operatorname{supp}\mu \cap \operatorname{supp}\nu),\\
&\Lambda(q) := \Lambda_{\mu,\nu}(q) = \Delta^q_{\mu,\nu}(\operatorname{supp}\mu \cap \operatorname{supp}\nu),\\
&B(q) := B_{\mu,\nu}(q) = \mathsf{Dim}^q_{\mu,\nu}(\operatorname{supp}\mu \cap \operatorname{supp}\nu).
\end{aligned} \tag{2.5}$$

It is also readily seen that

$$b(q) \leqslant B(q) \leqslant \Lambda(q).$$

In fact, Eq. (2.1) imply that

$$\begin{aligned}
&\dim_\mu(A) \leqslant \dim^0_{\mu,\nu}(A),\\
&\Delta_\mu(A) = \Delta^0_{\mu,\nu}(A),\\
&\mathsf{Dim}_\mu(A) = \mathsf{Dim}^0_{\mu,\nu}(A).
\end{aligned}$$

We are now ready to introduce new indices.

## 3. Multifractal box dimensions in a probability space

We now define multifractal box dimensions in a probability space. We first recall the definition multifractal box dimensions (see[1]). Let $\mu \in \mathscr{P}(\mathbf{R}^d)$ and $q \in \mathbf{R}$. For $E \subset \mathbf{R}^d$ and $\delta > 0$ write

$$S^q_{\mu,\delta}(E) := \sup\left\{\sum_i \mu(B(x_i,\delta))^q : (B(x_i,\delta))_i \text{ is a centered packing of } E\right\}. \tag{3.1}$$

The upper respectively lower multifractal $q$-box dimension $\overline{C}^q_\mu(E)$ and $\underline{C}^q_\mu(E)$ of $E$ (with respect to the measure $\mu$) is defined by

$$\overline{C}^q_\mu(E) := \limsup_{\delta\downarrow 0} \frac{\log S^q_{\mu,\delta}(E)}{-\log\delta}, \quad \underline{C}^q_\mu(E) := \liminf_{\delta\downarrow 0} \frac{\log S^q_{\mu,\delta}(E)}{-\log\delta}. \tag{3.2}$$

If $\overline{C}^q_\mu(E) = \underline{C}^q_\mu(E)$ we refer to the common value as the $q$-box dimension of $E$ (with respect to the measure $\mu$) and denote it by $C^q_\mu(E)$. There is another equally natural way to define $q$-box dimensions. For $q \in \mathbf{R}$ and $\delta > 0$ write

$$T^q_{\mu,\delta}(E) := \sup\left\{\sum_i \mu(B(x_i,\delta))^q : (B(x_i,\delta))_i \text{ is a centered covering of } E\right\} \tag{3.3}$$

and set

$$\overline{L}^q_\mu(E) := \limsup_{\delta\downarrow 0} \frac{\log T^q_{\mu,\delta}(E)}{-\log\delta}, \quad \underline{L}^q_\mu(E) := \liminf_{\delta\downarrow 0} \frac{\log T^q_{\mu,\delta}(E)}{-\log\delta}. \tag{3.4}$$

We will now define the generalization multifractal box dimensions in a probability space.

Let $\nu$ be a probability measure on $(\Omega, \mathscr{F})$. For $A \subset \Omega$ and $\delta > 0$ write

$$S^q_{\mu,\nu,\delta}(A) := \sup\left\{\sum_i \nu(C_i)^q : C_i \cap C_j = \emptyset,\ i \neq j,\ \mu(C_i) < \delta,\ C_i = u_n(\omega),\ \omega \in A\right\}. \tag{3.5}$$

The upper respectively lower multifractal $q$-box dimension $\overline{C}^q_{\mu,\nu}(A)$ and $\underline{C}^q_{\mu,\nu}(A)$ of $A$ (the measure $\nu$ respect to the measure $\mu$) in a probability space is defined by

$$\overline{C}^q_{\mu,\nu}(A) := \limsup_{\delta\downarrow 0} \frac{\log S^q_{\mu,\nu,\delta}(A)}{-\log\delta}, \quad \underline{C}^q_{\mu,\nu}(A) := \liminf_{\delta\downarrow 0} \frac{\log S^q_{\mu,\nu,\delta}(A)}{-\log\delta}. \tag{3.6}$$

The number $\overline{C}^q_{\mu,\nu}(A)$ is an obvious multifractal analogue of the upper $q$-box dimension $\overline{C}^q_{\mu}(A)$ of $A$ as given in [1], whereas $\underline{C}^q_{\mu,\nu}(A)$ is an obvious multifractal analogues of the lower $q$-box dimension $\underline{C}^q_{\mu}(A)$ of $A$ in a probability space. Also observe that

$$\overline{C}^0_{\mu,\nu}(A) = \overline{C}_{\mu}(A), \quad \underline{C}^0_{\mu,\nu}(A) = \underline{C}_{\mu}(A).$$

If $\overline{C}^q_{\mu,\nu}(A) = \underline{C}^q_{\mu,\nu}(A)$ we refer to the common value as the $q$-box dimension of $A$ (the measure $\nu$ with respect to the measure $\mu$) and denote it by $C^q_{\mu,\nu}(A)$.

There is another equally natural way to define $q$-box dimensions. For $q \in \mathbf{R}$ and $\delta > 0$ write

$$T^q_{\mu,\nu,\delta}(A) := \inf\left\{\sum_i \nu(C_i)^q : A \subset \cup_i C_i,\ \mu(C_i) < \delta,\ C_i = u_n(\omega),\ \omega \in A\right\} \tag{3.7}$$

and set

$$\overline{L}^q_{\mu,\nu}(A) := \limsup_{\delta\downarrow 0} \frac{\log T^q_{\mu,\nu,\delta}(A)}{-\log\delta}, \quad \underline{L}^q_{\mu,\nu}(A) := \liminf_{\delta\downarrow 0} \frac{\log T^q_{\mu,\nu,\delta}(A)}{-\log\delta}. \tag{3.8}$$

The next results summarize the important inequalities between $\overline{C}^q_{\mu,\nu}$, $\underline{C}^q_{\mu,\nu}$, $\overline{L}^q_{\mu,\nu}$, $\underline{L}^q_{\mu,\nu}$ and $\dim^q_{\mu,\nu}$, $\Delta^q_{\mu,\nu}$.

**Proposition 3.1.** *Let $\nu$ be a probability measure on $(\Omega, \mathscr{F})$ and $A \subset \Omega$, then*

(i) $\underline{L}^q_{\mu,\nu} \leqslant \underline{C}^q_{\mu,\nu}$ *for* $q \in \mathbf{R}$;
(ii) $\overline{L}^q_{\mu,\nu} \leqslant \overline{C}^q_{\mu,\nu}$ *for* $q \in \mathbf{R}$.

**Proof.** (i) Let $\{C_i\}_i \subset \mathscr{C}$ be a centered $\delta$-covering of $A$. It follows from the properties of $\mathscr{C}$ that we can suppose $C_i \cap C_j = \emptyset$ for $i \neq j$, hence $\{C_i\}_i$ is a $\delta$-packing of $A$, then by (3.5) and (3.7) we have

$$T^q_{\mu,\nu,\delta}(A) \leqslant \sum_i \nu(C_i)^q \leqslant S^q_{\mu,\nu,\delta}(A).$$

Taking logarithms and letting $\delta \downarrow 0$ yields $\underline{L}^q_{\mu,\nu} \leqslant \underline{C}^q_{\mu,\nu}$ by (3.6) and (3.8).

(ii) The proof of (ii) is similar to that of (i). $\square$

**Proposition 3.2.** $\overline{C}^q_{\mu,\nu} = \Delta^q_{\mu,\nu}$ *for* $q \in \mathbf{R}$ *and* $\nu \in \mathscr{P}(\Omega)$.

**Proof.** Put $t := \Delta^q_{\mu,\nu}(A)$. Let $A \subset \Omega$ and $\varepsilon > 0$. We may choose $0 < \delta_\varepsilon < 1$ such that

$$\widetilde{\mathscr{P}}^{q,t+\varepsilon}_{\mu,\nu,\delta}(A) < 1, \quad \text{for } 0 < \delta < \delta_\varepsilon.$$

Fix $0 < \delta < \delta_\varepsilon$ and let $\{C_i\}_i \subset \mathscr{C}, \frac{\delta}{2} < \mu(C_i) < \delta$ be a centered packing of $A$.

For $q < 1$, we have $\Delta^q_{\mu,\nu} > 0$, then $t + \varepsilon > 0$, hence

$$\sum_i \nu(C_i)^q = \delta^{-(t+\varepsilon)} \sum_i \nu(C_i)^q \delta^{t+\varepsilon} \leqslant \left(\frac{\delta}{2}\right)^{-(t+\varepsilon)} \sum_i \nu(C_i)^q \mu(C_i)^{t+\varepsilon} \leqslant \left(\frac{\delta}{2}\right)^{-(t+\varepsilon)} \widetilde{\mathscr{P}}^{q,t+\varepsilon}_{\mu,\nu,\delta}(A) \leqslant \left(\frac{\delta}{2}\right)^{-(t+\varepsilon)},$$

whence $S^q_{\mu,\nu,\delta}(A) \leqslant (\frac{\delta}{2})^{-(t+\varepsilon)}$. Taking logarithms then yields

$$\frac{\log S^q_{\mu,\nu,\delta}(A)}{-\log\delta} \leqslant (t+\varepsilon) - \frac{(t+\varepsilon)\log 2}{\log\delta}. \tag{3.9}$$

For $q \geqslant 1$, we have $\Delta^q_{\mu,\nu} < 0$, then $t + \varepsilon < 0$, hence

$$\sum_i \nu(C_i)^q = \delta^{-(t+\varepsilon)} \sum_i \nu(C_i)^q \delta^{t+\varepsilon} \leqslant \delta^{-(t+\varepsilon)} \sum_i \nu(C_i)^q \mu(C_i)^{t+\varepsilon} \leqslant \delta^{-(t+\varepsilon)} \widetilde{\mathscr{P}}^{q,t+\varepsilon}_{\mu,\nu,\delta}(A) \leqslant \delta^{-(t+\varepsilon)},$$

whence $S^q_{\mu,\nu,\delta}(A) \leqslant \delta^{-(t+\varepsilon)}$. Taking logarithms then yields

$$\frac{\log S^q_{\mu,\nu,\delta}(A)}{-\log\delta} \leqslant t+\varepsilon. \tag{3.10}$$

Letting $\delta \downarrow 0$ for (3.9) and (3) now yields $\overline{C}^q_{\mu,\nu}(A) \leqslant t+\varepsilon$ by (3.6), which completes the proof of $\overline{C}^q_{\mu,\nu}(A) \leqslant \Delta^q_{\mu,\nu}(A)$ since $\varepsilon > 0$ was arbitrary.

On the other hand, if $s > \overline{C}^q_{\mu,\nu}(A) > 0$ then $\forall\varepsilon > 0$, $s+\varepsilon > \limsup_{\delta\downarrow 0}\frac{\log S^q_{\mu,\nu,\delta}(A)}{-\log\delta}$, hence $S^q_{\mu,\nu,\delta}(A) < \delta^{-(s+\varepsilon)}$. Let $\{C_i\}_i$ be a $\delta$-packing of $A$. Thus

$$\sum_i \nu((C_i^q) < \delta^{-(s+\varepsilon)},$$

which implies that

$$\sum_i \nu(C_i)^q\mu(C_i)^{s+\varepsilon} < \delta^{s+\varepsilon}\sum_i \nu(C_i)^q < \delta^{s+\varepsilon}\delta^{-(s+\varepsilon)} = 1,$$

we have by (2.2)

$$\widetilde{\mathscr{P}}^{q,s+\varepsilon}_{\mu,\nu,\delta}(A) \leqslant 1,$$

hence from (2.4)

$$\Delta^q_{\mu,\nu}(A) \leqslant s+\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, it follows that $\Delta^q_{\mu,\nu}(A) \leqslant \overline{C}^q_{\mu,\nu}(A)$. Hence

$$\overline{C}^q_{\mu,\nu} = \Delta^q_{\mu,\nu}. \qquad \square$$

**Proposition 3.3.** $\dim^q_{\mu,\nu}(A) \leqslant \underline{L}^q_{\mu,\nu}(A)$ *for* $q \in \mathbf{R}$ *and* $\nu \in \mathscr{P}(\Omega)$.

**Proof.** In order to prove $\dim^q_{\mu,\nu}(A) \leqslant \underline{L}^q_{\mu,\nu}(A)$, it is thus sufficient to prove $\forall t > \underline{L}^q_{\mu,\nu}(A)$, $t \geqslant \dim^q_{\mu,\nu}(A)$, There is no harm in assuming $\underline{L}^q_{\mu,\nu}(A) < +\infty$.

We now must prove that

$$\mathscr{H}^{q,t}_{\mu,\nu}(A) = \sup_{B\subset A}\widetilde{\mathscr{H}}^{q,t}_{\mu,nu}(B) < +\infty$$

for all $B \subset A$. Since $t > \underline{L}^q_{\mu,\nu}(A) = \liminf_{\delta\downarrow 0}\frac{\log T^q_{\mu,\nu,\delta}(A)}{-\log\delta}$ there exists a sequence $(\delta_n)_n$ such that $\delta_n \downarrow 0$, $\delta_n \in (0,1)$ and

$$t > \frac{\log T^q_{\mu,\nu,\delta_n}(A)}{-\log\delta_n} \quad \text{for } n \in \mathbf{N},$$

which implies

$$\delta_n^{-t} > T^q_{\mu,\nu,\delta_n}(A).$$

For $n \in \mathbf{N}$ then there exists a centered $\delta_n$-covering $\{C_i\}_{i\in\mathbf{N}} \subset \mathscr{C}$ of $A$ satisfying

$$\delta_n^{-t} > \sum_i \nu(C_i)^q.$$

Let $n \in \mathbf{N}$ and put $I = \{i | C_i \cap B \neq \emptyset\}$. For $i \in I$ choose $\omega_i \in C_i \cap B$, then $\{u_n(\omega_i)\}_i$ is a centered $\delta_n$-covering of $B$, whence from (2.1)

$$\widetilde{\mathscr{H}}^{q,t}_{\mu,\nu,\delta_n}(B) \leqslant \sum_i \nu(u_n(\omega))^q\mu(u_n(\omega))^t \leqslant \delta_n^t\sum_i \nu(u_n(\omega))^q \leqslant \delta_n^t\sum_i \nu(C_i)^q \leqslant \delta_n^t\delta_n^{-t} = 1 < +\infty.$$

Letting $n \to \infty$ gives $\widetilde{\mathscr{H}}^{q,t}_{\mu,\nu}(B) \leqslant 1$ for all $B \subset A$. Whence $\widetilde{\mathscr{H}}^{q,t}_{\mu,\nu}(A) \leqslant 1 < +\infty$ and the proof is complete. $\square$

It follows immediately from Propositions 3.1–3.3 the following Theorem 3.4 hold.

**Theorem 3.4.** $\dim^q_{\mu,\nu} \leqslant \underline{L}^q_{\mu,\nu} = \underline{C}^q_{\mu,\nu} \leqslant \overline{L}^q_{\mu,\nu} = \overline{C}^q_{\mu,\nu} = \Delta^q_{\mu,\nu}$.

## 4. Multifractal dimension inequalities in a probability space

Let $\omega \in \Omega$, define the upper, lower local dimension of $\nu$ at a point $\omega$ respectively by

$$\overline{\alpha}_{\mu,\nu}(\omega) := \limsup_{n\to\infty} \frac{\log \nu(u_n(\omega))}{\log \mu(u_n(\omega))}, \quad \underline{\alpha}_{\mu,\nu}(\omega) := \liminf_{n\to\infty} \frac{\log \nu(u_n(\omega))}{\log \mu(u_n(\omega))}. \tag{4.1}$$

If $\overline{\alpha}_{\mu,\nu}(\omega)$ and $\underline{\alpha}_{\mu,\nu}(\omega)$ agree we refer to the common value as the local dimension of $\nu$ with respect to $\mu$ at $\omega$ and denote it by $\alpha_{\mu,\nu}(\omega)$. Put

$$M^q_{\mu,\nu,\delta} := S^q_{\mu,\nu,\delta}(\operatorname{supp}\mu \cap \operatorname{supp}\nu), \tag{4.2}$$

the lower and upper multifractal $q$-box dimensions $\underline{\tau}_{\mu,\nu}(q)$ and $\overline{\tau}_{\mu,\nu}(q)$ of $\nu$ with respect to $\mu$ are defined by

$$\underline{\tau}_{\mu,\nu}(q) := \liminf_{\delta\to 0} \frac{\log M^q_{\mu,\nu,\delta}}{-\log\delta}, \quad \overline{\tau}_{\mu,\nu}(q) := \limsup_{\delta\to 0} \frac{\log M^q_{\mu,\nu,\delta}}{-\log\delta}. \tag{4.3}$$

Recall that if $f : \mathbf{R} \to \mathbf{R}$ is a convex function and $x \in \mathbf{R}$, then we denote the left derivative and the right derivative of $f$ at $x$ by $D_-f(x)$ and $D_+f(x)$ respectively. We can now state our main results.

**Proposition 4.1.** *Let $\nu \in \mathscr{P}(\Omega)$ and $q, t \in \mathbf{R}$.*

(i) $\widetilde{\mathscr{H}}^{q,t}_{\mu,\nu} \leqslant \mathscr{H}^{q,t}_{\mu,\nu} \leqslant \mathscr{P}^{q,t}_{\mu,\nu} \leqslant \widetilde{\mathscr{P}}^{q,t}_{\mu,\nu}$;
(ii) $\dim^q_{\mu,\nu} \leqslant \mathsf{D}^q_{\mu,\nu} \leqslant \Delta^q_{\mu,\nu}$, *in particular* $b_{\mu,\nu} \leqslant B_{\mu,\nu} \leqslant \Lambda_{\mu,\nu}$. *Also* $\Lambda_{\mu,\nu} = \overline{\tau}_{\mu,\nu}$;
(iii) $b_{\mu,\nu}$ *is decreasing, and* $B_{\mu,\nu}$ *and* $\overline{\tau}_{\mu,\nu}$ *are convex and decreasing. Also* $b_{\mu,\nu}(1) = B_{\mu,\nu}(1) = \overline{\tau}_{\mu,\nu}(1) = 0$;
(iv) $D_-\overline{\tau}_{\mu,\nu}(1) \leqslant D_-B_{\mu,\nu}(1) \leqslant D_+B_{\mu,\nu}(1) \leqslant D_+\overline{\tau}_{\mu,\nu}(1)$.

**Proof.** Statements (i), (ii) follow from [3].

Proof of (iii): It follows from $\mathscr{H}^{q,t}_{\mu,\nu} \geqslant \mathscr{H}^{p,t}_{\mu,\nu}$ for $q \leqslant p$ and $\mathscr{H}^{q,t}_{\mu,\nu} \geqslant \mathscr{H}^{q,s}_{\mu,\nu}$ for $t \leqslant s$ that $b_{\mu,\nu}$ is decreasing. The proofs of $B_{\mu,\nu}$ and $\overline{\tau}_{\mu,\nu}$ decreasing are similar to the proof of $b_{\mu,\nu}$ decreasing.

We now prove that $B_{\mu,\nu}$ is convex as follows. Firstly, we may prove that $\widetilde{\mathscr{P}}^{q,t}_{\mu,\nu}$ is logarithmic convex, that is

$$\widetilde{\mathscr{P}}^{\alpha p+(1-\alpha)q,\alpha t+(1-\alpha)s}_{\mu,\nu}(A) \leqslant (\widetilde{\mathscr{P}}^{p,t}_{\mu,\nu}(A))^{\alpha} (\widetilde{\mathscr{P}}^{q,s}_{\mu,\nu}(A))^{1-\alpha} \tag{4.4}$$

for all $\alpha \in [0,1]$, $p, q, t, s \in \mathbf{R}$ and $A \subset \Omega$. In fact, let $\varepsilon$, $\delta > 0$, for all centered $\delta$-packing $\{C_i = u_n(\omega)\}_{i\in N}$ of $A$, we have

$$\begin{aligned}
\sum_i \nu(C_i)^{\alpha p+(1-\alpha)q} \mu(C_i)^{\alpha t+(1-\alpha)s} &= \sum_i ((\nu(C_i))^p(\mu(C_i))^t)^{\alpha}((\nu(C_i))^q(\mu(C_i))^s)^{1-\alpha} \text{ (by Hölder inequality)} \\
&\leqslant \left\{\sum_i (\nu(C_i))^p(\mu(C_i))^t\right\}^{\alpha} \left\{\sum_i (\nu(C_i))^q(\mu(C_i))^s\right\}^{1-\alpha} \quad \text{(by (2.2))} \\
&\leqslant (\widetilde{\mathscr{P}}^{p,t}_{\mu,\nu,\delta}(A))^{\alpha} (\widetilde{\mathscr{P}}^{q,s}_{\mu,\nu,\delta}(A))^{1-\alpha},
\end{aligned}$$

hence

$$\widetilde{\mathscr{P}}^{\alpha p+(1-\alpha)q,\alpha t+(1-\alpha)s}_{\mu,\nu}(A) \leqslant \widetilde{\mathscr{P}}^{\alpha p+(1-\alpha)q,\alpha t+(1-\alpha)s}_{\mu,\nu,\delta}(A) \leqslant (\widetilde{\mathscr{P}}^{p,t}_{\mu,\nu,\delta}(A))^{\alpha} (\widetilde{\mathscr{P}}^{q,s}_{\mu,\nu,\delta}(A))^{1-\alpha}, \quad \text{for all } \delta > 0.$$

Letting $\delta \to 0$ gives (4.4).

Let $p, q \in \mathbf{R}$, $\alpha \in [0,1]$ and $\varepsilon > 0$. Write $B_{\mu,\nu,A}(p) = t$ and $B_{\mu,\nu,A}(q) = s$. Clearly

$$\mathscr{P}^{q,s+\varepsilon}_{\mu,\nu}(A) = 0 = \mathscr{P}^{p,t+\varepsilon}_{\mu,\nu}(A).$$

We can thus choose coverings $(H_i)_{i\in\mathbf{N}}$ and $(K_i)_{i\in\mathbf{N}}$ of $A$ such that

$$\sum_i \widetilde{\mathscr{P}}^{p,t+\varepsilon}_{\mu,\nu}(H_i) \leqslant 1, \quad \sum_i \widetilde{\mathscr{P}}^{q,s+\varepsilon}_{\mu,\nu}(K_i) \leqslant 1.$$

For $n \in \mathbf{N}$, write $A_n = \bigcup_{i,j=1}^{n}(H_i \cap K_j)$. Fix $n \in \mathbf{N}$, we have

$$\begin{aligned}
\mathscr{P}_{\mu,\nu}^{\alpha p+(1-\alpha)q,\alpha t+(1-\alpha)s+\varepsilon}(A_n) &= \mathscr{P}_{\mu,\nu}^{\alpha p+(1-\alpha)q,\alpha(t+\varepsilon)+(1-\alpha)(s+\varepsilon)}\left(\bigcup_{i,j=1}^{n}(H_i \cap K_j)\right) \\
&\leqslant \sum\nolimits_{i,j=1}^{n} \mathscr{P}_{\mu,\nu}^{\alpha p+(1-\alpha)q,\alpha(t+\varepsilon)+(1-\alpha)(s+\varepsilon)}(H_i \cap K_j) \\
&\leqslant \sum_{i,j=1}^{n} \widetilde{\mathscr{P}}_{\mu,\nu}^{\alpha p+(1-\alpha)q,\alpha(t+\varepsilon)+(1-\alpha)(s+\varepsilon)}(H_i \cap K_j) \text{ (by (4.4))} \\
&\leqslant \sum_{i,j=1}^{n} \left(\widetilde{\mathscr{P}}_{\mu,\nu}^{p,t+\varepsilon}(H_i \cap K_j)\right)^{\alpha} \left(\widetilde{\mathscr{P}}_{\mu,\nu}^{q,s+\varepsilon}(H_i \cap K_j)\right)^{(1-\alpha)} \text{ (by Hölder inequality)} \\
&\leqslant \left(\sum_{i,j=1}^{n} \left(\widetilde{\mathscr{P}}_{\mu,\nu}^{p,t+\varepsilon}(H_i \cap K_j)\right)^{\alpha}\right) \left(\sum_{i,j=1}^{n} \left(\widetilde{\mathscr{P}}_{\mu,\nu}^{q,s+\varepsilon}(H_i \cap K_j)\right)^{(1-\alpha)}\right) \\
&\leqslant \left(\sum_{i,j=1}^{n} \left(\widetilde{\mathscr{P}}_{\mu,\nu}^{p,t+\varepsilon}(H_i)\right)^{\alpha}\right) \left(\sum_{i,j=1}^{n} \left(\widetilde{\mathscr{P}}_{\mu,\nu}^{q,s+\varepsilon}(K_j)\right)^{(1-\alpha)}\right) \\
&= \left(n\sum_{i=1}^{n} \left(\widetilde{\mathscr{P}}_{\mu,\nu}^{p,t+\varepsilon}(H_i)\right)^{\alpha}\right) \left(n\sum_{j=1}^{n} \left(\widetilde{\mathscr{P}}_{\mu,\nu}^{q,s+\varepsilon}(K_j)\right)^{(1-\alpha)}\right) \leqslant n^{\alpha} n^{1-\alpha} = n < +\infty,
\end{aligned}$$

hence

$$\mathbf{Dim}_{\mu,\nu}^{\alpha p+(1-\alpha)q}(A_n) \leqslant \alpha t + (1-\alpha)s + \varepsilon, \quad \text{for all } n \in \mathbf{N}.$$

Since $A \subseteq \cup_n A_n$, this implies that

$$\begin{aligned}
B_{\mu,\nu,A}(\alpha p + (1-\alpha)q) &= \mathbf{Dim}_{\mu,\nu}^{\alpha p+(1-\alpha)q}(A) \leqslant \mathbf{Dim}_{\mu,\nu}^{\alpha p+(1-\alpha)q}(\cup_n A_n) = \sup_n \mathbf{Dim}_{\mu,\nu}^{\alpha p+(1-\alpha)q}(A_n) \\
&\leqslant \alpha B_{\mu,\nu,A}(p) + (1-\alpha)B_{\mu,\nu,A}(q) + \varepsilon.
\end{aligned}$$

Since this is true for any $\varepsilon > 0$, convexity of $B_{\mu,\nu}$ is proved. □

It follows from logarithmic convexity $\widetilde{\mathscr{P}}_{\mu,\nu}^{q,t}$ that convexity of $\Delta_{\mu,\nu}$. Applying Proposition 3.2, we can have convexity of $\overline{\tau}_{\mu,\nu}$.

(iv) follows immediately from (ii) and (iii).

**Proposition 4.2.**

(i) *If $b_{\mu,\nu}$ is convex, then we have for* $\mathscr{H}_{\mu,\nu}^{q,b_{\mu,\nu}(q)}|\operatorname{supp}\mu \cap \operatorname{supp}\nu -$ a.e. $\omega$,
$$-D_+b_{\mu,\nu}(q) \leqslant \overline{\alpha}_{\mu,\nu}(\omega), \quad \underline{\alpha}_{\mu,\nu}(\omega) \leqslant -D_-b_{\mu,\nu}(q);$$

(ii) *We have for* $\mathscr{P}_{\mu,\nu}^{q,B_{\mu,\nu}(q)}|\operatorname{supp}\mu \cap \operatorname{supp}\nu -$ a.e. $\omega$,
$$-D_+B_{\mu,\nu}(q) \leqslant \overline{\alpha}_{\mu,\nu}(\omega), \quad \underline{\alpha}_{\mu,\nu}(\omega) \leqslant -D_-B_{\mu,\nu}(q).$$

**Proof.** (i) Proof of $-D_+b_{\mu,\nu}(q) \leqslant \overline{\alpha}_{\mu,\nu}(\omega)$ for $\mathscr{H}_{\mu,\nu}^{q,b_{\mu,\nu}(q)}|\operatorname{supp}\mu \cap \operatorname{supp}\nu -$ a.e. $\omega$ : write $a = D_+b_{\mu,\nu}(q)$ and let

$$F := \{\omega \in \operatorname{supp}\mu \cap \operatorname{supp}\nu : -a > \overline{\alpha}_{\mu,\nu}(\omega)\}.$$

To prove $-a \leqslant \overline{\alpha}_{\mu,\nu}(\omega)$ for $\mathscr{H}_{\mu,\nu}^{q,b_{\mu,\nu}(q)}|\operatorname{supp}\mu \cap \operatorname{supp}\nu -$ a.e. $\omega$, it suffices to prove that

$$\mathscr{H}_{\mu,\nu}^{q,b_{\mu,\nu}(q)}(F) = 0.$$

Fix $\varepsilon > 0$, put

$$E := \left\{\omega \in \operatorname{supp}\mu \cap \operatorname{supp}\nu : \liminf_{n\to\infty} \frac{\nu(u_n(\omega))}{\mu(u_n(\omega))^{-a-\varepsilon}} > 1\right\}.$$

It clearly suffices to prove that $\mathscr{H}_{\mu,\nu}^{q,b_{\mu,\nu}(q)}(E) = 0$ since $F \subset E$. In fact, $\forall \omega \in F$, from (4.1) we have

$$-a > \limsup_{n\to\infty} \frac{\log \nu(u_n(\omega))}{\log \mu(u_n(\omega))},$$

hence $\exists\varepsilon > 0$ such that

$$-a-\varepsilon > \limsup_{n\to\infty}\frac{\log\nu(u_n(\omega))}{\log\mu(u_n(\omega))} \Rightarrow (-a-\varepsilon)\log\mu(u_n(\omega)) < \log\nu(u_n(\omega) \Rightarrow \mu(u_n(\omega))^{-a-\varepsilon} < \nu(u_n(\omega))$$
$$\Rightarrow \frac{\nu(u_n(\omega))}{\mu(u_n(\omega))^{-a-\varepsilon}} > 1,$$

whence $\omega \in E$.

By the definition of right derivative $a = D_+B_{\mu,\nu}(q)$, there exists $h > 0$ such that $\frac{b_{\mu,\nu}(q+h)-b_{\mu,\nu}(q)}{h} < a+\varepsilon$, whence $b_{\mu,\nu}(q+h) < b_{\mu,\nu}(q) + h(a+\varepsilon)$, and we therefore deduce that

$$\mathscr{H}_{\mu,\nu}^{q+h,b_{\mu,\nu}(q)+h(a+\varepsilon)}(\operatorname{supp}\mu\cap\operatorname{supp}\nu) = 0. \tag{4.5}$$

Also observe by the equivalence theorem of limit inferior that for each $\omega \in E$ there exists a positive number $N$, such that for $n > N$

$$\frac{\nu(u_n(\omega))}{\mu(u_n(\omega))^{-a-\varepsilon}} > 1 \Rightarrow \nu(u_n(\omega))^{-h} < \mu(u_n(\omega))^{h(a+\varepsilon)} \Rightarrow \nu(u_n(\omega))^{h}\mu(u_n(\omega))^{h(a+\varepsilon)} > 1$$

hence

$$\nu(u_n(\omega))^{q}\mu(u_n(\omega))^{b_{\mu,\nu}(q)} \leqslant \nu(u_n(\omega))^{q+h}\mu(u_n(\omega))^{b_{\mu,\nu}(q)+h(a+\varepsilon)}. \tag{4.6}$$

It now follows easily from Eqs. (4.5) and (4.6) that:

$$\mathscr{H}_{\mu,\nu}^{q,b_{\mu,\nu}(q)}(E) \leqslant \mathscr{H}_{\mu,\nu}^{q+h,b_{\mu,\nu}(q)+h(a+\varepsilon)}(E) = 0.$$

Proof of $\underline{\alpha}_{\mu,\nu}(\omega) \leqslant - D_-b_{\mu,\nu}(q)$ for $\mathscr{H}_{\mu,\nu}^{q,b_{\mu,\nu}(q)}|\operatorname{supp}\mu\cap\operatorname{supp}\nu -$ a.e. $\omega$: write $a = D_-b_{\mu,\nu}(q)$ and let

$$F := \{\omega \in \operatorname{supp}\mu\cap\operatorname{supp}\nu : \underline{\alpha}_{\mu,\nu}(\omega) > -a\}.$$

To prove $\underline{\alpha}_{\mu,\nu}(\omega) \leqslant - D_-b_{\mu,\nu}(q)$ for $\mathscr{H}_{\mu,\nu}^{q,b_{\mu,\nu}(q)}|\operatorname{supp}\mu\cap\operatorname{supp}\nu -$ a.e. $\omega$, it suffices to prove that

$$\mathscr{H}_{\mu,\nu}^{q,b_{\mu,\nu}(q)}(F) = 0.$$

Fix $\varepsilon > 0$, put

$$E := \left\{\omega \in \operatorname{supp}\mu\cap\operatorname{supp}\nu : \limsup_{n\to\infty}\frac{\nu(u_n(\omega))}{\mu(u_n(\omega))^{-a+\varepsilon}} < 1\right\}.$$

It clearly suffices to prove that $\mathscr{H}_{\mu,\nu}^{q,b_{\mu,\nu}(q)}(E) = 0$ since $F \subset E$. In fact, $\forall\omega \in F$, from (4.1) we have

$$-a < \liminf_{n\to\infty}\frac{\log\nu(u_n(\omega))}{\log\mu(u_n(\omega))},$$

hence $\exists\varepsilon > 0$ such that

$$-a+\varepsilon < \liminf_{n\to\infty}\frac{\log\nu(u_n(\omega))}{\log\mu(u_n(\omega))} \Rightarrow (-a+\varepsilon)\log\mu(u_n(\omega)) > \log\nu(u_n(\omega) \Rightarrow \mu(u_n(\omega))^{-a-\varepsilon} > \nu(u_n(\omega))$$
$$\Rightarrow \frac{\nu(u_n(\omega))}{\mu(u_n(\omega))^{-a-\varepsilon}} < 1,$$

whence $\omega \in E$.

By the equivalent propositions of convexity of $b_{\mu,\nu}$, there exists $h > 0$ such that $\frac{b_{\mu,\nu}(q)-b_{\mu,\nu}(q-h)}{h} > a-\varepsilon$, whence $b_{\mu,\nu}(q-h) < b_{\mu,\nu}(q) - h(a-\varepsilon)$, and we therefore deduce that

$$\mathscr{H}_{\mu,\nu}^{q-h,b_{\mu,\nu}(q)-h(a-\varepsilon)}(\operatorname{supp}\mu\cap\operatorname{supp}\nu) = 0. \tag{4.7}$$

Next observe by the equivalence theorem of limit superior, that for each $\omega \in E$ there exist a positive number $N$ such that

$$\nu(u_n(\omega))^{q}\mu(u_n(\omega))^{b_{\mu,\nu}(q)} \leqslant \nu(u_n(\omega))^{q-h}\mu(u_n(\omega))^{b_{\mu,\nu}(q)-h(a-\varepsilon)} \quad \text{for } n > N. \tag{4.8}$$

It now follows easily from (4.7) and (4.8) that:

$$\mathscr{H}_{\mu,\nu}^{q,b_{\mu,\nu}(q)}(E) \leqslant \mathscr{H}_{\mu,\nu}^{q-h,b_{\mu,\nu}(q)-h(a-\varepsilon)}(E) = 0.$$

(ii) The proof of (ii) is similar to the proof of (i). $\square$

**Lemma 4.3.** *Let $\nu \in \mathscr{P}(\Omega)$ and $q \in \mathbf{R}$. For $\mathscr{H}_{\mu,\nu}^{q,B_{\mu,\nu}^q}|\mathrm{supp}\,\mu \cap \mathrm{supp}\,\nu$ – a.e. $\omega$, we have*

$$-D_+B_{\mu,\nu}(q) \leqslant \underline{\alpha}_{\mu,\nu}(\omega) \leqslant \overline{\alpha}_{\mu,\nu}(\omega) \leqslant -D_-B_{\mu,\nu}(q).$$

**Proof.** Proof of $-D_+B_{\mu,\nu}(q) \leqslant \underline{\alpha}_{\mu,\nu}(\omega)$ for $\mathscr{H}_{\mu,\nu}^{q,B_{\mu,\nu}^q}|\mathrm{supp}\,\mu \cap \mathrm{supp}\,\nu$ – a.e. $\omega$: write $A = D_+B_{\mu,\nu}(q)$. Fix $\varepsilon$, $\eta > 0$ and let

$$E := \left\{\omega \in \mathrm{supp}\,\mu \cap \mathrm{supp}\,\nu : \liminf_{n\to\infty} \frac{\nu(u_n(\omega))}{\mu(u_n(\omega))^{-A-\varepsilon}} > \eta\right\}.$$

It clearly suffices to prove that $\mathscr{H}_{\mu,\nu}^{q,B_{\mu,\nu}(q)}(E) = 0$.

By the definition of right derivative $A = D_+B_{\mu,\nu}(q)$, there exists $h > 0$ such that

$$\frac{B_{\mu,\nu}(q+h) - B_{\mu,\nu}(q)}{h} < A + \varepsilon,$$

whence $B_{\mu,\nu}(q+h) < B_{\mu,\nu}(q) + h(A+\varepsilon)$, and we therefore deduce that

$$\mathscr{P}_{\mu,\nu}^{q+h,B_{\mu,\nu}(q)+h(A+\varepsilon)}(\mathrm{supp}\,\mu \cap \mathrm{supp}\,\nu) = 0.$$

Also observe by the equivalence theorem of limit inferior, that for each $\omega \in E$ there exists a positive number $N$ such that

$$\nu(u_n(\omega))^q \mu(u_n(\omega))^{B_{\mu,\nu}(q)} \leqslant \eta^{-h} \nu(u_n(\omega))^{q+h} \mu(u_n(\omega))^{B_{\mu,\nu}(q)+h(A+\varepsilon)} \quad \text{for } n > N,$$

and we therefore deduce that

$$\mathscr{H}_{\mu,\nu}^{q,B_{\mu,\nu}(q)}(E) \leqslant \eta^{-h} \mathscr{H}_{\mu,\nu}^{q+h,B_{\mu,\nu}(q)+h(A+\varepsilon)}(E).$$

It now follows easily from

$$\mathscr{H}_{\mu,\nu}^{q+h,B_{\mu,\nu}(q)+h(A+\varepsilon)} \leqslant \mathscr{P}_{\mu,\nu}^{q+h,B_{\mu,\nu}(q)+h(A+\varepsilon)}$$

that

$$\mathscr{H}_{\mu,\nu}^{q,B_{\mu,\nu}(q)}(E) \leqslant \eta^{-h} \mathscr{P}_{\mu,\nu}^{q+h,B_{\mu,\nu}(q)+h(A+\varepsilon)}(E) = 0.$$

Proof of $\overline{\alpha}_{\mu,\nu}(\omega) \leqslant -D_-B_{\mu,\nu}(q)$ for $\mathscr{H}_{\mu,\nu}^{q,B_{\mu,\nu}(q)}|\mathrm{supp}\,\mu \cap \mathrm{supp}\,\nu$ – a.e. $\omega$ : write $A = D_-B_{\mu,\nu}(q)$. Fix $\varepsilon$, $k > 0$ and let

$$E := \left\{\omega \in \mathrm{supp}\,\mu \cap \mathrm{supp}\,\nu : \limsup_{n\to\infty} \frac{\nu(u_n(\omega))}{\mu(u_n(\omega))^{-A+\varepsilon}} < k\right\}.$$

It clearly suffices to prove that $\mathscr{H}_{\mu,\nu}^{q,B_{\mu,\nu}(q)}(E) = 0$.

By the equivalent propositions of convexity of $B_{\mu,\upsilon}$ (by Proposition 4.1), there exists $h > 0$ such that $\frac{B_{\mu,\nu}(q)-B_{\mu,\nu}(q-h)}{h} > A - \varepsilon$, whence $B_{\mu,\nu}(q-h) < B_{\mu,\nu}(q) - h(A-\varepsilon)$, and we therefore deduce that

$$\mathscr{P}_{\mu,\nu}^{q-h,B_{\mu,\nu}(q)-h(A-\varepsilon)}(\mathrm{supp}\,\mu \cap \mathrm{supp}\,\nu) = 0. \tag{4.9}$$

By the equivalence theorem of limit superior for each $\omega \in E$, there exists a positive number $N$ such that

$$\nu(u_n(\omega))^q \mu(u_n(\omega))^{B_{\mu,\nu}(q)} \leqslant k^h \nu(u_n(\omega))^{q-h} \mu(u_n(\omega))^{B_{\mu,\nu}(q)-h(A-\varepsilon)} \quad \text{for } n > N,$$

and we therefore deduce that

$$\mathscr{H}_{\mu,\nu}^{q,B_{\mu,\nu}(q)}(E) \leqslant \eta^{-h} \mathscr{H}_{\mu,\nu}^{q+h,B_{\mu,\nu}(q)+h(A+\varepsilon)}(E).$$

It now follows, from Proposition 4.1 and Eq. (4.9), that

$$\mathscr{H}_{\mu,\nu}^{q,B_{\mu,\nu}(q)}(E) \leqslant k^h \mathscr{P}_{\mu,\nu}^{q-h,B_{\mu,\nu}(q)-h(A-\varepsilon)}(E) = 0. \qquad \square$$

**Proposition 4.4.** *Write $t = b_{\mu,\nu}(q)$ and $A_\pm = D_\pm B_{\mu,\nu}(q)$. Assume that $b_{\mu,\nu}(q) = B_{\mu,\nu}(q)$. For $\mathscr{H}_{\mu,\nu}^{q,t}|\mathrm{supp}\,\mu \cap \mathrm{supp}\,\nu$ – a.e. $\omega$ we have*

$$-A_+ \leqslant \underline{\alpha}_{\mu,\nu}(\omega) \leqslant \overline{\alpha}_{\mu,\nu}(\omega) \leqslant -A_-. \tag{4.10}$$

**Proof.** Inequality (4.10) follows from Lemma 4.3.

Since $b_{\mu,\nu}(1) = B_{\mu,\nu}(1) = 0$ (cf. Proposition 4.1) and $\mathscr{H}_{\mu,\nu}^{1,0}$ is equivalent $\nu$, Proposition 4.4 yields the following corollary by setting $q = 1$.

**Corollary 4.5.** *Let $\nu$ be a probability measure on $\Omega$.*

(i) *We have*

$$-D_+B_{\mu,\nu}(1) \leqslant \underline{\alpha}_{\mu,\nu}(\omega) \leqslant \overline{\alpha}_{\mu,\nu}(\omega) \leqslant -D_-B_{\mu,\nu}(1), \quad \text{for } \nu-\text{a.e. } \omega; \tag{4.11}$$

(ii) *Write $\tau = \overline{\tau}_{\mu,\nu}$, since $-D_+\tau\ (1) \leqslant -D_+B_{\mu,\nu}(1)$ and $-D_-B_{\mu,\gamma}(1) \leqslant -D_-\tau(1)$ (by Proposition 4.1), (4.11) implies that*

$$-D_+\tau(1) \leqslant \underline{\alpha}_{\mu,\nu}(\omega) \leqslant \overline{\alpha}_{\mu,\nu}(\omega) \leqslant -D_-\tau(1) \quad \text{for } \nu-\text{a.e. } \omega. \tag{4.12}$$

## 5. Conclusion

The present research is about the multifractal formalism. The conceptions of the multifractal spectrum and all kinds of multifractal dimensions have been developed by previous a few of authors which include mathematician and physicist. In this paper, we establish the conceptions for lower and upper multifractal box dimensions in a probability space, investigate the relation between the multifractal box dimension and multifractal Hausdorff dimension, multifractal packing dimension in a probability space. Furthermore, we explore the dimension inequalities of multifractal Hausdorff measures and multifractal packing measures in a probability space.

We note that some scholar such as El Naschie [7–17], Ord et al. [18,19], and Nottale [20] have achieved many valuable results on the same subject and application. Our paper is relevant to their work published in Chaos, Solitons and Fractals. It is particularly relevant in physics for relation between the dimensions of *E*-infinity theory and sphere packing problem researched in high energy physics [1–18]. Therefore researches concerning fractals and the multifractal formalism in a probability space is very significant.

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