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Basic bilateral very well-poised series and Shukla's ${}_8\psi_8$-summation formula

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Abstract

By means of Jackson's q-Dougall–Dixon formula on terminating very well-poised hypergeometric series, the linearization lemma on the factorial fractions with well-poised parameter-pairs will be established. It will be applied to prove the generalized well-poised ${}_8\psi_8$-series and ${}_{10}\psi_{10}$-series identities. The linearization method will further be employed to present a constructive proof of Milne's transformation formula from nonterminating very well-poised bilateral series to terminating multiple unilateral series.

Keywords: Basic hypergeometric series; Jackson's summation formula; Watson's transformation

1. Introduction and notation

For two complex x and q, define the shifted factorial by

$$(x;q)_0 = 1 \quad \text{and} \quad (x;q)_n = (1-x)(1-xq)\cdots\left(1-xq^{n-1}\right) \quad \text{for } n \in \mathbb{N}.$$

When $|q| < 1$, the shifted factorial of infinite order is well defined

$$(x;q)_\infty = \prod_{k=0}^{\infty}\left(1-xq^k\right) \quad \text{and} \quad (x;q)_n = \frac{(x;q)_\infty}{(xq^n;q)_\infty} \quad \text{for } n \in \mathbb{Z}.$$

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Its product and fraction forms are abbreviated compactly to

$$[a,b,\dots,c;q]_n=(a;q)_n(b;q)_n\cdots(c;q)_n, \tag{1a}$$

$$\left[\begin{matrix}a,\ b,\ \dots,\ c\\ \alpha,\ \beta,\ \dots,\ \gamma\end{matrix}\,\middle|\,q\right]_n=\frac{(a;q)_n(b;q)_n\cdots(c;q)_n}{(\alpha;q)_n(\beta;q)_n\cdots(\gamma;q)_n}. \tag{1b}$$

Following Bailey [3] and Slater [17], the unilateral and bilateral basic hypergeometric series are defined, respectively, by

$${}_{1+r}\phi_s\left[\begin{matrix}a_0,\ a_1,\ \dots,\ a_r\\ b_1,\ \dots,\ b_s\end{matrix}\,\middle|\,q;z\right]=\sum_{n=0}^{\infty}z^n\left[\begin{matrix}a_0,\ a_1,\ \dots,\ a_r\\ q,\ b_1,\ \dots,\ b_s\end{matrix}\,\middle|\,q\right]_n\left\{(-1)^nq^{\binom{n}{2}}\right\}^{s-r}, \tag{2a}$$

$${}_{r}\psi_s\left[\begin{matrix}a_1,\ a_2,\ \dots,\ a_r\\ b_1,\ b_2,\ \dots,\ b_s\end{matrix}\,\middle|\,q;z\right]=\sum_{n=-\infty}^{\infty}z^n\left[\begin{matrix}a_1,\ a_2,\ \dots,\ a_r\\ b_1,\ b_2,\ \dots,\ b_s\end{matrix}\,\middle|\,q\right]_n\left\{(-1)^nq^{\binom{n}{2}}\right\}^{s-r}. \tag{2b}$$

Obviously, the unilateral series may be considered as a special case of the corresponding bilateral one with one of the denominator parameters equal to q. Throughout the paper, the base q will be confined to $|q|<1$ for nonterminating q-series.

One of the deepest results in the theory of basic hypergeometric series is the very well-poised bilateral ${}_6\psi_6$-series identity discovered by Bailey [4] (see also [17, §7.1] and [10, II-33]):

$${}_6\psi_6\left[\begin{matrix}q\sqrt{a},\ -q\sqrt{a},\ b,\ c,\ d,\ e\\ \sqrt{a},\ -\sqrt{a},\ qa/b,\ qa/c,\ qa/d,\ qa/e\end{matrix}\,\middle|\,q;\frac{qa^2}{bcde}\right]=\Omega[a:b,c,d,e], \tag{3}$$

where $|qa^2/bcde|<1$ for convergence and $\Omega[a:b,c,d,e]$ stands for the factorial fraction

$$\Omega[a:b,c,d,e]=\left[\begin{matrix}q,qa,q/a,qa/bc,qa/bd,qa/be,qa/cd,qa/ce,qa/de\\ qa^2/bcde,qa/b,qa/c,qa/d,qa/e,q/b,q/c,q/d,q/e\end{matrix}\,\middle|\,q\right]_\infty. \tag{4}$$

There are several proofs of this important identity up to now. For the most elementary proof, refer to the forthcoming paper by Chu [8], where the reformulated Abel's lemma on summation by parts will be employed.

In 1950, M. Jackson [11, Eq. 3.1] established a transformation, which expresses a well-poised ${}_8\psi_8$-series in terms of three balanced ${}_4\phi_3$-series. By specializing Jackson's formula, Shukla [18, Eq. 4.1] obtained, in 1958, the following generalization of Bailey's identity (3):

$${}_8\psi_8\left[\begin{matrix}q\sqrt{a},\ -q\sqrt{a},\ b,\ c,\ d,\ e,\ qw,\ qa/w\\ \sqrt{a},\ -\sqrt{a},\ qa/b,\ qa/c,\ qa/d,\ qa/e,\ a/w,\ w\end{matrix}\,\middle|\,q;\frac{a^2}{bcde}\right] \tag{5a}$$

$$=\Omega[a:b,c,d,e]\left\{1-\frac{(1-bc/a)(1-bd/a)(1-be/a)}{(1-bw/a)(1-b/w)(1-bcde/a^2)}\right\}\frac{(1-w/b)(1-bw/a)}{(1-w)(1-w/a)} \tag{5b}$$

subject to the convergence condition $|a^2/bcde|<1$. The existence of this summation formula has been pointed out earlier by Slater and Lakin [15] even though they did not state it explicitly.

By means of Jackson's q-Dougall–Dixon formula on terminating very well-poised hypergeometric series, we shall first establish the linearization lemma on the factorial fractions with well-poised parameter-pairs. It will then be applied in the second and third sections respectively to prove the generalized well-poised ${}_8\psi_8$-series and ${}_{10}\psi_{10}$-series identities. More generally, the linearization method will be combined with the induction principle in the fourth section to present a constructive proof of Milne's beautiful transformation formula from nonterminating very well-poised bilateral series to terminating multiple unilateral series. Finally, in the fifth section, an alternative double series expression will be derived for the very well-poised ${}_{10}\psi_{10}$-series, which may also be considered as a common generalization of both bilateral series identities displayed in (3) and (5a)–(5b).

2. The well-poised ${}_8\psi_8$-series identity

First of all, we show that two pairs of well-poised parameters can be linearized by the following useful reduction formula.

Lemma 1 *(Linearization of well-poised pairs).*

$$\left[\begin{matrix} b, & d, & q^m w, qa/w \\ qa/b, & qa/d, & q^{1-m}a/w, w \end{matrix}\,\Big|\, q\right]_k$$

$$= \left[\begin{matrix} d, & d/a, & w/b, & bw/a \\ d/b, & bd/a, & w, & w/a \end{matrix}\,\Big|\, q\right]_m \tag{6a}$$

$$\times \sum_{\ell=0}^{m} \frac{1-q^{2\ell-m}b/d}{1-q^{-m}b/d} q^{\ell} \left[\begin{matrix} q^{\ell}b, & q^{m-\ell}d \\ q^{1-\ell}a/b, & q^{1+\ell-m}a/d \end{matrix}\,\Big|\, q\right]_k \tag{6b}$$

$$\times \left[\begin{matrix} q^{-m}b/d, b, b/a, w/d, q^{1-m}a/dw, q^{-m} \\ q, q^{1-m}/d, q^{1-m}a/d, q^{1-m}b/w, bw/a, qb/d \end{matrix}\,\Big|\, q\right]_{\ell}. \tag{6c}$$

Proof. By means of the relation on shifted factorial fractions

$$\left[\begin{matrix} q^{\ell}b, & q^{m-\ell}d \\ q^{1-\ell}a/b, & q^{1+\ell-m}a/d \end{matrix}\,\Big|\, q\right]_k \left[\begin{matrix} b, & b/a \\ q^{1-m}/d, & q^{1-m}a/d \end{matrix}\,\Big|\, q\right]_{\ell}$$

$$= \left[\begin{matrix} q^{\ell}b, & q^{1+\ell-m-k}/d \\ q^{\ell-k}b/a, & q^{1+\ell-m}a/d \end{matrix}\,\Big|\, q\right]_k \left[\begin{matrix} b, & b/a \\ q^{1-m}/d, & q^{1-m}a/d \end{matrix}\,\Big|\, q\right]_{\ell} \left(q^{m-1}bd/a\right)^k$$

$$= \left[\begin{matrix} q^{k}b, & q^{-k}b/a \\ q^{1-m-k}/d, & q^{1-m+k}a/d \end{matrix}\,\Big|\, q\right]_{\ell} \left[\begin{matrix} b, & q^{1-m-k}/d \\ q^{-k}b/a, & q^{1-m}a/d \end{matrix}\,\Big|\, q\right]_k \left(q^{m-1}bd/a\right)^k$$

$$= \left[\begin{matrix} q^{k}b, & q^{-k}b/a \\ q^{1-m-k}/d, & q^{1-m+k}a/d \end{matrix}\,\Big|\, q\right]_{\ell} \left[\begin{matrix} b, & q^{m}d \\ qa/b, & q^{1-m}a/d \end{matrix}\,\Big|\, q\right]_k,$$

we can see that (6b)–(6c) equals $\left[\begin{smallmatrix} b, q^m d \\ qa/b, q^{1-m}a/d \end{smallmatrix}\,|\, q\right]_k$ times the very well-poised series:

$${}_8\phi_7\left[\begin{matrix} q^{-m}b/d, q\sqrt{q^{-m}b/d}, -q\sqrt{q^{-m}b/d}, q^k b, q^{-k}b/a, w/d, q^{1-m}a/dw, q^{-m} \\ \sqrt{q^{-m}b/d}, -\sqrt{q^{-m}b/d}, q^{1-m-k}/d, q^{1-m+k}a/d, q^{1-m}b/w, bw/a, qb/d \end{matrix}\,\Big|\, q; q\right]$$

$$= \left[\begin{matrix} d/b, & bd/a, & w, & w/a \\ d, & d/a, & w/b, & bw/a \end{matrix}\,\Big|\, q\right]_m \left[\begin{matrix} d, & q^{1-m}a/d, & q^m w, & qa/w \\ qa/d, & q^m d, & q^{1-m}a/w, & w \end{matrix}\,\Big|\, q\right]_k,$$

where we have applied Jackson's q-Dougall–Dixon formula [10, II-22]:

$${}_8\phi_7\left[\begin{matrix} a, & q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d, & e, & q^{-m} \\ & \sqrt{a}, & -\sqrt{a}, & qa/b, & qa/c, & qa/d, & qa/e, & q^{m+1}a \end{matrix}\,\Big|\, q; q\right] \tag{7a}$$

$$= \left[\begin{matrix} qa, qa/bc, qa/bd, qa/cd \\ qa/b, qa/c, qa/d, qa/bcd \end{matrix}\,\Big|\, q\right]_m \quad \text{for} \quad q^{1+m}a^2 = bcde. \tag{7b}$$

We have therefore reduced (6b)–(6c) to the following factorial fractions:

$$\left[\begin{matrix} d/b, & bd/a, & w, & w/a \\ d, & d/a, & w/b, & bw/a \end{matrix}\,\Big|\, q\right]_m \left[\begin{matrix} b, & d, & q^m w, & qa/w \\ qa/b, & qa/d, & q^{1-m}a/w, & w \end{matrix}\,\Big|\, q\right]_k.$$

In the last equation, canceling the first factorial fraction of order m by that on the right-hand side of (6a), we see that the second factorial fraction of order k coincides with the left member of (6a). This completes the proof of Lemma 1. □

Applying the linearization formula displayed in Lemma 1 and then interchanging the summation order, we can manipulate the following well-poised bilateral series:

$$\begin{aligned}
&{}_8\psi_8\left[\begin{matrix} q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d, & e, & q^m w, & qa/w \\ \sqrt{a}, & -\sqrt{a}, & qa/b, & qa/c, & qa/d, & qa/e, & q^{1-m}a/w, & w \end{matrix}\,\Big|\, q; \frac{q^{1-m}a^2}{bcde}\right] \\
&= \sum_{k=-\infty}^{+\infty} \frac{1-q^{2k}a}{1-a} \left[\begin{matrix} c, & e \\ qa/c, & qa/e \end{matrix}\,\Big|\, q\right]_k \left(\frac{q^{1-m}a^2}{bcde}\right)^k \left[\begin{matrix} b, & d, & q^m w, qa/w \\ qa/b, & qa/d, & q^{1-m}a/w, w \end{matrix}\,\Big|\, q\right]_k \\
&= \left[\begin{matrix} d, & d/a, & w/b, & bw/a \\ w/a, & w, & bd/a, & d/b \end{matrix}\,\Big|\, q\right]_m \\
&\quad\times \sum_{\ell=0}^{m} \frac{1-q^{2\ell-m}b/d}{1-q^{-m}b/d} \left[\begin{matrix} q^{-m}b/d, b, b/a, w/d, q^{1-m}a/dw, q^{-m} \\ q, q^{1-m}/d, q^{1-m}a/d, q^{1-m}b/w, bw/a, qb/d \end{matrix}\,\Big|\, q\right]_\ell q^\ell \\
&\quad\times {}_6\psi_6\left[\begin{matrix} q\sqrt{a}, & -q\sqrt{a}, & q^\ell b, & c, & q^{m-\ell}d, & e \\ \sqrt{a}, & -\sqrt{a}, & q^{1-\ell}a/b, & qa/c, & q^{1-m+\ell}a/d, & qa/e \end{matrix}\,\Big|\, q; \frac{q^{1-m}a^2}{bcde}\right],
\end{aligned}$$

where we assume $|q^{1-m}a^2/bcde| < 1$ for convergence. Evaluating the last series by means of Bailey's very well-poised bilateral ${}_6\psi_6$-series identity (3) and simplifying the result, we get

$$\begin{aligned}
&{}_6\psi_6\left[\begin{matrix} q\sqrt{a}, & -q\sqrt{a}, & q^\ell b, & c, & q^{m-\ell}d, & e \\ \sqrt{a}, & -\sqrt{a}, & q^{1-\ell}a/b, & qa/c, & q^{1-m+\ell}a/d, & qa/e \end{matrix}\,\Big|\, q; \frac{q^{1-m}a^2}{bcde}\right] \\
&= \Omega\left[a : q^\ell b, c, q^{m-\ell}d, e\right] \\
&= \Omega[a : b, c, d, e] \times \left[\begin{matrix} bd/a, & cd/a, & de/a \\ d, & d/a, & bcde/a^2 \end{matrix}\,\Big|\, q\right]_m \\
&\quad\times \left(\frac{a}{ce}\right)^\ell \left[\begin{matrix} bc/a, & be/a, & q^{1-m}/d, & q^{1-m}a/d \\ b, & b/a, & q^{1-m}a/cd, & q^{1-m}a/de \end{matrix}\,\Big|\, q\right]_\ell
\end{aligned}$$

which leads us consequently to the following expression:

$$\begin{aligned}
&{}_8\psi_8\left[\begin{matrix} q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d, & e, & q^m w, & qa/w \\ \sqrt{a}, & -\sqrt{a}, & qa/b, & qa/c, & qa/d, & qa/e, & q^{1-m}a/w, & w \end{matrix}\,\Big|\, q; \frac{q^{1-m}a^2}{bcde}\right] && \text{(8a)} \\
&= \Omega[a : b, c, d, e] \left[\begin{matrix} w/b, & bw/a, & cd/a, & de/a \\ w, & w/a, & d/b, & bcde/a^2 \end{matrix}\,\Big|\, q\right]_m && \text{(8b)} \\
&\quad\times {}_8\phi_7\left[\begin{matrix} q^{-m}b/d, q\sqrt{q^{-m}b/d}, -q\sqrt{q^{-m}b/d}, bc/a, be/a, w/d, q^{1-m}a/dw, q^{-m} \\ \sqrt{q^{-m}b/d}, -\sqrt{q^{-m}b/d}, q^{1-m}a/cd, q^{1-m}a/de, q^{1-m}b/w, bw/a, qb/d \end{matrix}\,\Big|\, q; \frac{qa}{ce}\right]. && \text{(8c)}
\end{aligned}$$

Recall Watson's transformation [10, III-17] on terminating very well-poised series:

$$ {}_8\phi_7\left[\begin{matrix} a, & q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d, & e, & q^{-n} \\ & \sqrt{a}, & -\sqrt{a}, & qa/b, & qa/c, & qa/d, & qa/e, & q^{n+1}a \end{matrix} \,\middle|\, q; \frac{q^{2+n}a^2}{bcde}\right] \tag{9a} $$

$$ = \left[\begin{matrix} qa, qa/de \\ qa/d, qa/e \end{matrix} \,\middle|\, q\right]_n {}_4\phi_3\left[\begin{matrix} q^{-n}, & d, & e, & qa/bc \\ & qa/b, & qa/c, & q^{-n}de/a \end{matrix} \,\middle|\, q; q\right]. \tag{9b} $$

The ${}_8\phi_7$-series in the identity (8c) can be reformulated in terms of the balanced series:

$$ {}_8\phi_7\left[\begin{matrix} q^{-m}b/d, q\sqrt{q^{-m}b/d}, -q\sqrt{q^{-m}b/d}, bc/a, be/a, w/d, q^{1-m}a/dw, q^{-m} \\ \sqrt{q^{-m}b/d}, -\sqrt{q^{-m}b/d}, q^{1-m}a/cd, q^{1-m}a/de, q^{1-m}b/w, bw/a, qb/d \end{matrix} \,\middle|\, q; \frac{qa}{ce}\right] $$
$$ = \left[\begin{matrix} d/b, bcde/a^2 \\ cd/a, de/a \end{matrix} \,\middle|\, q\right]_m {}_4\phi_3\left[\begin{matrix} q^{-m}, & bc/a, & bd/a, & be/a \\ & bw/a, & q^{1-m}b/w, & bcde/a^2 \end{matrix} \,\middle|\, q; q\right]. $$

Substituting the last relation into (8c), we derive finally from (8a)–(8c) the following ${}_8\psi_8$-series identity.

Theorem 2 *(The first extension of Shukla's ${}_8\psi_8$-summation formula). For $m \in \mathbb{N}_0$, suppose that $|q^{1-m}a^2/bcde| < 1$. There holds the bilateral series identity*:

$$ {}_8\psi_8\left[\begin{matrix} q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d, & e, & q^m w, & qa/w \\ \sqrt{a}, & -\sqrt{a}, & qa/b, & qa/c, & qa/d, & qa/e, & q^{1-m}a/w, & w \end{matrix} \,\middle|\, q; \frac{q^{1-m}a^2}{bcde}\right] \tag{10a} $$

$$ = \Omega[a:b,c,d,e]\left[\begin{matrix} w/b, & bw/a \\ w, & w/a \end{matrix} \,\middle|\, q\right]_m {}_4\phi_3\left[\begin{matrix} q^{-m}, bc/a, bd/a, be/a \\ q^{1-m}b/w, bw/a, bcde/a^2 \end{matrix} \,\middle|\, q; q\right]. \tag{10b} $$

For $m = 0, 1$, this result reduces respectively to Bailey's ${}_6\psi_6$-series identity (3) and Shukla's ${}_8\psi_8$-summation formula (5a)–(5b).

3. The well-poised ${}_{10}\psi_{10}$-series identity

We can further generalize Theorem 2 by adding two extra pairs of well-poised parameters. Applying Lemma 1 to factorial fraction

$$ \left[\begin{matrix} c, & e, & q^n v, & qa/v \\ qa/c, & qa/e, & q^{1-n}a/v, & v \end{matrix} \,\middle|\, q\right]_k, $$

we can reformulate the ${}_{10}\psi_{10}$-series as follows:

$$ {}_{10}\psi_{10}\left[\begin{matrix} q\sqrt{a}, -q\sqrt{a}, b, c, d, e, q^m u, qa/u, q^n v, qa/v \\ \sqrt{a}, -\sqrt{a}, qa/b, qa/c, qa/d, qa/e, q^{1-m}a/u, u, q^{1-n}a/v, v \end{matrix} \,\middle|\, q; \frac{q^{1-m-n}a^2}{bcde}\right] $$
$$ = \sum_{k=-\infty}^{+\infty} \frac{1-q^{2k}a}{1-a}\left[\begin{matrix} b, d, q^m u, qa/u \\ qa/b, qa/d, q^{1-m}a/u, u \end{matrix} \,\middle|\, q\right]_k \left(\frac{q^{1-m-n}a^2}{bcde}\right)^k $$
$$ \times \left[\begin{matrix} c, e, q^n v, qa/v \\ qa/c, qa/e, q^{1-n}a/v, v \end{matrix} \,\middle|\, q\right]_k $$
$$ = \left[\begin{matrix} c, & c/a, & v/e, & ev/a \\ v, & v/a, & c/e, & ce/a \end{matrix} \,\middle|\, q\right]_n $$
$$ \times \sum_{\ell=0}^{n}\left[\begin{matrix} q^{-n}e/c, q\sqrt{q^{-n}e/c}, -q\sqrt{q^{-n}e/c}, e, e/a, v/c, q^{1-n}a/cv, q^{-n} \\ q, \sqrt{q^{-n}e/c}, -\sqrt{q^{-n}e/c}, q^{1-n}/c, q^{1-n}a/c, q^{1-n}e/v, ev/a, qe/c \end{matrix} \,\middle|\, q\right]_\ell q^\ell $$

$$\times\, {}_8\psi_8\left[\begin{matrix} q\sqrt{a}, -q\sqrt{a}, b, q^{n-\ell}c, d, q^{\ell}e, q^m u, qa/u \\ \sqrt{a}, -\sqrt{a}, qa/b, q^{1-n+\ell}a/c, qa/d, q^{1-\ell}a/e, q^{1-m}a/u, u \end{matrix}\,\Big|\, q; \frac{q^{1-m-n}a^2}{bcde}\right]$$

provided that $|q^{1-m-n}a^2/bcde| < 1$ for convergence. Evaluating the last ${}_8\psi_8$-series by Theorem 2 and then simplifying the result, we have

$$\begin{aligned}
&{}_8\psi_8\left[\begin{matrix} q\sqrt{a}, & -q\sqrt{a}, & b, & q^{n-\ell}c, & d, & q^{\ell}e, & q^m u, & qa/u \\ \sqrt{a}, & -\sqrt{a}, & qa/b, & q^{1-n+\ell}a/c, & qa/d, & q^{1-\ell}a/e, & q^{1-m}a/u, & u \end{matrix}\,\Big|\, q; \frac{q^{1-m-n}a^2}{bcde}\right] \\
&\quad = \Omega\big[a : b, q^{n-\ell}c, d, q^{\ell}e\big]\left[\begin{matrix} u/b, bu/a \\ u, u/a \end{matrix}\,\Big|\, q\right]_m {}_4\phi_3\left[\begin{matrix} q^{-m}, q^{n-\ell}bc/a, bd/a, q^{\ell}be/a \\ q^{1-m}b/u, bu/a, q^n bcde/a^2 \end{matrix}\,\Big|\, q; q\right] \\
&\quad = \Omega[a : b, c, d, e]\left[\begin{matrix} u/b, bu/a \\ u, u/a \end{matrix}\,\Big|\, q\right]_m \left(\frac{a}{bd}\right)^{\ell} \left[\begin{matrix} be/a, & de/a, & q^{1-n}/c, & q^{1-n}a/c \\ e, & e/a, & q^{1-n}a/bc, & q^{1-n}a/cd \end{matrix}\,\Big|\, q\right]_{\ell} \\
&\qquad \times \left[\begin{matrix} bc/a, cd/a, ce/a \\ c, c/a, bcde/a^2 \end{matrix}\,\Big|\, q\right]_n {}_4\phi_3\left[\begin{matrix} q^{-m}, q^{n-\ell}bc/a, bd/a, q^{\ell}be/a \\ q^{1-m}b/u, bu/a, q^n bcde/a^2 \end{matrix}\,\Big|\, q; q\right].
\end{aligned}$$

This leads us to the following simplified relation:

$$ {}_{10}\psi_{10}\left[\begin{matrix} q\sqrt{a}, -q\sqrt{a}, b, c, d, e, q^m u, qa/u, q^n v, qa/v \\ \sqrt{a}, -\sqrt{a}, qa/b, qa/c, qa/d, qa/e, q^{1-m}a/u, u, q^{1-n}a/v, v \end{matrix}\,\Big|\, q; \frac{q^{1-m-n}a^2}{bcde}\right] \tag{11a}$$

$$ = \Omega[a : b, c, d, e]\left[\begin{matrix} u/b, bu/a \\ u, u/a \end{matrix}\,\Big|\, q\right]_m \left[\begin{matrix} v/e, ev/a, bc/a, cd/a \\ v, v/a, c/e, bcde/a^2 \end{matrix}\,\Big|\, q\right]_n \tag{11b}$$

$$ \times \sum_{\ell=0}^{n} \frac{1-q^{2\ell-n}e/c}{1-q^{-n}e/c} \left[\begin{matrix} q^{-n}e/c, be/a, de/a, v/c, q^{1-n}a/cv, q^{-n} \\ q, q^{1-n}a/bc, q^{1-n}a/cd, q^{1-n}e/v, ev/a, qe/c \end{matrix}\,\Big|\, q\right]_{\ell} \left(\frac{qa}{bd}\right)^{\ell} \tag{11c}$$

$$ \times\, {}_4\phi_3\left[\begin{matrix} q^{-m}, & q^{n-\ell}bc/a, & bd/a, & q^{\ell}be/a \\ & q^{1-m}b/u, & bu/a, & q^n bcde/a^2 \end{matrix}\,\Big|\, q; q\right]. \tag{11d}$$

Writing the last ${}_4\phi_3$-series explicitly

$$\begin{aligned}
&{}_4\phi_3\left[\begin{matrix} q^{-m}, q^{n-\ell}bc/a, bd/a, q^{\ell}be/a \\ q^{1-m}b/u, bu/a, q^n bcde/a^2 \end{matrix}\,\Big|\, q; q\right] \\
&\quad = \sum_{k=0}^{m} \left[\begin{matrix} q^{-m}, q^{n-\ell}bc/a, bd/a, q^{\ell}be/a \\ q, q^{1-m}b/u, bu/a, q^n bcde/a^2 \end{matrix}\,\Big|\, q\right]_k q^k,
\end{aligned}$$

exchanging the summation order between (11c) and (11d) and then applying the two factorial relations

$$\begin{aligned}
\big(q^{\ell}be/a; q\big)_k &= (be/a; q)_k \frac{(q^k be/a; q)_{\ell}}{(be/a; q)_{\ell}}, \\
\big(q^{n-\ell}bc/a; q\big)_k &= \big(q^n bc/a; q\big)_k \frac{(q^{1-n}a/bc; q)_{\ell}}{(q^{1-n-k}a/bc; q)_{\ell}} q^{-k\ell},
\end{aligned}$$

we find that (11c)–(11d) is equal to the following double sum:

$$\sum_{k=0}^{m}\left[\begin{matrix} q^{-m}, & q^{n}bc/a, & bd/a, & be/a \\ q, & q^{1-m}b/u, & bu/a, & q^{n}bcde/a^2 \end{matrix}\,\Big|\, q\right]_k q^k$$
$$\times\, {}_8\phi_7\left[\begin{matrix} q^{-n}e/c, q\sqrt{q^{-n}e/c}, -q\sqrt{q^{-n}e/c}, de/a, q^{k}be/a, v/c, q^{1-n}a/cv, q^{-n} \\ \sqrt{q^{-n}e/c}, -\sqrt{q^{-n}e/c}, q^{1-n}a/cd, q^{1-n-k}a/bc, q^{1-n}e/v, ev/a, qe/c \end{matrix}\,\Big|\, q; \frac{q^{1-k}a}{bd}\right].$$

By means of Watson's transformation (9a)–(9b), the last ${}_8\phi_7$-series may be expressed as:

$$\begin{aligned} &{}_8\phi_7\left[\begin{matrix} q^{-n}e/c, q\sqrt{q^{-n}e/c}, -q\sqrt{q^{-n}e/c}, de/a, q^{k}be/a, v/c, q^{1-n}a/cv, q^{-n} \\ \sqrt{q^{-n}e/c}, -\sqrt{q^{-n}e/c}, q^{1-n}a/cd, q^{1-n-k}a/bc, q^{1-n}e/v, ev/a, qe/c \end{matrix}\,\Big|\, q; \frac{q^{1-k}a}{bd}\right] \\ &\quad = \left[\begin{matrix} c/e, & q^{k}bcde/a^2 \\ q^{k}bc/a, & cd/a \end{matrix}\,\Big|\, q\right]_n {}_4\phi_3\left[\begin{matrix} q^{-n}, q^{k}be/a, ce/a, de/a \\ q^{1-n}e/v, ev/a, q^{k}bcde/a^2 \end{matrix}\,\Big|\, q; q\right]. \end{aligned}$$

Substituting the last two expressions into (11c)–(11d) and taking into account of the relation

$$\left[\begin{matrix} c/e, q^{k}bcde/a^2 \\ q^{k}bc/a, cd/a \end{matrix}\,\Big|\, q\right]_n = \left[\begin{matrix} c/e, bcde/a^2 \\ bc/a, cd/a \end{matrix}\,\Big|\, q\right]_n \times \left[\begin{matrix} bc/a, q^{n}bcde/a^2 \\ q^{n}bc/a, bcde/a^2 \end{matrix}\,\Big|\, q\right]_k,$$

we can finally combine (11a)–(11b) with (11c)–(11d) together and get the resulting transformation

$${}_{10}\psi_{10}\left[\begin{matrix} q\sqrt{a}, -q\sqrt{a}, b, c, d, e, q^{m}u, qa/u, q^{n}v, qa/v \\ \sqrt{a}, -\sqrt{a}, qa/b, qa/c, qa/d, qa/e, q^{1-m}a/u, u, q^{1-n}a/v, v \end{matrix}\,\Big|\, q; \frac{q^{1-m-n}a^2}{bcde}\right] \tag{12a}$$

$$= \Omega[a:b,c,d,e]\left[\begin{matrix} u/b, bu/a \\ u, u/a \end{matrix}\,\Big|\, q\right]_m \left[\begin{matrix} v/e, ev/a \\ v, v/a \end{matrix}\,\Big|\, q\right]_n \tag{12b}$$

$$\times \sum_{k=0}^{m} q^k \left[\begin{matrix} q^{-m}, bc/a, bd/a, be/a \\ q, q^{1-m}b/u, bu/a, bcde/a^2 \end{matrix}\,\Big|\, q\right]_k$$
$$\times\, {}_4\phi_3\left[\begin{matrix} q^{-n}, q^{k}be/a, ce/a, de/a \\ q^{1-n}e/v, ev/a, q^{k}bcde/a^2 \end{matrix}\,\Big|\, q; q\right]. \tag{12c}$$

For $DEF = q^{1-n}ABC$, recalling Sears' transformation (cf. [10, III-15]) on terminating balanced series

$${}_4\phi_3\left[\begin{matrix} q^{-n}, & A, B, C \\ & D, E, F \end{matrix}\,\Big|\, q; q\right] = A^n \left[\begin{matrix} E/A, F/A \\ E, F \end{matrix}\,\Big|\, q\right]_n {}_4\phi_3\left[\begin{matrix} q^{-n}, A, D/B, D/C \\ D, q^{1-n}A/E, q^{1-n}A/F \end{matrix}\,\Big|\, q; q\right] \tag{13}$$

we can further reformulate the ${}_4\phi_3$-series in (12c) as

$$\begin{aligned} &{}_4\phi_3\left[\begin{matrix} q^{-n}, q^{k}be/a, ce/a, de/a \\ q^{1-n}e/v, ev/a, q^{k}bcde/a^2 \end{matrix}\,\Big|\, q; q\right] \\ &\quad = q^{-nk}\left[\begin{matrix} v/b, & bv/a \\ v/e, & ev/a \end{matrix}\,\Big|\, q\right]_n \left[\begin{matrix} qb/v, & q^{n}bv/a \\ bv/a, & q^{1-n}b/v \end{matrix}\,\Big|\, q\right]_k \\ &\quad\quad \times\, {}_4\phi_3\left[\begin{matrix} q^{-n}, q^{k}bc/a, q^{k}bd/a, q^{k}be/a \\ q^{1-n+k}b/v, q^{k}bv/a, q^{k}bcde/a^2 \end{matrix}\,\Big|\, q; q\right]. \end{aligned}$$

Substituting this last expression into (12c), we derive the following bilateral series identity.

Theorem 3 *(The second extension of Shukla's ${}_8\psi_8$-summation formula). For $m, n \in \mathbb{N}_0$, suppose that $|q^{1-m-n}a^2/bcde| < 1$. There holds the bilateral series identity*:

$$
{}_{10}\psi_{10}\left[\begin{matrix} q\sqrt{a}, -q\sqrt{a}, b, c, d, e, q^m u, qa/u, q^n v, qa/v \\ \sqrt{a}, -\sqrt{a}, qa/b, qa/c, qa/d, qa/e, q^{1-m}a/u, u, q^{1-n}a/v, v \end{matrix} \,\middle|\, q; \frac{q^{1-m-n}a^2}{bcde}\right] \tag{14a}
$$

$$
= \Omega[a:b,c,d,e]\left[\begin{matrix} u/b, & bu/a \\ u, & u/a \end{matrix} \,\middle|\, q\right]_m \left[\begin{matrix} v/b, & bv/a \\ v, & v/a \end{matrix} \,\middle|\, q\right]_n \tag{14b}
$$

$$
\times \sum_{k=0}^{m} \left[\begin{matrix} q^{-m}, q^n bv/a, qb/v, bc/a, bd/a, be/a \\ q, q^{1-m}b/u, bu/a, q^{1-n}b/v, bv/a, bcde/a^2 \end{matrix} \,\middle|\, q\right]_k q^{k-nk} \tag{14c}
$$

$$
\times {}_4\phi_3\left[\begin{matrix} q^{-n}, & q^k bc/a, & q^k bd/a, & q^k be/a \\ & q^{1-n+k}b/v, & q^k bv/a, & q^k bcde/a^2 \end{matrix} \,\middle|\, q; q\right]. \tag{14d}
$$

We remark that for $n = 0$, this theorem reduces to Theorem 2.

4. Milne's generalization

Both transformation formulae examined in the last two sections are special cases of the following more general one which expresses a nonterminating very well-poised bilateral series in terms of a terminating multiple unilateral series.

Theorem 4. (Milne [13, Theorem 1.7]) *For n-pairs of complex parameters $\{x_\kappa, y_\kappa\}$ satisfying the finite condition*:

$$
qa/x_\kappa y_\kappa = q^{-N_\kappa} \quad \text{with } N_\kappa \in \mathbb{N}_0 \text{ for } 1 \leqslant \kappa \leqslant n \quad \text{and} \quad N := \sum_{\kappa=1}^{n} N_\kappa \tag{15}
$$

suppose that $|q^{1-N}a^2/bcde| < 1$. Then there holds the bilateral series identity:

$$
{}_{2n+6}\psi_{6+2n}\left[\begin{matrix} q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d, & e, & \{x_\iota, y_\iota\}_{\iota=1}^n \\ \sqrt{a}, & -\sqrt{a}, & qa/b, & qa/c, & qa/d, & qa/e, & \{qa/x_\iota, qa/y_\iota\}_{\iota=1}^n \end{matrix} \,\middle|\, q; \frac{q^{1-N}a^2}{bcde}\right] \tag{16a}
$$

$$
= \Omega[a:b,c,d,e]\prod_{\iota=1}^{n}\left[\begin{matrix} x_\iota, x_\iota/a, qb/y_\iota, qa/by_\iota \\ x_\iota/b, bx_\iota/a, q/y_\iota, qa/y_\iota \end{matrix} \,\middle|\, q\right]_\infty \tag{16b}
$$

$$
\times \sum_{\tilde{m}} \frac{(qa/x_n y_n; q)_{m_n}}{(q;q)_{m_n}} \left[\begin{matrix} bc/a, bd/a, be/a \\ qb/x_n, qb/y_n, bcde/a^2 \end{matrix} \,\middle|\, q\right]_{M_n} q^{M_n} \tag{16c}
$$

$$
\times \prod_{\jmath=1}^{n-1} \frac{(qa/x_\jmath y_\jmath; q)_{m_\jmath}}{(q;q)_{m_\jmath}} \left[\begin{matrix} bx_{\jmath+1}/a, by_{\jmath+1}/a \\ qb/x_\jmath, qb/y_\jmath \end{matrix} \,\middle|\, q\right]_{M_\jmath} \left(\frac{qa}{x_{\jmath+1}y_{\jmath+1}}\right)^{M_\jmath}, \tag{16d}
$$

where the multiple sum runs over $\tilde{m} = (m_1, m_2, \ldots, m_n) \in \mathbb{N}_0^n$ with their partial sums denoted by $M_\kappa := \sum_{\iota=1}^{\kappa} m_\iota$ for $\kappa = 1, 2, \ldots, n$.

This important result is first established by Milne [13] who has ingeniously incorporated algebraic manipulation with the analytic continuation method devised by Askey and Ismail [2] for verifying Bailey's very well-poised bilateral ${}_6\psi_6$-series identity (3). The following two facts can at least exemplify its importance.

First, specifying the parameters with

$$b \to bq^{-M} \quad \left\{ \begin{array}{lcl} x_\iota & \to & q^{2M}az_\iota/b \\ y_\iota & \to & q^{1-M}b/z_\iota \\ N_\iota & \to & M \end{array} \right\} \quad \text{for } \iota = 1, 2, \ldots, n,$$

and then letting $M \to \infty$, we recover from Theorem 4 the following multivariate partition identity due to Andrews.

Corollary 5. (Andrews [1, §6], see Milne [13, Theorem 3.1] also.)

$$\prod_{k=1}^{n} \frac{1}{(z_k;q)_\infty} = \sum_{\tilde{m}\in\mathbb{N}_0^n} q^{\binom{M_n}{2}} \prod_{\iota=1}^{n} \frac{z_\iota^{m_\iota} q^{\binom{m_\iota}{2}}}{(q;q)_{m_\iota}(z_\iota;q)_{M_\iota}}.$$

Then letting $e = a/b$ in Theorem 4, we see that the corresponding multiple sum reduces to one. Exchanging b and c, the result may be stated as the following bilateral series formula.

Corollary 6. (Chu [6, Theorem 2]) *For n-pairs of complex parameters $\{x_\kappa, y_\kappa\}$ satisfying the conditions of Theorem 4, there holds the following bilateral series identity*:

$$\begin{aligned} &{}_{2n+6}\psi_{6+2n}\left[\begin{matrix} q\sqrt{a}, & -q\sqrt{a}, & b, & d, & c, & a/c, & \{x_\kappa, y_\kappa\}_{\kappa=1}^n \\ \sqrt{a}, & -\sqrt{a}, & qa/b, & qa/d, & qa/c, & qc, & \{qa/x_\kappa, qa/y_\kappa\}_{\kappa=1}^n \end{matrix} \,\middle|\, q; \frac{q^{1-N}a}{bd}\right] \\ &= \left(\frac{a}{c}\right)^n \left[\begin{matrix} q,\ q,\ qa,\ q/a,\ qa/bc,\ qa/cd,\ qc/b,\ qc/d \\ qa/c,\ qc/a,\ qc,\ q/c,\ q/b,\ q/d,\ qa/b,\ qa/d \end{matrix} \,\middle|\, q\right]_\infty \\ &\quad \times \prod_{\kappa=1}^{n} \left[\begin{matrix} qc/x_\kappa, qc/y_\kappa \\ qa/x_\kappa, qa/y_\kappa \end{matrix} \,\middle|\, q\right]_{N_\kappa} \end{aligned}$$

provided that $|q^{1-N}a/bd| < 1$ for convergence.

This bilateral series identity contains several known summation formulae appeared in [5,9, 12,14] as special cases, which concern classical and basic hypergeometric series of Karlsson–Minton-type with integral parameter differences.

Based on the linearization method developed in the last two sections, we present now a constructive proof for Milne's transformation formula. Another proof via the Saalschütz chain reactions can be found in Chu [7], where the iteration technique possesses more directness and generality. For other multidimensional generalizations of Shukla's formula, refer to Schlosser [16].

Proof of Theorem 4. For $n = 1, 2$, it is not hard to check that this transformation reduces respectively to Theorems 2 and 3. Suppose that the transformation stated in the theorem is true when there are n-pairs of extra parameters $\{x_k, y_k\}$. In order to prove the theorem, we have to verify the formula for $(n+1)$-pairs of $\{x_k, y_k\}$.

Introduce a new pair of $\{x_{n+1}, y_{n+1}\}$ with $qa/x_{n+1}y_{n+1} = q^{-\lambda}$ and $\lambda \in \mathbb{N}_0$. The corresponding ${}_{2n+8}\psi_{8+2n}$-series can be transformed by Lemma 6 into the following:

$$
{}_{2n+8}\psi_{8+2n}\left[\begin{matrix} q\sqrt{a}, -q\sqrt{a}, b, c, d, e, & \{x_\iota, y_\iota\}_{\iota=1}^{n+1} \\ \sqrt{a}, -\sqrt{a}, qa/b, qa/c, qa/d, qa/e, & \{qa/x_\iota, qa/y_\iota\}_{\iota=1}^{n+1} \end{matrix} \,\middle|\, q; \frac{q^{1-N-\lambda}a^2}{bcde}\right] \tag{17a}
$$

$$
= \sum_{k=-\infty}^{+\infty} \left[\begin{matrix} q\sqrt{a}, -q\sqrt{a}, b, d, \{x_\iota, y_\iota\}_{\iota=1}^{n} \\ \sqrt{a}, -\sqrt{a}, qa/b, qa/d, \{qa/x_\iota, qa/y_\iota\}_{\iota=1}^{n} \end{matrix} \,\middle|\, q\right]_k \left(\frac{q^{1-N-\lambda}a^2}{bcde}\right)^k \tag{17b}
$$

$$
\times \left[\begin{matrix} c, & e, & x_{n+1}, & y_{n+1} \\ qa/c, & qa/e, & qa/x_{n+1}, & qa/y_{n+1} \end{matrix} \,\middle|\, q\right]_k \tag{17c}
$$

$$
= \left[\begin{matrix} c, & c/a, & qa/ex_{n+1}, & qe/x_{n+1} \\ c/e, & ce/a, & qa/x_{n+1}, & q/x_{n+1} \end{matrix} \,\middle|\, q\right]_\lambda \tag{17d}
$$

$$
\times \sum_{\ell=0}^{\lambda} \left[\begin{matrix} q^{-\lambda}e/c, q\sqrt{q^{-\lambda}e/c}, -q\sqrt{q^{-\lambda}e/c}, e, e/a, q^{-\lambda}, qa/cx_{n+1}, qa/cy_{n+1} \\ q, \sqrt{q^{-\lambda}e/c}, -\sqrt{q^{-\lambda}e/c}, q^{1-\lambda}/c, q^{1-\lambda}a/c, qe/c, qe/x_{n+1}, qe/y_{n+1} \end{matrix} \,\middle|\, q\right]_\ell q^\ell \tag{17e}
$$

$$
\times {}_{2n+6}\psi_{6+2n}\left[\begin{matrix} q\sqrt{a}, -q\sqrt{a}, b, q^{\lambda-\ell}c, d, q^\ell e, \{x_\iota, y_\iota\} \\ \sqrt{a}, -\sqrt{a}, qa/b, q^{1-\lambda+\ell}a/c, qa/d, q^{1-\ell}a/e, \{qa/x_\iota, qa/y_\iota\} \end{matrix} \,\middle|\, q; \frac{q^{1-N-\lambda}a^2}{bcde}\right]. \tag{17f}
$$

According to the induction hypothesis, the last ${}_{2n+6}\psi_{6+2n}$-series can be expressed as the following multiple sum:

$$
{}_{2n+6}\psi_{6+2n}\left[\begin{matrix} q\sqrt{a}, -q\sqrt{a}, b, q^{\lambda-\ell}c, d, q^\ell e, \{x_\iota, y_\iota\} \\ \sqrt{a}, -\sqrt{a}, qa/b, q^{1-\lambda+\ell}a/c, qa/d, q^{1-\ell}a/e, \{qa/x_\iota, qa/y_\iota\} \end{matrix} \,\middle|\, q; \frac{q^{1-N-\lambda}a^2}{bcde}\right] \tag{18a}
$$

$$
= \Omega\left[a : b, cq^{\lambda-\ell}, d, eq^\ell\right] \prod_{\iota=1}^{n} \left[\begin{matrix} x_\iota, x_\iota/a, qb/y_\iota, qa/by_\iota \\ x_\iota/b, bx_\iota/a, q/y_\iota, qa/y_\iota \end{matrix} \,\middle|\, q\right]_\infty \tag{18b}
$$

$$
\times \sum_{\tilde{m}} \frac{(qa/x_n y_n; q)_{m_n}}{(q; q)_{m_n}} \left[\begin{matrix} q^{\lambda-\ell}bc/a, bd/a, q^\ell be/a \\ qb/x_n, qb/y_n, q^\lambda bcde/a^2 \end{matrix} \,\middle|\, q\right]_{M_n} q^{M_n} \tag{18c}
$$

$$
\times \prod_{\jmath=1}^{n-1} \frac{(qa/x_\jmath y_\jmath; q)_{m_\jmath}}{(q; q)_{m_\jmath}} \left[\begin{matrix} bx_{\jmath+1}/a, by_{\jmath+1}/a \\ qb/x_\jmath, qb/y_\jmath \end{matrix} \,\middle|\, q\right]_{M_\jmath} \left(\frac{qa}{x_{\jmath+1}y_{\jmath+1}}\right)^{M_\jmath}. \tag{18d}
$$

We need further to reduce the Ω-function

$$
\begin{aligned}
\Omega\left[a : b, q^{\lambda-\ell}c, d, q^\ell e\right] = \Omega[a : b, c, d, e] &\left[\begin{matrix} bc/a, cd/a, ce/a \\ c, c/a, bcde/a^2 \end{matrix} \,\middle|\, q\right]_\lambda \\
&\times \left[\begin{matrix} be/a, & de/a, & q^{1-\lambda}/c, & q^{1-\lambda}a/c \\ e, & e/a, & q^{1-\lambda}a/bc, & q^{1-\lambda}a/cd \end{matrix} \,\middle|\, q\right]_\ell \left(\frac{a}{bd}\right)^\ell
\end{aligned}
$$

and to separate the factorial fractions indexed by ℓ:

$$
\begin{aligned}
\left[\begin{matrix} q^{\lambda-\ell}bc/a, bd/a, q^\ell be/a \\ qb/x_n, qb/y_n, q^\lambda bcde/a^2 \end{matrix} \,\middle|\, q\right]_{M_n} = &\left[\begin{matrix} q^{1-\lambda}a/bc, & q^{M_n}be/a \\ q^{1-\lambda-M_n}a/bc, & be/a \end{matrix} \,\middle|\, q\right]_\ell q^{-\ell M_n} \\
&\times \left[\begin{matrix} q^\lambda bc/a, bd/a, be/a \\ qb/x_n, qb/y_n, q^\lambda bcde/a^2 \end{matrix} \,\middle|\, q\right]_{M_n}.
\end{aligned}
$$

Keeping in mind of these two relations just displayed, substituting (18) into (17f), and then changing the summation order between ℓ and $\tilde{m}$ in (17), we derive finally the transformation formula:

$$
\begin{aligned}
&{}_{2n+8}\psi_{8+2n}\left[\begin{matrix} q\sqrt{a}, -q\sqrt{a}, b, c, d, e, & \{x_\imath, y_\imath\}_{\imath=1}^{n+1} \\ \sqrt{a}, -\sqrt{a}, qa/b, qa/c, qa/d, qa/e, & \{qa/x_\imath, qa/y_\imath\}_{\imath=1}^{n+1} \end{matrix} \,\Big|\, q; \frac{q^{1-N-\lambda}a^2}{bcde}\right] \\
&= \left[\begin{matrix} bc/a, cd/a, qa/ex_{n+1}, qe/x_{n+1} \\ bcde/a^2, c/e, qa/x_{n+1}, q/x_{n+1} \end{matrix} \,\Big|\, q\right]_\lambda \prod_{\imath=1}^{n} \left[\begin{matrix} x_\imath, x_\imath/a, qb/y_\imath, qa/by_\imath \\ x_\imath/b, bx_\imath/a, q/y_\imath, qa/y_\imath \end{matrix} \,\Big|\, q\right]_\infty \\
&\quad \times \Omega[a:b,c,d,e] \sum_{\tilde{m}} \frac{(qa/x_ny_n;q)_{m_n}}{(q;q)_{m_n}} \left[\begin{matrix} q^\lambda bc/a, bd/a, be/a \\ qb/x_n, qb/y_n, q^\lambda bcde/a^2 \end{matrix} \,\Big|\, q\right]_{M_n} q^{M_n} \\
&\quad \times \prod_{\jmath=1}^{n-1} \frac{(qa/x_\jmath y_\jmath;q)_{m_\jmath}}{(q;q)_{m_\jmath}} \left[\begin{matrix} bx_{\jmath+1}/a, by_{\jmath+1}/a \\ qb/x_\jmath, qb/y_\jmath \end{matrix} \,\Big|\, q\right]_{M_\jmath} \left(\frac{qa}{x_{\jmath+1}y_{\jmath+1}}\right)^{M_\jmath} \\
&\quad \times {}_8\phi_7\left[\begin{matrix} \frac{q^{-\lambda}e}{c}, \sqrt{\frac{q^{2-\lambda}e}{c}}, -\sqrt{\frac{q^{2-\lambda}e}{c}}, \frac{de}{a}, q^{-\lambda}, \frac{q^{M_n}be}{a}, \frac{qa}{cx_{n+1}}, \frac{qa}{cy_{n+1}} \\ \sqrt{\frac{q^{-\lambda}e}{c}}, -\sqrt{\frac{q^{-\lambda}e}{c}}, \frac{q^{1-\lambda}a}{cd}, \frac{qe}{c}, \frac{q^{1-\lambda-M_n}a}{bc}, \frac{qe}{x_{n+1}}, \frac{qe}{y_{n+1}} \end{matrix} \,\Big|\, q; \frac{q^{1-M_n}a}{bd}\right].
\end{aligned}
$$

For the last very well-poised series, apply Watson's transformation (9a)–(9b):

$$
\begin{aligned}
&{}_8\phi_7\left[\begin{matrix} \frac{q^{-\lambda}e}{c}, \sqrt{\frac{q^{2-\lambda}e}{c}}, -\sqrt{\frac{q^{2-\lambda}e}{c}}, \frac{de}{a}, q^{-\lambda}, \frac{q^{M_n}be}{a}, \frac{qa}{cx_{n+1}}, \frac{qa}{cy_{n+1}} \\ \sqrt{\frac{q^{-\lambda}e}{c}}, -\sqrt{\frac{q^{-\lambda}e}{c}}, \frac{q^{1-\lambda}a}{cd}, \frac{qe}{c}, \frac{q^{1-\lambda-M_n}a}{bc}, \frac{qe}{x_{n+1}}, \frac{qe}{y_{n+1}} \end{matrix} \,\Big|\, q; \frac{q^{1-M_n}a}{bd}\right] \\
&= \left[\begin{matrix} q^{1-\lambda}e/c, ce/a \\ \frac{qe}{x_{n+1}}, \frac{qe}{y_{n+1}} \end{matrix} \,\Big|\, q\right]_\lambda {}_4\phi_3\left[\begin{matrix} q^{-\lambda}, q^{1-M_n-\lambda}a^2/bcde, \frac{qa}{cx_{n+1}}, \frac{qa}{cy_{n+1}} \\ q^{1-M_n-\lambda}a/bc, q^{1-\lambda}a/cd, q^{1-\lambda}a/ce \end{matrix} \,\Big|\, q; q\right]
\end{aligned}
$$

and then further Sears' transformation (13) on the reversal of the above displayed balanced ${}_4\phi_3$-series:

$$
\begin{aligned}
&{}_4\phi_3\left[\begin{matrix} q^{-\lambda}, q^{1-M_n-\lambda}a^2/bcde, qa/cx_{n+1}, qa/cy_{n+1} \\ q^{1-M_n-\lambda}a/bc, q^{1-\lambda}a/cd, q^{1-\lambda}a/ce \end{matrix} \,\Big|\, q; q\right] \\
&= {}_4\phi_3\left[\begin{matrix} q^{-\lambda}, q^{M_n}bc/a, cd/a, ce/a \\ q^{M_n}bcde/a^2, qc/x_{n+1}, qc/y_{n+1} \end{matrix} \,\Big|\, q; q\right] \\
&\quad \times \left(\frac{q^\lambda c}{y_{n+1}}\right)^\lambda \left[\begin{matrix} q^{M_n}bcde/a^2, qc/x_{n+1}, qa/cx_{n+1} \\ q^{M_n}bc/a, cd/a, ce/a \end{matrix} \,\Big|\, q\right]_\lambda \\
&= {}_4\phi_3\left[\begin{matrix} q^{-\lambda}, q^{M_n}bc/a, q^{M_n}bd/a, q^{M_n}be/a \\ q^{M_n}bcde/a^2, q^{1+M_n}b/x_{n+1}, q^{1+M_n}b/y_{n+1} \end{matrix} \,\Big|\, q; q\right] \\
&\quad \times \left(\frac{-c}{b}\right)^\lambda \left[\begin{matrix} q^{M_n}bcde/a^2, \frac{q^{1+M_n}b}{x_{n+1}}, \frac{q^{1+M_n}b}{y_{n+1}} \\ q^{M_n}bc/a, cd/a, ce/a \end{matrix} \,\Big|\, q\right]_\lambda q^{\binom{\lambda}{2}-\lambda M_n}.
\end{aligned}
$$

Unifying the last three expressions and canceling the common factorial factors by the following three relations:

$$
\frac{(q^\lambda bc/a;q)_{M_n}}{(q^\lambda bcde/a^2;q)_{M_n}} \left[\begin{matrix} bc/a, q^{M_n}bcde/a^2 \\ bcde/a^2, q^{M_n}bc/a \end{matrix} \,\Big|\, q\right]_\lambda = \frac{(bc/a;q)_{M_n}}{(bcde/a^2;q)_{M_n}},
$$

$$\begin{aligned}
&\big[q^{1+M_n}b/x_{n+1}, q^{1+M_n}b/y_{n+1}; q\big]_\lambda \\
&\quad = \left[\begin{matrix} bx_{n+1}/a, by_{n+1}/a \\ qb/x_{n+1}, qb/y_{n+1} \end{matrix} \,\middle|\, q\right]_{M_n} [qb/x_{n+1}, qb/y_{n+1}; q]_\lambda, \\
&\left[\begin{matrix} q^{1-\lambda}e/c, qa/ex_{n+1}, qb/x_{n+1}, qb/y_{n+1} \\ c/e, qe/y_{n+1}, q/x_{n+1}, qa/x_{n+1} \end{matrix} \,\middle|\, q\right]_\lambda \\
&\quad = \left(\frac{-b}{c}\right)^\lambda \left[\begin{matrix} qb/y_{n+1}, qa/by_{n+1} \\ q/y_{n+1}, qa/y_{n+1} \end{matrix} \,\middle|\, q\right]_\lambda q^{-\binom{\lambda}{2}}
\end{aligned}$$

we obtain the following simplified transformation

$$\begin{aligned}
&{}_{2n+8}\psi_{8+2n}\left[\begin{matrix} q\sqrt{a}, -q\sqrt{a}, b, c, d, e, & \{x_\iota, y_\iota\}_{\iota=1}^{n+1} \\ \sqrt{a}, -\sqrt{a}, qa/b, qa/c, qa/d, qa/e, & \{qa/x_\iota, qa/y_\iota\}_{\iota=1}^{n+1} \end{matrix} \,\middle|\, q; \frac{q^{1-N-\lambda}a^2}{bcde}\right] \\
&\quad = \Omega[a:b,c,d,e]\left[\begin{matrix} qb/y_{n+1}, qa/by_{n+1} \\ q/y_{n+1}, qa/y_{n+1} \end{matrix} \,\middle|\, q\right]_\lambda \prod_{\iota=1}^{n}\left[\begin{matrix} x_\iota, x_\iota/a, qb/y_\iota, qa/by_\iota \\ x_\iota/b, bx_\iota/a, q/y_\iota, qa/y_\iota \end{matrix} \,\middle|\, q\right]_\infty \\
&\qquad \times \sum_{\tilde{m}} \sum_{\ell=0}^{\lambda} \frac{(qa/x_{n+1}y_{n+1}; q)_\ell}{(q;q)_\ell} \left[\begin{matrix} bc/a, bd/a, be/a \\ qb/x_{n+1}, qb/y_{n+1}, bcde/a^2 \end{matrix} \,\middle|\, q\right]_{\ell+M_n} q^{\ell+M_n} \\
&\qquad \times \prod_{\jmath=1}^{n} \frac{(qa/x_\jmath y_\jmath; q)_{m_\jmath}}{(q;q)_{m_\jmath}} \left[\begin{matrix} bx_{\jmath+1}/a, by_{\jmath+1}/a \\ qb/x_\jmath, qb/y_\jmath \end{matrix} \,\middle|\, q\right]_{M_\jmath} \left(\frac{qa}{x_{\jmath+1}y_{\jmath+1}}\right)^{M_\jmath}.
\end{aligned}$$

Recalling the finite condition $q^{1+\lambda}a/x_{n+1}y_{n+1} = 1$, we conclude that the last transformation is exactly the case of Theorem 4 with the $(n+1)$-pairs of extra well-poised parameters $\{x_k, y_k\}$ thanks to

$$\left[\begin{matrix} qb/y_{n+1}, qa/by_{n+1} \\ q/y_{n+1}, qa/y_{n+1} \end{matrix} \,\middle|\, q\right]_\lambda = \left[\begin{matrix} x_{n+1}, x_{n+1}/a, qb/y_{n+1}, qa/by_{n+1} \\ x_{n+1}/b, bx_{n+1}/a, q/y_{n+1}, qa/y_{n+1} \end{matrix} \,\middle|\, q\right]_\infty.$$

Applying the induction principle, this completes the proof of Theorem 4. □

5. Another well-poised ${}_{10}\psi_{10}$-series identity

By linearizing different pairs of factorial fractions, we will derive another well-poised ${}_{10}\psi_{10}$-series identity which can also be considered as a common generalization of both Bailey's ${}_6\psi_6$-series (3) and Shukla's formula (5a)–(5b).

Applying Lemma 1 to factorial fraction

$$\left[\begin{matrix} q^m u, & qa/u, & q^n v, & qa/v \\ q^{1-m}a/u, & u, & q^{1-n}a/v, & v \end{matrix} \,\middle|\, q\right]_k$$

we can manipulate the ${}_{10}\psi_{10}$-series as follows:

$$\begin{aligned}
&{}_{10}\psi_{10}\left[\begin{matrix} q\sqrt{a}, -q\sqrt{a}, b, c, d, e, q^m u, qa/u, q^n v, qa/v \\ \sqrt{a}, -\sqrt{a}, qa/b, qa/c, qa/d, qa/e, q^{1-m}a/u, u, q^{1-n}a/v, v \end{matrix} \,\middle|\, q; \frac{q^{1-m-n}a^2}{bcde}\right] \\
&\quad = \sum_{k=-\infty}^{+\infty} \frac{1-q^{2k}a}{1-a} \left[\begin{matrix} b, c, d, e \\ qa/b, qa/c, qa/d, qa/e \end{matrix} \,\middle|\, q\right]_k
\end{aligned}$$

$$\times\left(\frac{q^{1-m-n}a^2}{bcde}\right)^k\left[\begin{matrix}q^m u, qa/u, q^n v, qa/v\\ q^{1-m}a/u, u, q^{1-n}a/v, v\end{matrix}\Big| q\right]_k$$

$$=\left[\begin{matrix}q^m u, & q^m u/a, & uv/qa, & qv/u\\ v, & v/a, & q^{m-1}u^2/a, & q^{1+m}\end{matrix}\Big| q\right]_n\sum_{i=0}^n\frac{1-q^{1-m-n+2i}a/u^2}{1-q^{1-m-n}a/u^2}$$

$$\times\left[\begin{matrix}q^{1-m-n}a/u^2, q^{-m}v/u, q^{1-m-n}a/uv, qa/u, q/u, q^{-n}\\ q, q^{2-n}a/uv, q^{1-m-n}/u, q^{1-m-n}a/u, q^{2-m}a/u^2, qv/u\end{matrix}\Big| q\right]_i q^i$$

$$\times{}_8\psi_8\left[\begin{matrix}q\sqrt{a}, -q\sqrt{a}, b, c, d, e, q^{m+n-i}u, q^{1+i}a/u\\ \sqrt{a}, -\sqrt{a}, qa/b, qa/c, qa/d, qa/e, q^{-i}u, q^{1+i-m-n}a/u\end{matrix}\Big| q; \frac{q^{1-m-n}a^2}{bcde}\right]$$

provided that $|q^{1-m-n}a^2/bcde|<1$ for convergence. By means of Theorem 2, evaluating the last ${}_8\psi_8$-series

$${}_8\psi_8\left[\begin{matrix}q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d, & e, & q^{m+n-i}u, & q^{1+i}a/u\\ \sqrt{a}, & -\sqrt{a}, & qa/b, & qa/c, & qa/d, & qa/e, & q^{-i}u, & q^{1+i-m-n}a/u\end{matrix}\Big| q; \frac{q^{1-m-n}a^2}{bcde}\right]$$

$$=\Omega[a:b,c,d,e]\left[\begin{matrix}q^{-i}u/b, q^{-i}bu/a\\ q^{-i}u, q^{-i}u/a\end{matrix}\Big| q\right]_{m+n}$$

$$\times{}_4\phi_3\left[\begin{matrix}q^{-m-n}, bc/a, bd/a, be/a\\ q^{1+i-m-n}b/u, q^{-i}bu/a, bcde/a^2\end{matrix}\Big| q; q\right]$$

and then invoking the relation on factorial fractions

$$\left[\begin{matrix}q^{-i}u/b, q^{-i}bu/a\\ q^{-i}u, q^{-i}u/a\end{matrix}\Big| q\right]_{m+n}=\left[\begin{matrix}u/b, bu/a\\ u, u/a\end{matrix}\Big| q\right]_{m+n}$$

$$\times\left[\begin{matrix}qb/u, qa/bu, q^{1-m-n}/u, q^{1-m-n}a/u\\ q^{1-m-n}b/u, q^{1-m-n}a/bu, q/u, qa/u\end{matrix}\Big| q\right]_i,$$

we find the following simplified expression:

$${}_{10}\psi_{10}\left[\begin{matrix}q\sqrt{a}, -q\sqrt{a}, & b, c, d, e, q^m u, qa/u, q^n v, qa/v\\ \sqrt{a}, -\sqrt{a}, & qa/b, qa/c, qa/d, qa/e, q^{1-m}a/u, u, q^{1-n}a/v, v\end{matrix}\Big| q; \frac{q^{1-m-n}a^2}{bcde}\right] \tag{19a}$$

$$=\Omega[a:b,c,d,e]\left[\begin{matrix}u/b, & bu/a\\ u, & u/a\end{matrix}\Big| q\right]_{m+n}\left[\begin{matrix}q^m u, q^m u/a, uv/qa, qv/u\\ v, v/a, q^{m-1}u^2/a, q^{1+m}\end{matrix}\Big| q\right]_n \tag{19b}$$

$$\times\sum_{i=0}^n\frac{1-q^{1-m-n+2i}\frac{a}{u^2}}{1-q^{1-m-n}\frac{a}{u^2}}\left[\begin{matrix}q^{1-m-n}\frac{a}{u^2}, \frac{qa}{bu}, \frac{qb}{u}, q^{-m}\frac{v}{u}, q^{1-m-n}\frac{a}{uv}, q^{-n}\\ q, q^{1-m-n}\frac{a}{bu}, q^{1-m-n}\frac{b}{u}, q^{2-n}\frac{a}{uv}, \frac{qv}{u}, q^{2-m}\frac{a}{u^2}\end{matrix}\Big| q\right]_i q^i \tag{19c}$$

$$\times{}_4\phi_3\left[\begin{matrix}q^{-m-n}, & bc/a, & bd/a, & be/a\\ & q^{1+i-m-n}b/u, & q^{-i}bu/a, & bcde/a^2\end{matrix}\Big| q; q\right]. \tag{19d}$$

Writing the last ${}_4\phi_3$-series explicitly as the finite sum

$$\sum_{\ell=0}^{m+n}\left[\begin{matrix}q^{-m-n}, & bc/a, & bd/a, & be/a\\ q, & q^{1+i-m-n}b/u, & q^{-i}bu/a, & bcde/a^2\end{matrix}\Big| q\right]_\ell q^\ell$$

exchanging the summation order between (19c) and (19d), and then applying the factorial fraction relation

$$\frac{1}{[q^{1+i-m-n}b/u, q^{-i}bu/a; q]_\ell} = \frac{q^{i\ell}}{[q^{1-m-n}b/u, bu/a; q]_\ell}\left[\begin{matrix} q^{1-\ell}a/bu, q^{1-m-n}b/u \\ qa/bu, q^{1+\ell-m-n}b/u \end{matrix}\,\middle|\, q\right]_i,$$

we find that (19c)–(19d) is equal to the following double sum:

$$\sum_{\ell=0}^{m+n}\left[\begin{matrix} q^{-m-n}, & bc/a, & bd/a, & be/a \\ q, & q^{1-m-n}b/u, & bu/a, & bcde/a^2 \end{matrix}\,\middle|\, q\right]_\ell q^\ell$$

$$\times\, {}_8\phi_7\left[\begin{matrix} q^{1-m-n}\frac{a}{u^2}, q^{\frac{3-m-n}{2}}\frac{\sqrt{a}}{u}, -q^{\frac{3-m-n}{2}}\frac{\sqrt{a}}{u}, \frac{qb}{u}, q^{1-\ell}\frac{a}{bu}, q^{-m}\frac{v}{u}, q^{1-m-n}\frac{a}{uv}, q^{-n} \\ q^{\frac{1-m-n}{2}}\frac{\sqrt{a}}{u}, -q^{\frac{1-m-n}{2}}\frac{\sqrt{a}}{u}, q^{1-m-n}\frac{a}{bu}, q^{1+\ell-m-n}\frac{b}{u}, q^{2-n}\frac{a}{uv}, q\frac{v}{u}, q^{2-m}\frac{a}{u^2} \end{matrix}\,\middle|\, q; q^{1+\ell}\right].$$

The last ${}_8\phi_7$-series may be reformulated by means of (9a)–(9b) as:

$${}_8\phi_7\left[\begin{matrix} q^{1-m-n}\frac{a}{u^2}, q^{\frac{3-m-n}{2}}\frac{\sqrt{a}}{u}, -q^{\frac{3-m-n}{2}}\frac{\sqrt{a}}{u}, \frac{qb}{u}, q^{1-\ell}\frac{a}{bu}, q^{-m}\frac{v}{u}, q^{1-m-n}\frac{a}{uv}, q^{-n} \\ q^{\frac{1-m-n}{2}}\frac{\sqrt{a}}{u}, -q^{\frac{1-m-n}{2}}\frac{\sqrt{a}}{u}, q^{1-m-n}\frac{a}{bu}, q^{1+\ell-m-n}\frac{b}{u}, q^{2-n}\frac{a}{uv}, q\frac{v}{u}, q^{2-m}\frac{a}{u^2} \end{matrix}\,\middle|\, q; q^{1+\ell}\right]$$

$$= \left[\begin{matrix} q^{m-1}u^2/a, bv/a \\ q^m bu/a, uv/qa \end{matrix}\,\middle|\, q\right]_n {}_4\phi_3\left[\begin{matrix} q^{-n}, & q^{-m}v/u, & qb/u, & q^\ell bv/a \\ & qv/u, & q^{1+\ell-m-n}b/u, & bv/a \end{matrix}\,\middle|\, q; q\right].$$

Substituting the last two expressions into (19c)–(19d) and then combining the result with (19a)–(19b), we derive the following simplified transformation formula.

Theorem 7 *(The third extension of Shukla's ${}_8\psi_8$-summation formula). For $m, n \in \mathbb{N}_0$, suppose that $|q^{1-m-n}a^2/bcde| < 1$. There holds the bilateral series identity*:

$${}_{10}\psi_{10}\left[\begin{matrix} q\sqrt{a}, -q\sqrt{a}, b, c, d, e, q^m u, qa/u, q^n v, qa/v \\ \sqrt{a}, -\sqrt{a}, qa/b, qa/c, qa/d, qa/e, q^{1-m}a/u, u, q^{1-n}a/v, v \end{matrix}\,\middle|\, q; \frac{q^{1-m-n}a^2}{bcde}\right] \tag{20a}$$

$$= \Omega[a : b, c, d, e]\frac{(u/b; q)_{m+n}}{(q; q)_{m+n}}\left[\begin{matrix} q, bu/a \\ u, u/a \end{matrix}\,\middle|\, q\right]_m \left[\begin{matrix} qv/u, bv/a \\ v, v/a \end{matrix}\,\middle|\, q\right]_n \tag{20b}$$

$$\times \sum_{\ell=0}^{m+n}\left[\begin{matrix} q^{-m-n}, & bc/a, & bd/a, & be/a \\ q, & q^{1-m-n}b/u, & bu/a, & bcde/a^2 \end{matrix}\,\middle|\, q\right]_\ell q^\ell \tag{20c}$$

$$\times\, {}_4\phi_3\left[\begin{matrix} q^{-n}, & q^{-m}v/u, & qb/u, & q^\ell bv/a \\ & qv/u, & q^{1+\ell-m-n}b/u, & bv/a \end{matrix}\,\middle|\, q; q\right]. \tag{20d}$$

We remark that for $n = 0$, this theorem also reduces to Theorem 2. However, we have failed to show the equivalence between Theorem 2 and Theorem 7. Another problem is to find a right approach to iterate Theorem 7 for obtaining Milne-like transformation.

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