# Communications in Algebra

## Weak Global Dimension of Coherent Rings

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# WEAK GLOBAL DIMENSION OF COHERENT RINGS

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*In this article, we study the weak global dimension of coherent rings in terms of the left FP-injective resolutions of modules. Let R be a left coherent ring and $\mathscr{FI}$ the class of all FP-injective left R-modules. It is shown that $wD(R) \leq n$ $(n \geq 1)$ if and only if every nth $\mathscr{FI}$-syzygy of a left R-module is FP-injective; and $wD(R) \leq n$ $(n \geq 2)$ if and only if every $(n-2)$th $\mathscr{FI}$-syzygy in a minimal $\mathscr{FI}$-resolution of a left R-module has an FP-injective cover with the unique mapping property. Some results for the weak global dimension of commutative coherent rings are also given.*

*Key Words:*   *FP*-injective dimension; (Pre)cover; Syzygy; Weak global dimension.

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## 1.  INTRODUCTION

We first recall some known notions and facts needed in the article.

Let $R$ be a ring. A left $R$-module $M$ is called *FP-injective* (or *absolutely pure*) (Megibben, 1970; Stenström, 1970) if $\mathrm{Ext}^1(N, M) = 0$ for any finitely presented left $R$-module $N$. The *FP-injective dimension of M*, denoted by *FP-id(M)*, is defined to be the smallest non-negative integer $n$ such that $\mathrm{Ext}^{n+1}(F, M) = 0$ for any finitely presented left $R$-module $F$. If no such $n$ exists, set $FP\text{-}id(M) = \infty$.

Let $\mathscr{C}$ be a class of $R$-modules and $M$ an $R$-module. Following Enochs (1981), we say that a homomorphism $\phi : C \to M$ is a *$\mathscr{C}$-precover* if $C \in \mathscr{C}$ and the abelian group homomorphism $\mathrm{Hom}(C', \phi) : \mathrm{Hom}(C', C) \to \mathrm{Hom}(C', M)$ is surjective for each $C' \in \mathscr{C}$. A $\mathscr{C}$-precover $\phi : C \to M$ is said to be a *$\mathscr{C}$-cover* if every endomorphism $g : C \to C$ such that $\phi g = \phi$ is an isomorphism. A $\mathscr{C}$-cover $\phi : C \to M$ is said to have the *unique mapping property* (Ding, 1996) if for any homomorphism $f : C' \to M$ with $C' \in \mathscr{C}$, there is a unique homomorphism $g : C' \to C$ such that $\phi g = f$. Dually we have the definitions of a $\mathscr{C}$-(pre)envelope (with the

unique mapping property). $\mathscr{C}$-covers ($\mathscr{C}$-envelopes) may not exist in general, but if they exist, they are unique up to isomorphism.

In what follows, we write $\mathscr{FI}$ for the class of all *FP*-injective left *R*-modules. It has been recently proven that every left *R*-module has an *FP*-injective (pre)cover over a left coherent ring *R* (see Pinzon, 2005), so every left *R*-module *N* has a *left $\mathscr{FI}$-resolution*, that is, there is a $\mathrm{Hom}(\mathscr{FI}, -)$ exact complex $\cdots \to F_1 \to F_0 \to N \to 0$ (not necessarily exact) with each $F_i$ *FP*-injective.

Let $K_0 = N$, $K_1 = \ker(F_0 \to N)$, $K_i = \ker(F_{i-1} \to F_{i-2})$ for $i \geq 2$. The *n*th kernel $K_n$ ($n \geq 0$) is called the *n*th $\mathscr{FI}$-*syzygy* of *N*.

A left $\mathscr{FI}$-resolution $\cdots \to F_1 \to F_0 \to N \to 0$ is said to be *minimal* if each $F_i \to K_i$ is an *FP*-injective cover.

It is well known that every left *R*-module has an injective (pre)cover if and only if *R* is a left Noetherian ring (see Enochs, 1981). Enochs and Jenda investigated the global dimension of a left Noetherian ring using the left injective resolutions of left *R*-modules (see Enochs and Jenda, 1985, 1991, 2000; Jenda, 1986). The main purpose of this article is to generalize their results to the case when *R* is a left coherent ring. We shall study the weak global dimension of a left coherent ring *R* in terms of the left *FP*-injective resolutions of left *R*-modules. Let *R* be a left coherent ring. It is shown that $wD(R) \leq n$ ($n \geq 1$) if and only if every *n*th $\mathscr{FI}$-syzygy of a left *R*-module is *FP*-injective if and only if every $(n-1)$th $\mathscr{FI}$-syzygy of a left *R*-module has an *FP*-injective cover which is a monomorphism; and $wD(R) \leq n$ ($n \geq 2$) if and only if every $(n-2)$th $\mathscr{FI}$-syzygy in a minimal left $\mathscr{FI}$-resolution of a left *R*-module has an *FP*-injective cover with the unique mapping property. If *R* is a commutative coherent ring, we prove that $wD(R) \leq n$ ($n \geq 2$) if and only if every $(n-2)$th $\mathscr{FI}$-syzygy in a minimal left $\mathscr{FI}$-resolution of a pure-injective *R*-module has an (*FP*-)injective cover with the unique mapping property if and only if $FP\text{-}id(R^+ \otimes M) \leq n - 2$ for any *R*-module *M* if and only if $fd(\mathrm{Hom}(R^+, M)) \leq n - 2$ for any (pure-injective) *R*-module *M*.

Throughout this article, *R* is an associative ring with identity and all modules are unitary. $wD(R)$ stands for the weak global dimension of a ring *R*. $_RM$ denotes a left *R*-module. For an *R*-module *M*, the character module $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is denoted by $M^+$, and $fd(M)$ is the flat dimension of *M*. Let *M* and *N* be two *R*-modules. $\mathrm{Hom}(M, N)$ (resp., $M \otimes N$) means $\mathrm{Hom}_R(M, N)$ (resp., $M \otimes_R N$), and similarly $\mathrm{Ext}^n(M, N)$ denotes $\mathrm{Ext}_R^n(M, N)$ for an integer $n \geq 1$. For unexplained concepts and notations, we refer the reader to Enochs and Jenda (2000), Rotman (1979), and Xu (1996).

## 2. WEAK GLOBAL DIMENSION OF COHERENT RINGS

We begin with the following lemmas.

**Lemma 2.1.** *Let R be a left coherent ring and M a left R-module. Then FP-$id(M) \leq n$ ($n \geq 0$) if and only if for every left $\mathscr{FI}$-resolution $\cdots \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to N \to 0$ of any left R-module N, $\mathrm{Hom}(M, F_n) \to \mathrm{Hom}(M, K_n)$ is an epimorphism, where $K_n$ is the nth $\mathscr{FI}$-syzygy of N.*
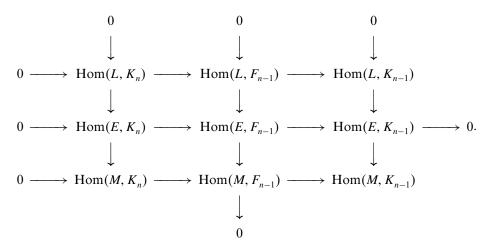
*Proof.* The proof is modelled on that of Jenda (1986, Lemma 2.2).

We proceed by induction on $n$. Let $n = 0$. If $M$ is $FP$-injective, then it is clear that $\text{Hom}(M, F_0) \to \text{Hom}(M, K_0)$ is an epimorphism. Conversely, put $N = M$. Then $\text{Hom}(M, F_0) \to \text{Hom}(M, M)$ is an epimorphism. Thus $M$ is isomorphic to a direct summand of $F_0$, and so $M$ is $FP$-injective.

Let $n \geq 1$. By Göbel and Trlifaj (2006, Theorem 4.1.6), there is an exact sequence $0 \to M \to E \to L \to 0$ with $E$ $FP$-injective and $\text{Ext}^1(L, G) = 0$ for all $FP$-injective left $R$-modules $G$. Thus we have the following exact commutative diagrams:

$$
\begin{array}{ccccc}
\text{Hom}(E, F_n) & \longrightarrow & \text{Hom}(E, K_n) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \\
\text{Hom}(M, F_n) & \longrightarrow & \text{Hom}(M, K_n) & & \\
\downarrow & & & & \\
0 & & & &
\end{array}
$$

and

$$
\begin{array}{ccccccc}
& & 0 & & 0 & & 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 \to & \text{Hom}(L, K_n) & \to & \text{Hom}(L, F_{n-1}) & \to & \text{Hom}(L, K_{n-1}) & \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \to & \text{Hom}(E, K_n) & \to & \text{Hom}(E, F_{n-1}) & \to & \text{Hom}(E, K_{n-1}) & \to 0. \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \to & \text{Hom}(M, K_n) & \to & \text{Hom}(M, F_{n-1}) & \to & \text{Hom}(M, K_{n-1}) & \\
& & & \downarrow & & & \\
& & & 0 & & &
\end{array}
$$

Thus $FP\text{-}id(M) \leq n$ if and only if $FP\text{-}id(L) \leq n - 1$ by Stenström (1970, Lemma 3.1) if and only if $\text{Hom}(L, F_{n-1}) \to \text{Hom}(L, K_{n-1})$ is an epimorphism by induction if and only if $\text{Hom}(E, K_n) \to \text{Hom}(M, K_n)$ is an epimorphism by the second diagram if and only if $\text{Hom}(M, F_n) \to \text{Hom}(M, K_n)$ is an epimorphism by the first diagram. □

**Lemma 2.2.** *Let $R$ be a left coherent ring and $M$ a left $R$-module. If $\text{Ext}^1(F, M) = 0$ for all $FP$-injective left $R$-modules $F$, then $M$ has an $FP$-injective cover $E \to M$ with $E$ injective.*

*Proof.* Let $f : F \to M$ be an $FP$-injective cover of $M$. Consider the short exact sequence $0 \to F \xrightarrow{i} E \to L \to 0$ with $E$ injective. Note that $L$ is $FP$-injective by Stenström (1970, Lemma 3.1). So there exists $g : E \to M$ such that $gi = f$ since

$\text{Ext}^1(L, M) = 0$. Thus there exists $h : E \to F$ such that $fh = g$ since $f$ is a cover. Therefore $fhi = f$, and hence $hi$ is an isomorphism. It follows that $F$ is injective. $\square$

We are now in a position to prove the following theorem.

**Theorem 2.3.** *The following are equivalent for a left coherent ring $R$ and $n \geq 1$:*

(1) $wD(R) \leq n$;
(2) *Every $n$th $\mathcal{FI}$-syzygy of a left $R$-module is FP-injective*;
(3) *Every $n$th $\mathcal{FI}$-syzygy of a left $R$-module has FP-injective dimension $\leq 1$*;
(4) *Every $(n-1)$th $\mathcal{FI}$-syzygy of a left $R$-module has an FP-injective cover which is a monomorphism*;
(5) *For every left $\mathcal{FI}$-resolution $\cdots \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to N \to 0$ of a left $R$-module $N$, $\text{Hom}(M, F_n) \to \text{Hom}(M, K_n)$ is an epimorphism for any pure-injective left $R$-module $M$, where $K_n$ is the $n$th $\mathcal{FI}$-syzygy of $N$*;
(6) *$wD(R) < \infty$ and every left $\mathcal{FI}$-resolution $\cdots \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to N \to 0$ of a left $R$-module $N$ is exact at $F_i$, $i \geq n-1$, where $F_{-1} = N$*;
(7) *$wD(R) < \infty$ and every $n$th $\mathcal{FI}$-syzygy of a left $R$-module $N$ has an FP-injective cover which is an epimorphism*;
*Moreover, if $n \geq 2$, then the above conditions are equivalent to*:
(8) *Every $(n-2)$th $\mathcal{FI}$-syzygy in a minimal left $\mathcal{FI}$-resolution of a left $R$-module has an FP-injective cover with the unique mapping property.*

*Proof.* $(1) \Rightarrow (2)$ Let $K_n$ be an $n$th $\mathcal{FI}$-syzygy of a left $R$-module. Then *FP-$id(K_n) \leq wD(R) \leq n$* by Stenström (1970, Theorem 3.3), and so $\text{Hom}(K_n, F_n) \to \text{Hom}(K_n, K_n)$ is an epimorphism by Lemma 2.1. Therefore, $K_n$ is *FP*-injective.

$(2) \Rightarrow (3)$ is trivial.

$(3) \Rightarrow (4)$ Let $f : F_{n-1} \to K_{n-1}$ be an *FP*-injective precover of $K_{n-1}$ and $K_n = \ker(f)$. Then *FP-$id(K_n) \leq 1$* by (3). So $\text{im}(f)$ is *FP*-injective by Stenström (1970, Lemma 3.1). Thus the inclusion $\text{im}(f) \to K_{n-1}$ is an *FP*-injective cover which is a monomorphism.

$(4) \Rightarrow (2)$ Let $\cdots \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to N \to 0$ be any left $\mathcal{FI}$-resolution of a left $R$-module $N$ and $K_{n-1} = \ker(F_{n-2} \to F_{n-3})$, $K_n = \ker(F_{n-1} \to F_{n-2}) = \ker(F_{n-1} \to K_{n-1})$. Since $K_{n-1}$ has a monic *FP*-injective cover $E \to K_{n-1}$ by (4), we have $K_n \oplus E \cong F_{n-1}$ by Enochs and Jenda (2000, Lemma 8.6.3). Hence $K_n$ is *FP*-injective.

$(2) \Rightarrow (5)$ For any left $\mathcal{FI}$-resolution $\cdots \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to N \to 0$ of a left $R$-module $N$, $F_n \to K_n$ is a split epimorphism by (2). So $\text{Hom}(M, F_n) \to \text{Hom}(M, K_n)$ is an epimorphism for any (pure-injective) left $R$-module $M$.

$(5) \Rightarrow (1)$ Let $M$ be any right $R$-module. Then $M^+$ is pure-injective. Hence *FP-$id(M^+) \leq n$* by (5) and Lemma 2.1. Thus $fd(M) \leq n$ by Fieldhouse (1972, Theorem 2.1), and so $wD(R) \leq n$.

$(1) \Rightarrow (6)$   Since $FP\text{-}id(_RR) \leq wD(R) \leq n$, $\mathrm{Hom}(_RR, F_n) \to \mathrm{Hom}(_RR, K_n)$ is an epimorphism by Lemma 2.1, that is, $F_n \to K_n$ is an epimorphism. It follows that $F_n \to F_{n-1} \to F_{n-2}$ is exact. In addition, $FP\text{-}id(_RR) \leq k$ for any $k \geq n+1$ by (1). So $F_k \to F_{k-1} \to F_{k-2}$ is exact, and hence (6) holds.

$(6) \Rightarrow (7)$   is clear since $F_n \to F_{n-1} \to F_{n-2}$ is exact.

$(7) \Rightarrow (1)$   Note that $FP\text{-}id(_RR) \leq n$ by (7) and Lemma 2.1. Thus $wD(R) = FP\text{-}id(_RR) \leq n$ by Stenström (1970, Proposition 3.5) since $wD(R) < \infty$.

$(4) \Rightarrow (8)$   Suppose that $F_{n-3} \to F_{n-4} \to \cdots \to F_1 \to F_0 \to N \to 0$ is a partial minimal left $\mathscr{FI}$-resolution of a left $R$-module $N$ with $K_{n-2} = \ker(F_{n-3} \to F_{n-4})$. Let $f : F_{n-2} \to K_{n-2}$ be the $FP$-injective cover of $K_{n-2}$. Then $K_{n-1} = \ker(F_{n-2} \to F_{n-3})$ has a monic $FP$-injective cover $i : F_{n-1} \to K_{n-1}$ by (4). Note that $\mathrm{Ext}^1(G, K_{n-1}) = 0$ for any $FP$-injective left $R$-module $G$ by Wakamatsu's Lemma (see Xu, 1996, Lemma 2.1.1). Thus $F_{n-1}$ is injective by Lemma 2.2. But $K_{n-1}$ has no nonzero injective submodule by Xu (1996, Corollary 1.2.8). Thus $F_{n-1} = 0$, and hence $\mathrm{Hom}(F, K_{n-1}) \cong \mathrm{Hom}(F, F_{n-1}) = 0$ for any $FP$-injective left $R$-module $F$. Therefore, the exact sequence $0 \to K_{n-1} \to F_{n-2} \xrightarrow{f} K_{n-2}$ yields the desired exactness of $0 \to \mathrm{Hom}(F, F_{n-2}) \xrightarrow{f_*} \mathrm{Hom}(F, K_{n-2}) \to 0$.

$(8) \Rightarrow (2)$   Let $\cdots \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to N \to 0$ be any left $\mathscr{FI}$-resolution of a left $R$-module $N$ with $K_n = \ker(F_{n-1} \to F_{n-2})$. By (8), $N$ has a left $\mathscr{FI}$-resolution of the form $0 \to E_{n-2} \to E_{n-3} \to \cdots \to E_1 \to E_0 \to N \to 0$. Thus by Enochs and Jenda (2000, Corollary 8.6.4), we have $K_n \oplus F_{n-2} \oplus E_{n-3} \oplus \cdots \cong F_{n-1} \oplus E_{n-2} \oplus F_{n-3} \oplus \cdots$. Therefore, $K_n$ is $FP$-injective, as required.   $\square$

It is known that $R$ is a left Noetherian ring with left global dimension at most 2 if and only if every left $R$-module has an injective cover with the unique mapping property (see Enochs and Jenda, 2000, Remark 8.4.18); $R$ is a left Noetherian and left hereditary ring if and only if every left $R$-module has an injective cover which is a monomorphism (see Enochs and Jenda, 1991, Corollary 3.4 or Rada and Saorin, 1998, Corollary 4.12). Now we have the following corollary.

**Corollary 2.4.**   *Let $R$ be a ring. Then*:

(1) *$R$ is left coherent and $wD(R) \leq 2$ if and only if every left $R$-module has an FP-injective cover with the unique mapping property*;
(2) *$R$ is left semihereditary if and only if every left $R$-module has an FP-injective cover which is a monomorphism.*

*Proof.*   (1)   The necessity follows from Theorem 2.3. For the sufficiency, we shall prove that the class of $FP$-injective left $R$-modules is closed under direct limits. Let $\{C_i, \varphi^i_j\}$ be a direct system of left $R$-modules with each $C_i$ $FP$-injective. By hypothesis, $\varinjlim C_i$ has an $FP$-injective cover $\alpha : E \to \varinjlim C_i$ with the unique mapping property. Let $\alpha_i : C_i \to \varinjlim C_i$ satisfy $\alpha_i = \alpha_j \varphi^i_j$ whenever $i \leq j$. Then there exists $f_i : C_i \to E$ such that $\alpha_i = \alpha f_i$ for any $i$. It follows that $\alpha f_i = \alpha f_j \varphi^i_j$, and so $f_i = f_j \varphi^i_j$ whenever $i \leq j$. Therefore, by the definition of direct limits, there exists $\beta : \varinjlim C_i \to E$ such that $f_i = \beta \alpha_i$. So $(\alpha \beta)\alpha_i = \alpha(\beta \alpha_i) = \alpha f_i = \alpha_i$ for any $i$. Therefore $\alpha\beta = 1_{\varinjlim C_i}$

by the definition of direct limits, and hence $\varinjlim C_i$ is a direct summand of $E$. So $\varinjlim C_i$ is $FP$-injective. Thus $R$ is a left coherent ring by Stenström (1970, Theorem 3.2), and so $wD(R) \leq 2$ by Theorem 2.3.

(2)   The necessity comes from Theorem 2.3. Conversely, every left $R$-module has an $FP$-injective cover with the unique mapping property since every left $R$-module has a monic $FP$-injective cover. So $R$ is left coherent by (1), and hence $wD(R) \leq 1$ by Theorem 2.3. Thus $R$ is left semihereditary. $\qquad\square$

**Remark 2.5.**  (1) Corollary 2.4 (1) may be viewed as the dual of Asensio Mayor and Martínez Hernández (1993, Proposition 2.1) which says that a ring $R$ is left coherent and $wD(R) \leq 2$ if and only if every left $R$-module has a flat envelope with the unique mapping property.

(2)   Corollary 2.4 (2) has been proven in Chen and Ding (1996) and Rada and Saorin (1998) in a different way.

Next we shall study the weak global dimension of commutative coherent rings.

**Lemma 2.6.**  *Let $R$ be a commutative ring. Then $R$ is a coherent ring if and only if* $\mathrm{Hom}(R^+, E)$ *is flat for any injective $R$-module $E$.*

*Proof.*   If $R$ is coherent, then $\mathrm{Hom}(R^+, E)$ is flat by Matlis (1982, Theorem 1) (for $R^+$ is injective).

Conversely, we shall show that any direct product $\Pi R$ of $R$ is flat. Indeed, since $R$ is a pure submodule of $R^{++}$, $\Pi R$ is a pure submodule of $\Pi R^{++}$ by Cheatham and Stone (1981, Lemma 1(2)). Note that $\Pi R^{++} \cong \Pi \mathrm{Hom}(R^+ \otimes R, \mathbb{Q}/\mathbb{Z}) \cong \Pi \mathrm{Hom}(R^+, R^+) \cong \mathrm{Hom}(R^+, \Pi R^+)$. Thus $\Pi R^{++}$ is flat by hypothesis, and hence $\Pi R$ is flat. So $R$ is coherent. $\qquad\square$

**Theorem 2.7.**  *The following are equivalent for a commutative coherent ring $R$ and an integer $n \geq 2$:*

(1) $wD(R) \leq n$;
(2) *Every $(n-2)$th $\mathcal{FI}$-syzygy in a minimal left $\mathcal{FI}$-resolution of a pure-injective $R$-module has an FP-injective cover with the unique mapping property*;
(3) *Every $(n-2)$th $\mathcal{FI}$-syzygy in a minimal left $\mathcal{FI}$-resolution of a pure-injective $R$-module has an injective cover with the unique mapping property*;
(4) *$FP\text{-}id(R^+ \otimes M) \leq n - 2$ for any $R$-module $M$*;
(5) *$fd(\mathrm{Hom}(R^+, M)) \leq n - 2$ for any (pure-injective) $R$-module $M$*;
(6) *$wD(R) < \infty$ and $FP\text{-}id(\mathrm{Hom}(R^+, E)) \leq n$ for any injective $R$-module $E$*;
(7) *$wD(R) < \infty$ and $fd(R^+ \otimes M) \leq n$ for any flat $R$-module $M$*.

*Proof.*   (1) $\Rightarrow$ (2) follows from Theorem 2.3.

(2) $\Rightarrow$ (3)   Let $F_{n-3} \to \cdots \to F_1 \to F_0 \to M \to 0$ be a partial minimal left $\mathcal{FI}$-resolution of a pure-injective $R$-module $M$. By (2), $K_{n-2} = \ker(F_{n-3} \to F_{n-4})$ has an $FP$-injective cover $f : F_{n-2} \to K_{n-2}$ with the unique mapping property.

Suppose $n = 2$. There is a pure exact sequence $0 \to F_0 \overset{i}{\to} E \to L \to 0$ with $E$ injective. Since $M$ is pure-injective, there exists $g : E \to M$ such that $gi = f$. It is easy to check that $g : E \to M$ is an $FP$-injective precover of $M$. So $F_0$ is injective by Enochs and Jenda (2000, Proposition 5.1.2).

Suppose $n > 2$. Then $\mathrm{Ext}^1(G, K_{n-2}) = 0$ for any $FP$-injective $R$-module $G$ by Wakamatsu's lemma. So $F_{n-2}$ is injective by Lemma 2.2.

$(3) \Rightarrow (4)$　Let $M$ be any $R$-module and $F_{n-3} \to \cdots \to F_1 \to F_0 \to M^+ \to 0$ be a minimal left $\mathscr{FI}$-resolution of $M^+$. Then $K_{n-2} = \ker(F_{n-3} \to F_{n-4})$ has an injective cover $F_{n-2} \to K_{n-2}$ with the unique mapping property by (3). Thus we get the following exact sequence

$$0 \to \mathrm{Hom}(R^+, F_{n-2}) \to \mathrm{Hom}(R^+, F_{n-3}) \to \cdots$$
$$\to \mathrm{Hom}(R^+, F_0) \to \mathrm{Hom}(R^+, M^+) \to 0.$$

Since each $\mathrm{Hom}(R^+, F_i)$ is flat by Lemma 2.6, $fd(\mathrm{Hom}(R^+, M^+)) \le n - 2$. Note that $\mathrm{Hom}(R^+, M^+) \cong (R^+ \otimes M)^+$. Thus, by Fieldhouse (1972, Theorem 2.2), we have

$$FP - id(R^+ \otimes M) = fd((R^+ \otimes M)^+) = fd(\mathrm{Hom}(R^+, M^+)) \le n - 2.$$

$(4) \Rightarrow (1)$　Let $M$ be a finitely presented $R$-module. Then $R^+ \otimes M \cong (\mathrm{Hom}(M, R))^+$ by Colby (1975, Lemma 2), and so

$$fd(\mathrm{Hom}(M, R)) = FP - id(\mathrm{Hom}(M, R)^+) = FP - id(R^+ \otimes M) \le n - 2$$

by (4) and Fieldhouse (1972, Theorem 2.1). Therefore, $wD(R) \le n$ by Jones and Teply (1982, Lemma 4).

$(1) \Rightarrow (5)$　Let $M$ be any $R$-module. Then there is an exact sequence $0 \to M \to E^0 \to E^1$ with $E^0$ and $E^1$ injective. So we obtain the exact sequence

$$0 \to \mathrm{Hom}(R^+, M) \to \mathrm{Hom}(R^+, E^0) \to \mathrm{Hom}(R^+, E^1) \to L \to 0.$$

Thus $fd(\mathrm{Hom}(R^+, M)) \le n - 2$ since $fd(L) \le n$ by (1) and each $\mathrm{Hom}(R^+, E^i)$ is flat by Lemma 2.6.

$(5) \Rightarrow (4)$　For any $R$-module $M$, we have $FP\text{-}id(R^+ \otimes M) = fd((R^+ \otimes M)^+) = fd(\mathrm{Hom}(R^+, M^+)) \le n - 2$ by (5).

$(1) \Rightarrow (6)$　Let $E$ be any injective $R$-module. Then $(\mathrm{Hom}(R^+, E))^+$ is injective since $\mathrm{Hom}(R^+, E)$ is flat by Lemma 2.6. Thus there exists a split exact sequence $0 \to (\mathrm{Hom}(R^+, E))^+ \to \Pi R^+$. Note that $fd(R^+) \le n$ by (1), and so $fd(\Pi R^+) \le n$. Thus $fd(\mathrm{Hom}(R^+, E))^+ \le n$, and hence $FP\text{-}id(\mathrm{Hom}(R^+, E)) = fd(\mathrm{Hom}(R^+, E))^+ \le n$.

$(6) \Rightarrow (7)$　Let $M$ be any flat $R$-module. Then we have $FP\text{-}id((R^+ \otimes M)^+) = FP\text{-}id(\mathrm{Hom}(R^+, M^+)) \le n$ by (6). Thus $fd(R^+ \otimes M) = FP\text{-}id((R^+ \otimes M)^+) \le n$.

$(7) \Rightarrow (1)$　Note that $FP\text{-}id(R) = fd(R^+) = fd(R^+ \otimes R) \le n$ by (7). So $wD(R) = FP\text{-}id(R) \le n$ by Stenström (1970, Proposition 3.5) since $wD(R) < \infty$. $\quad\square$

Letting $n = 2$ in Theorem 2.7, we have the following corollary.

**Corollary 2.8.** *The following are equivalent for a commutative coherent ring $R$:*

(1) $wD(R) \leq 2$;
(2) *Every pure-injective $R$-module has an $(FP\text{-})$injective cover with the unique mapping property*;
(3) $R^+ \otimes M$ *is FP-injective for any $R$-module $M$*;
(4) $\mathrm{Hom}(R^+, M)$ *is flat for any $(pure\text{-}injective)$ $R$-module $M$.*

We conclude the article with the following corollary.

**Corollary 2.9.** *Let $R$ be a commutative ring. Then $R$ is coherent and $wD(R) \leq 2$ if and only if $\mathrm{Hom}(R^+, M)$ is flat for any $(pure\text{-}injective)$ $R$-module $M$.*

*Proof.* The result follows from Lemma 2.6 and Corollary 2.8. $\qquad\square$

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## REFERENCES

Asensio Mayor, J., Martínez Hernández, J. (1993). On flat and projective envelopes. *J. Algebra* 160:434–440.

Cheatham, T. J., Stone, D. R. (1981). Flat and projective character modules. *Proc. Amer. Math. Soc.* 81(2):175–177.

Chen, J. L., Ding, N. Q. (1996). A note on existence of envelopes and covers. *Bull. Austral. Math. Soc.* 54:383–390.

Colby, R. R. (1975). Rings which have flat injective modules. *J. Algebra* 35:239–252.

Ding, N. Q. (1996). On envelopes with the unique mapping property. *Comm. Algebra* 24(4):1459–1470.

Enochs, E. E. (1981). Injective and flat covers, envelopes and resolvents. *Israel J. Math.* 39:189–209.

Enochs, E. E., Jenda, O. M. G. (1985). Balanced functors applied to modules. *J. Algebra* 92:303–310.

Enochs, E. E., Jenda, O. M. G. (1991). Resolvents and dimensions of modules and rings. *Arch. Math.* 56:528–532.

Enochs, E. E., Jenda, O. M. G. (2000). *Relative Homological Algebra*. Berlin-New York: Walter de Gruyter.

Fieldhouse, D. J. (1972). Character modules, dimension and purity. *Glasgow Math. J.* 13:144–146.

Göbel, R., Trlifaj, J. (2006). *Approximations and Endomorphism Algebras of Modules*. GEM 41. Berlin-New York: Walter de Gruyter.

Jenda, O. M. G. (1986). Injective and flat dimensions. *Math. Japonica* 31(2):205–209.

Jones, M. F., Teply, M. L. (1982). Coherent rings of finite weak global dimension. *Comm. Algebra* 10(5):493–503.

Matlis, E. (1982). Commutative coherent rings. *Canad. J. Math.* 34(6):1240–1244.

Megibben, C. (1970). Absolutely pure modules. *Proc. Amer. Math. Soc.* 26:561–566.

Pinzon, K. R. (2005). Absolutely Pure Modules. University of Kentucky, Thesis.

Rada, J., Saorin, M. (1998). Rings characterized by (pre)envelopes and (pre)covers of their modules. *Comm. Algebra* 26(3):899–912.

Rotman, J. J. (1979). *An Introduction to Homological Algebra*. New York: Academic Press.

Stenström, B. (1970). Coherent rings and *FP*-injective modules. *J. London Math. Soc.* 2: 323–329.

Xu, J. (1996). *Flat Covers of Modules*. Lecture Notes in Math. 1634. Berlin-Heidelberg-New York: Springer-Verlag.