Nonlinear Dyn (2009) 56: 213–221
DOI 10.1007/s11071-008-9393-8

ORIGINAL PAPER

# Approximate stationary solution and stochastic stability for a class of differential equations with parametric colored noise

**Huiqing Zhang · Yong Xu · Wei Xu**

Received: 18 February 2008 / Accepted: 1 July 2008 / Published online: 20 August 2008


**Abstract** This paper aims to study a class of differential equations with parametric Gaussian colored noise. We present the general framework to get the solvability conditions of the approximate stationary probability density function, which is determined by the Fokker-Planck-Kolmogorov (FPK) equations. These equations are derived using the stochastic averaging method and the operator theory with the perturbation technique. An illustrative example is proposed to demonstrate the procedure of our proposed method. The analytical expression of approximate stationary probability density function is obtained. Numerical simulation is carried out to verify the analytical results and excellent agreement can be easily found. The FPK equation for the probability density function of order $\varepsilon^0$ is used to examine the almost-sure stability for the amplitude process. Finally, the stability in probability of the amplitude process is investigated by Lin and Cai's method.

H. Zhang · Y. Xu · W. Xu (✉)
Department of Applied Mathematics, Northwestern Polytechnical University, Xi'an 710072, China
e-mail: weixu@nwpu.edu.cn

H. Zhang
e-mail: huiqingzhang@nwpu.edu.cn

Y. Xu
e-mail: hsux3@163.com

**Keywords** Gaussian colored noise · Stochastic averaging · Operator theory · Fokker-Planck-Kolmogorov (FPK) · Stationary probability density · Stochastic stability

## 1 Introduction

In this paper, we consider a class of differential equations with parametric colored noise of the form

$$\ddot{x} + w_0^2 x + \varepsilon^2 \gamma g(\dot{x}) + \varepsilon^2 f(x) p(w_0 t) = \varepsilon h(x, \dot{x}) u(t) \tag{1}$$

where $x$ is the displacement transverse to the ideal orbit of the particle, $w_0$ is natural frequency, $\varepsilon$ is a small parameter, $u(t)$ is an exponentially correlated Gaussian noise of vanishing mean, i.e. $\langle u(t)\rangle = 0$, $\langle u(t)u(s)\rangle = \frac{D}{\tau} e^{-\frac{|t-s|}{\tau}}$. The functions $f$, $g$ and $h$ are generally analytic, $p(w_0 t)$ is a periodic function in $t$ and dots represent differentiation with respect to time $t$.

The deterministic case of (1) appears in many fields of applied science and practical problems, such as particle accelerators, colliding particle beams in high-energy accelerators [1–7] and rotor dynamics [8–18]. The so-called beam–beam interaction model has been examined by many authors for both deterministic and random cases or for both real and complex cases (see [8–18] and references therein). For example, Mahmoud [14] studied model (1) under additive random noise using stochastic averaging method, Xu et al. [16] examined it in the presence of random narrow-band excitation using revised method of multiple scale, and Xu et al. [15] explored the complex case of model (1) with the random broadband excitation by using the standard stochastic averaging.

The stochastic averaging method was proposed by Stratonovich [19] and it has been proven to be a useful technique for analyzing certain classes of random differential equations [20–23]. Another version of stochastic averaging is based on a perturbation theoretic approach to the FPK equation [24–27]. Recently, the authors investigated the complex beam–beam interaction models under additive Gaussian colored noise using the stochastic averaging method and perturbation technique [17]. Up to now, many results of random dynamics have been obtained under white noise especially Gaussian white noise, but little is known about colored noise [17, 28, 29]. For more knowledge about colored noise we refer the readers to [30, 31].

In this paper, model (1) in the presence of parametric Gaussian colored noise is studied by using the method of the second version of stochastic averaging combined with the perturbation technique. We obtain the approximate stationary probability density function of order $\varepsilon^1$ and we consider different kinds of stochastic stability for a special case of (1).

The paper is organized as following. In Sect. 2, we give an outline of our technique which is based on the stochastic averaging and perturbation theory. In Sect. 3, choosing $g(\dot{x}) = \dot{x}^3$, $f(x) = x^3$ and $p(w_0 t) = \cos w_0 t$, we derive the approximate stationary probability density function of order $\varepsilon^1$. The stochastic stability of the amplitude process is examined. We carry out the numerical simulation to verify the theoretical results and excellent agreement can be easily found. Finally, Sect. 4 contains our conclusions.

## 2 Outline of the analytical technique

Assume the process $u(t)$ is a continuous Markovian process driven by Gaussian white noise $\Gamma(t)$, that is

$$\dot{u} = -\frac{1}{\tau}u + \frac{\sqrt{D}}{\tau}\Gamma(t), \tag{2}$$

where $\langle\Gamma(t)\rangle = 0$, $\langle\Gamma(t)\Gamma(s)\rangle = 2\delta(t-s)$.

The Fokker-Planck equation corresponding to (2) can be written as [32]

$$\frac{\partial P}{\partial t} = \frac{1}{\tau}\frac{\partial}{\partial u}[uP] + \frac{D}{\tau^2}\frac{\partial^2}{\partial u^2}P. \tag{3}$$

The Fokker-Planck operator $L_u$ and its formal adjoint operator are given by

$$L_u = \frac{1}{\tau}\frac{\partial}{\partial u}u + \frac{D}{\tau^2}\frac{\partial^2}{\partial u^2}, \tag{4}$$

$$L_u^* = -\frac{1}{\tau}u\frac{\partial}{\partial u} + \frac{D}{\tau^2}\frac{\partial^2}{\partial u^2}. \tag{5}$$

Equation (3) has a stationary, zero-mean, Gaussian, Markov solution process whose corresponding probability density function is given as:

$$P_s(u) = \sqrt{\frac{\tau}{2\pi D}}e^{-\frac{\tau}{2D}u^2}. \tag{6}$$

If $Q^*(u)$ is an eigenfunction of $L_u^*$ corresponding to the eigenvalue $\lambda$, then $Q(u) = P_s(u)Q^*(u)$ is an eigenfunction of $L_u$ corresponding to the same eigenvalue.

Now we get the eigenfunction equation for $Q^*(u)$:

$$Q^{*''} - \frac{\tau}{D}uQ^{*'} + \frac{\tau^2\gamma}{D}Q^* = 0, \tag{7}$$

and the eigenfunction and corresponding eigenvalue can be obtained as

$$Q_n^*(u) = \left(2^n n!\right)^{-\frac{1}{2}} H_n\left(\sqrt{\frac{\tau}{2D}}u\right), \tag{8}$$

$$\lambda_n = \frac{n}{\tau}, \tag{9}$$

where $H_n(u)$ is a Hermite polynomial.

The joint response process $(x, \dot{x})$ can be transformed into $(a, \phi)$ according to the relationships

$$x(t) = a(t)\cos\phi(t), \qquad \dot{x}(t) = -w_0 a(t)\sin\phi(t). \tag{10}$$

Therefore, the equation of motion for the amplitude $a(t)$ and phase $\phi(t)$ processes yields

$$\begin{aligned}\dot{a} &= \varepsilon^2\frac{\gamma}{w_0}\sin\phi g(-w_0 a\sin\phi) \\ &\quad + \varepsilon^2\frac{1}{w_0}f(a\cos\phi)\sin\phi p(w_0 t) \\ &\quad - \varepsilon\frac{1}{w_0}h(a\cos\phi, -w_0 a\sin\phi)\sin\phi u,\end{aligned} \tag{11}$$

$$\dot{\phi} = w_0 + \varepsilon^2 \frac{\gamma}{w_0 a} g(-w_0 a \sin\phi)\cos\phi + \varepsilon^2 \frac{1}{w_0 a} f(a\cos\phi)\cos\phi p(w_0 t) - \varepsilon \frac{1}{w_0 a} h(a\cos\phi, -w_0 a\sin\phi)\cos\phi u. \tag{12}$$

The process $(a, \phi, u)$ is a Markovian process, whose probability density $p_\varepsilon$ is the solution of the following Fokker-Planck equation

$$\begin{aligned}\frac{\partial p_\varepsilon}{\partial t} = {} & \frac{\partial}{\partial a}\Big[-\varepsilon^2 \frac{\gamma}{w_0}\sin\phi g(-w_0 a\sin\phi) \\ & - \varepsilon^2 \frac{1}{w_0} f(a\cos\phi)\sin\phi p(w_0 t) \\ & + \varepsilon \frac{1}{w_0} h(a\cos\phi, -w_0 a\sin\phi)\sin\phi u\Big] p_\varepsilon \\ & + \frac{\partial}{\partial\phi}\Big[-w_0 - \varepsilon^2 \frac{\gamma}{w_0 a} g(-w_0 a\sin\phi)\cos\phi \\ & - \varepsilon^2 \frac{1}{w_0 a} f(a\cos\phi)\cos\phi p(w_0 t) \\ & + \varepsilon \frac{1}{w_0 a} h(a\cos\phi, -w_0 a\sin\phi)\cos\phi u\Big] p_\varepsilon \\ & + \frac{1}{\tau}\frac{\partial}{\partial u}[u p_\varepsilon] + \frac{D}{\tau^2}\frac{\partial^2}{\partial u^2} p_\varepsilon. \end{aligned} \tag{13}$$

It is well known that for the limit $\varepsilon \to 0$, the process $a(t)$ is a slow variable, whereas the process $\phi(t)$ is a fast variable, and thus $a(t)$ will not change significantly over a time interval of the order $O(1)$. To investigate the response over a time interval of order $O(\varepsilon^2)$ as in [25], the time variable is scaled as $T = \varepsilon^2 t$, where $T$ is a slow time scale.

Let

$$L_0 = -w_0 \frac{\partial}{\partial\phi} + L_u, \tag{14}$$

$$L_1 = \frac{u}{w_0}\sin\phi \frac{\partial}{\partial a} h + \frac{u}{w_0 a}\frac{\partial}{\partial\phi}(h\cos\phi), \tag{15}$$

$$\begin{aligned} L_2 = {} & \frac{\partial}{\partial a}\Big[-\frac{\gamma}{w_0}\sin\phi g - \frac{1}{w_0} f\sin\phi p(w_0 t)\Big] \\ & + \frac{\partial}{\partial\phi}\Big[-\frac{\gamma}{w_0 a} g\cos\phi - \frac{1}{w_0 a} f\cos\phi p(w_0 t)\Big] \\ & - \frac{\partial}{\partial T}, \end{aligned} \tag{16}$$

then (13) can be rewritten as

$$\big(L_0 + \varepsilon L_1 + \varepsilon^2 L_2\big) p_\varepsilon = 0. \tag{17}$$

A solution of (17) is to be obtained in terms of a perturbation expansion of the parameter $\varepsilon$ in the following form:

$$p_\varepsilon = p_0 + \varepsilon p_1 + \varepsilon^2 p_2 + \cdots. \tag{18}$$

Inserting (18) into (17) and identifying terms of equal power of $\varepsilon$ leads to the sequence of equations:

$$\varepsilon^0: \quad L_0 p_0 = 0, \tag{19}$$

$$\varepsilon^1: \quad L_0 p_1 = -L_1 p_0, \tag{20}$$

$$\varepsilon^2: \quad L_0 p_2 = -L_1 p_1 - L_2 p_0, \tag{21}$$

$\ldots$

Each term of expansion (18) is required to be $2\pi$-periodic in the variable $\phi$, integrable over $a \in [0, \infty]$, and assumed to satisfy natural boundary conditions for the variable $u$. The normalization condition of $p_\varepsilon$ is equivalent to

$$\int_0^\infty da \int_0^{2\pi} d\phi \int du\, p_0 = 1, \tag{22}$$

$$\int_0^\infty da \int_0^{2\pi} d\phi \int du\, p_i = 0, \quad (i \geq 1). \tag{23}$$

Then a solvability condition for equation $L_0 p = q$ can be presented as [25]

$$\frac{1}{2\pi}\int_0^{2\pi} d\phi \int q\, du = 0, \tag{24}$$

which imposes that the average of $q$ over the variables $\phi$ and $u$ is zero.

Now we begin to find the expression of $p_0$ in expansion (18). Expand $p_0$ along the eigenfunction of the operator $L_u$, i.e.

$$p_0 = p_0^{(0)} Q_0 + p_1^{(0)} Q_1 + \cdots. \tag{25}$$

Applying operator $L_0$ on (25), we deduce that

$$\Big(-w_0 \frac{\partial}{\partial\phi} - \frac{n}{\tau}\Big) p_n^{(0)} = 0. \tag{26}$$

Since the only periodic non-zero coefficient in expansion (25) is

$$p_0^{(0)} = p_0^{(0)}(a), \tag{27}$$

one immediately obtains

$$p_0 = p_0^{(0)}(a)Q_0(u), \tag{28}$$

where $p_0^{(0)}(a)$ will be found from subsequent solvability conditions.

From (20), it is easy to see that $p_1$ is a solution of the equation

$$\left(-w_0\frac{\partial}{\partial\phi} + L_u\right)p_1 = -\left[\frac{u}{w_0}\sin\phi\frac{\partial}{\partial a}\left(hp_0^{(0)}\right) + \frac{u}{w_0 a}\frac{\partial}{\partial\phi}\left(hp_0^{(0)}\cos\phi\right)\right]Q_0(u). \tag{29}$$

Expand $p_1$ along eigenfunction $Q_n$ of operator $L_u$:

$$p_1 = p_0^{(1)}Q_0 + p_1^{(1)}Q_1 + \cdots. \tag{30}$$

Inserting (30) into (29), we get

$$\left(-w_0\frac{\partial}{\partial\phi} - \frac{1}{\tau}\right)p_1^{(1)} = -\frac{1}{w_0}\sqrt{\frac{D}{\tau}}\left[\sin\phi\frac{\partial}{\partial a}\left(hp_0^{(0)}\right) + \frac{1}{a}\frac{\partial}{\partial\phi}\left(hp_0^{(0)}\cos\phi\right)\right], \tag{31}$$

$$\left(-w_0\frac{\partial}{\partial\phi} - \frac{n}{\tau}\right)p_n^{(1)} = 0, \quad n \neq 1. \tag{32}$$

Then the only non-zero periodic solutions are $p_0^{(1)}(a,\phi) = p_0^{(1)}(a)$ and $p_1^{(1)}(a,\phi)$.

So $p_1$ takes the form

$$p_1 = p_0^{(1)}(a)Q_0(u) + p_1^{(1)}(a,\phi)Q_1(u), \tag{33}$$

which satisfies the normality condition (23) if $\int p_0^{(1)}(a)\,da = 0$.

Then the expression for $p_0$ can be determined by that of $p_1(a,\phi,u)$ and solvability condition

$$\frac{1}{2\pi}\int_0^{2\pi}d\phi\int du\,(L_2p_0 + L_1p_1) = 0. \tag{34}$$

## 3 Example

### 3.1 The approximate stationary probability density

Choosing $g(\dot{x}) = \dot{x}^3$, $f(x) = x^3$, $p(w_0t) = \cos w_0t$ and $h(x,\dot{x}) = w_0\sigma x$, (1) becomes

$$\ddot{x} + w_0^2x + \varepsilon^2\gamma\dot{x}^3 + \varepsilon^2x^3\cos(w_0t) = \varepsilon w_0\sigma x u(t). \tag{35}$$

From (31)–(33), we get

$$p_1(a,\phi,u) = p_0^{(1)}(a)Q_0(u) - \frac{\tau\sigma}{1+4w_0^2\tau^2}\sqrt{\frac{D}{\tau}} \times \left(\frac{1}{2}\frac{\partial}{\partial a}\left(ap_0^{(0)}\right) - p_0^{(0)}\right) \times (2w_0\tau\cos 2\phi - \sin 2\phi)Q_1(u). \tag{36}$$

Next, one determines the expression of $p_0^{(0)}(a)$.

Considering

$$\frac{1}{2\pi}\int_0^{2\pi}d\phi\int du\,L_2p_0 = -\frac{\partial}{\partial T}p_0^{(0)} + \frac{3\gamma w_0^2}{8}\frac{\partial}{\partial a}\left[a^3p_0^{(0)}\right], \tag{37}$$

$$\frac{1}{2\pi}\int_0^{2\pi}d\phi\int du\,L_1p_1 = \frac{D\sigma^2}{2(1+4w_0^2\tau^2)}\left[\frac{\partial^2}{\partial a^2}\left(\frac{1}{2}a^2p_0^{(0)}\right) - \frac{\partial}{\partial a}\left(\frac{3}{2}ap_0^{(0)}\right)\right], \tag{38}$$

we obtain that $p_0^{(0)}(a)$ satisfies the following FPK equation by using the solvability condition (34):

$$\frac{\partial}{\partial T}p_0^{(0)}(a) = -\frac{\partial}{\partial a}\left[\varphi(a)p_0^{(0)}(a)\right] + \frac{1}{2}\frac{\partial^2}{\partial a^2}\left[\psi^2(a)p_0^{(0)}(a)\right], \tag{39}$$

where

$$\varphi(a) = -\frac{3\gamma w_0^2}{8}a^3 + \frac{3D\sigma^2}{4(1+4w_0^2\tau^2)}a, \tag{40}$$

$$\psi^2(a) = \frac{D\sigma^2}{2(1+4w_0^2\tau^2)}a^2. \tag{41}$$

Suppose (39) has reflecting boundaries; then stationary solution can be given as

$$p_0^{(0)}(a) = N_1 a\exp\left[-\frac{3\gamma w_0^2(1+4w_0^2\tau^2)}{4D\sigma^2}a^2\right]. \tag{42}$$

To determine the expression of $p_0^{(1)}$, we have first to get $p_2$.

After tedious calculation, we obtain

$$\begin{aligned}
&L_1 p_1 \\
&= \sigma\bigg[\sin\phi\cos\phi\left(\frac{\partial}{\partial a}\big(ap_0^{(1)}(a)\big) - 2p_0^{(1)}(a)\right)uQ_0(u) \\
&\quad - \frac{\tau\sigma}{1+4w_0^2\tau^2}\sqrt{\frac{D}{\tau}} \\
&\quad \times \frac{\partial}{\partial a}\left(\frac{a}{2}\frac{\partial}{\partial a}\big(ap_0^{(0)}(a)\big) - ap_0^{(0)}(a)\right) \\
&\quad \times \left(w_0\tau\sin 2\phi\cos 2\phi - \frac{1}{2}\sin^2 2\phi\right)uQ_1(u) \\
&\quad + \frac{\tau\sigma}{1+4w_0^2\tau^2}\sqrt{\frac{D}{\tau}}\left(\frac{1}{2}\frac{\partial}{\partial a}\big(ap_0^{(0)}(a)\big) - p_0^{(0)}(a)\right) \\
&\quad \times \big(2w_0\tau(\sin 2\phi + \sin 4\phi) + \cos 2\phi \\
&\quad + \cos 4\phi\big)uQ_1(u)\bigg],
\end{aligned} \tag{43}$$

$$\begin{aligned}
&L_2 p_0 \\
&= \bigg[\frac{\partial}{\partial a}\bigg(\gamma w_0^2 a^3\sin^4\phi \\
&\quad - \frac{1}{w_0}a^3\cos^3\phi\sin\phi\cos w_0 t\bigg)p_0^{(0)}(a) \\
&\quad + \frac{\partial}{\partial\phi}\bigg(\gamma w_0^2 a^2\sin^3\phi\cos\phi \\
&\quad - \frac{1}{w_0}a^2\cos^4\phi\cos w_0 t\bigg)p_0^{(0)}(a) \\
&\quad + \frac{\partial}{\partial a}\big(\varphi(a)p_0^{(0)}(a)\big) \\
&\quad - \frac{1}{2}\frac{\partial^2}{\partial a^2}\big(\psi^2(a)p_0^{(0)}(a)\big)\bigg]Q_0(u).
\end{aligned} \tag{44}$$

Expanding $p_2$ along the eigenfunction of $L_u$, we have

$$p_2 = p_0^{(2)}Q_0 + p_1^{(2)}Q_1 + \cdots. \tag{45}$$

In order to get the expression of $p_1^{(2)}$, noting that

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi}d\phi\int du\, L_1 p_2 \\
&= \frac{\sigma}{2\pi}\int_0^{2\pi}d\phi\int du\, u\sin\phi\cos\phi\frac{\partial}{\partial a}\big(ap_1^{(2)}\big)Q_1(u),
\end{aligned} \tag{46}$$

and inserting (45) into (44) leads to the following equation:

$$\begin{aligned}
&\left(-w_0\frac{\partial}{\partial\phi} - \frac{1}{\tau}\right)p_1^{(2)} \\
&= -\sigma\sqrt{\frac{D}{\tau}}\left[\sin\phi\cos\phi\left(\frac{\partial}{\partial a}\big(ap_0^{(1)}(a)\big) - 2p_0^{(1)}(a)\right)\right].
\end{aligned} \tag{47}$$

Through solving the above equation, we get

$$\begin{aligned}
p_1^{(2)} &= -\frac{\tau\sigma}{2(1+4w_0^2\tau^2)}\sqrt{\frac{D}{\tau}} \\
&\quad \times \left(\frac{\partial}{\partial a}\big(ap_0^{(1)}(a)\big) - 2p_0^{(1)}(a)\right) \\
&\quad \times (2w_0\tau\cos 2\phi - \sin 2\phi).
\end{aligned} \tag{48}$$

Substituting (48) into (46) yields the following equality:

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi}d\phi\int du\, L_1 p_2 \\
&= \frac{D\sigma^2}{8(1+4w_0^2\tau^2)}\sqrt{\frac{D}{\tau}}\frac{\partial}{\partial a} \\
&\quad \times \left(a\left(\frac{\partial}{\partial a}\big(ap_0^{(1)}(a)\big) - 2p_0^{(1)}(a)\right)\right).
\end{aligned} \tag{49}$$

On the other hand, one can get

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi}d\phi\int du\, L_2 p_1 \\
&= \frac{\gamma}{2}\frac{\partial}{\partial a}\big[ap_0^{(1)}(a)\big] - \frac{\partial p_0^{(1)}(a)}{\partial T}.
\end{aligned} \tag{50}$$

Using the solvability condition, we can see that $p_0^{(1)}(a)$ satisfies the following FPK equation:

$$\begin{aligned}
&\frac{\partial p_0^{(1)}(a)}{\partial T} \\
&= -\frac{\partial}{\partial a}\big[\varphi_1(a)p_0^{(1)}(a)\big] \\
&\quad + \frac{1}{2}\frac{\partial^2}{\partial a^2}\big[\psi_1^2(a)p_0^{(1)}(a)\big],
\end{aligned} \tag{51}$$

where

$$\varphi_1(a) = \frac{3\gamma w_0^2}{8}a^3 - \frac{3D\sigma^2}{8(1+4w_0^2\tau^2)}a, \tag{52}$$

$$\psi_1^2(a) = \frac{D\sigma^2}{4(1+4w_0^2\tau^2)} a^2. \tag{53}$$

The stationary solution with reflecting boundaries can be obtained as

$$p_0^{(1)}(a) = N_2 a \exp\left[-\frac{3\gamma w_0^2(1+4w_0^2\tau^2)}{2D\sigma^2} a^2\right]. \tag{54}$$

Now, we get a kind of approximate stationary probability density function of order $\varepsilon$

$$\begin{aligned} p_s(a,\phi,u) &= \frac{3\gamma w_0^2(1+4w_0^2\tau^2)}{2D\sigma^2}\sqrt{\frac{\tau}{2\pi D}} \\ &\quad \times a \exp\left[-\frac{3\gamma w_0^2(1+4w_0^2\tau^2)}{4D\sigma^2} a^2\right] \exp\left[-\frac{\tau}{2D} u^2\right] \\ &\quad + \varepsilon \left\{ \frac{9\gamma^2 \tau w_0^4 (1+4w_0^2\tau^2)}{8D^3\sigma^3\sqrt{2\pi}} \sqrt{\frac{\tau}{D}} a^3 \right. \\ &\quad \times \exp\left[-\frac{3\gamma w_0^2(1+4w_0^2\tau^2)}{2D\sigma^2} a^2\right] \\ &\quad \left. \times (2w_0\tau\cos 2\phi - \sin 2\phi) u \right\} \exp\left[-\frac{\tau}{2D} u^2\right]. \end{aligned} \tag{55}$$

### 3.2 Numerical analysis

Now we carry out the numerical simulation to verify the theoretical results. Without loss of generality, we take the stationary probability density function (55) up to order $\varepsilon^0$, and obviously it is possible to obtain the marginal probability density distribution of the amplitude $a(t)$ as

$$\begin{aligned} p_s(a) &= \frac{3\gamma w_0^2(1+4w_0^2\tau^2)}{2D\sigma^2} a \\ &\quad \times \exp\left[-\frac{3\gamma w_0^2(1+4w_0^2\tau^2)}{4D\sigma^2} a^2\right]. \end{aligned} \tag{56}$$

Then one can represent the first and second moments for the amplitude $a(t)$ as

$$Ea = \sqrt{\frac{D\pi\sigma^2}{3\gamma w_0^2(1+4w_0^2\tau^2)}}, \tag{57}$$

$$Ea^2 = \frac{4D\sigma^2}{3\gamma w_0^2(1+4w_0^2\tau^2)}, \tag{58}$$

**Table 1** $w_0 = 1.0,\ D = 1.0,\ \sigma = 3/2,\ \tau = 1/2$

| $\gamma$ | Second moment of $a(t)$ | |
|---|---|---|
| | Analytical results (58) | Numerical results (60) |
| 0.5 | 3.0 | 3.000210 |
| 1.0 | 1.5 | 1.499351 |
| 1.5 | 1.0 | 0.999064 |
| 2.0 | 0.75 | 0.750010 |
| 2.5 | 0.6 | 0.590007 |
| 3.0 | 0.5 | 0.510853 |
| 10.0 | 0.15 | 0.151006 |
| 12.0 | 0.125 | 0.124999 |

**Table 2** $w_0 = 1.0,\ D = 1.0,\ \sigma = 1.0,\ \gamma = 1/3$

| $\tau$ | Second moment of $a(t)$ | |
|---|---|---|
| | Analytical results (58) | Numerical results (60) |
| 0.25 | 3.2 | 3.200001 |
| 0.5 | 2.0 | 1.999936 |
| 1.0 | 0.8 | 0.790922 |
| 2.0 | 0.235294 | 0.235291 |
| 5.0 | 0.039604 | 0.039607 |
| 10.0 | 0.009975 | 0.009966 |

$$\mathrm{Var}\, a = \frac{D\sigma^2(4-\pi)}{3\gamma w_0^2(1+4w_0^2\tau^2)}. \tag{59}$$

Noting equations in (10), the amplitude $a(t)$ can be computed as

$$a(t) = \sqrt{x(t)^2 + \dot{x}(t)^2/w_0^2}. \tag{60}$$

By means of the 6th Runge–Kutta routine (with IMSL), we integrated (35) numerically and the IMSL routines were used to generate Gaussian random values, which was then applied to generate the Gaussian colored noise $u(t)$ due to [36]. Based on one thousand random values of $u(t)$ and appropriate initial conditions, we obtain the numerical results for second moments of $a(t)$ by (60).

Take $\varepsilon = 0.01$. Tables 1 and 2 show the analytical (58) and numerical (60) results for different values of $\gamma$ at $\tau = 1/2$, and different values of $\tau$ at $\gamma = 1/3$, respectively. Obviously one can find the excellent agreement between the two results.

### 3.3 Stochastic stability for the amplitude process

For a stochastic process $x(t;x_0,t_0)$, $x(t_0)=x_0$ and $t\ge t_0$, $\|x(t;x_0,t_0)\|$ is a suitable norm of $x(t;x_0,t_0)$. The almost-sure stability (also called the Lyapunov stability with probability 1) for $x(t;x_0,t_0)$ is defined in such a way that the trivial solution $x(t;x_0,t_0)=0$ is said to be almost-sure stability if, for every pair of $\varepsilon_1,\varepsilon_2>0$, there exists a $\delta(\varepsilon_1,\varepsilon_2,t_0)>0$ such that

$$\mathrm{Prob}\left\{\bigcup_{\|x_0\|\le\delta}\left[\sup_{t\ge t_0}\|x(t;x_0,t_0)\|\ge\varepsilon_1\right]\right\}\le\varepsilon_2, \tag{61}$$

where $x_0$ is deterministic.

The trivial solution is said to be stable in probability if, for every pair of $\varepsilon_1,\varepsilon_2>0$, there exists a $\delta(\varepsilon_1,\varepsilon_2,t_0)>0$ such that

$$\mathrm{Prob}\left[\|x(t;x_0,t_0)\|\ge\varepsilon_1\right]\le\varepsilon_2, \tag{62}$$

provided $\|x_0\|\le\delta$, where $x_0$ is deterministic.

It can be seen that (61) implies (62). The two types of stability conditions become equivalent if the system is linear [33–35].

#### 3.3.1 Almost-sure stability

Corresponding to the FPK equation (39), the $It\hat{o}$ stochastic differential equation for the amplitude process $a(T)$ of order $O(\varepsilon^0)$ can be obtained

$$da(T)=\varphi(a)\,dT+\psi(a)\,dW(T), \tag{63}$$

where $W(T)$ is the Wiener process of unit intensity.

The linearization of (63) becomes

$$da(T)=\varphi^*(a)\,dT+\psi^*(a)\,dW(T), \tag{64}$$

where $\varphi^*(a)=\frac{3D\sigma^2}{4(1+4w_0^2\tau^2)}a$, $\psi^*(a)=\sqrt{\frac{D\sigma^2}{2(1+4w_0^2\tau^2)}}a$.

The solution for (64) can be presented as

$$a(T)=a(T_0)\exp\left\{\left[\frac{\varphi^*(a)}{a}-\frac{1}{2}\left(\frac{\psi^*(a)}{a}\right)^2\right]T+\frac{\psi^*(a)}{a}W(T)\right\}, \tag{65}$$

where $\frac{\varphi^*(a)}{a}-\frac{1}{2}(\frac{\psi^*(a)}{a})^2=\frac{D\sigma^2}{2(1+4w_0^2\tau^2)}>0$ means that $a=0$ is not almost-sure stable [28, 33–35].

#### 3.3.2 Stability in probability

It is known that the trivial solution is said to be asymptotically stable in probability if and only if it is an exit or an attractive natural boundary and the other boundary is an entrance or a repulsive natural boundary [33].

Since $\varphi(0)=0$ and $\psi(0)=0$, the boundary at $a=0$ is a trap. The diffusion exponent, drift exponent and character value are defined, respectively, as:

$\alpha_l$ is the diffusion exponent of $a=0$ if

$$\psi^2(a)=O|a-0|^{\alpha_l},\quad \alpha_l\ge 0,\ \text{as } a\to 0. \tag{66}$$

$\beta_l$ is the drift exponent of $a=0$ if

$$\varphi(a)=O|a-0|^{\beta_l},\quad \beta_l\ge 0,\ \text{as } a\to 0. \tag{67}$$

$c_l$ is the character value of $a=0$, given by

$$c_l=\lim_{a\to 0^+}\frac{2\varphi(a)(a-0)^{\alpha_l-\beta_l}}{\psi^2(a)}. \tag{68}$$

It is easy to see that

$$\alpha_l=2,\qquad \beta_l=1,\qquad c_l=3, \tag{69}$$

which implies that $a=0$ is a repulsive natural boundary.

Since $\varphi(+\infty)$ is unbounded, the diffusion exponent, drift exponent and character value are defined, respectively, as:

$\alpha_r$ is the diffusion exponent of $a=+\infty$ if

$$\psi^2(a)=O|a|^{\alpha_r},\quad \alpha_r\ge 0,\ \text{as } a\to+\infty. \tag{70}$$

$\beta_r$ is the drift exponent of $a=+\infty$ if

$$\varphi(a)=O|a|^{\beta_r},\quad \beta_r\ge 0,\ \text{as } a\to+\infty. \tag{71}$$

$c_r$ is the character value of $a=+\infty$, given by

$$c_r=-\lim_{a\to+\infty}\frac{2\varphi(a)|a|^{\alpha_r-\beta_r}}{\psi^2(a)}, \tag{72}$$

which leads to that $\alpha_r=2,\beta_r=3,c_r=\frac{3\gamma w_0^2}{8}$ for $\gamma\neq 0$ and $\alpha_r=2,\beta_r=1,c_r=0$ for $\gamma\neq 0$; the boundary $a=+\infty$ should be investigated in two separate cases in view of the parameter $\gamma$.

Case 1: $\gamma>0$

Under this condition, we have

$$\beta_r>\alpha_r-1,\qquad \varphi(+\infty)<0,\qquad \beta_r>1, \tag{73}$$

which means that the right boundary at $a = +\infty$ is an entrance natural boundary.

Case 2: $\gamma < 0$

This conditions lead to

$$\beta_r > \alpha_r - 1, \qquad \varphi(+\infty) > 0, \qquad \beta_r > 1, \tag{74}$$

which means that the right boundary at $a = +\infty$ is an exit natural boundary.

Case 3: $\gamma = 0$

We have

$$\beta_r = \alpha_r - 1, \qquad \beta_r = 1, \qquad c_r > -\beta_r, \tag{75}$$

which means that the right boundary at $a = +\infty$ is a repulsive natural boundary.

Finally, we get the conclusion that $a = 0$ is not asymptotically stable in probability. Particularly, when $\gamma = 0$, the conclusion matches the fact that the stability conditions are equivalent for linear system.

## 4 Conclusions

In this paper, a class of random differential equations with parametric colored noise is considered. The method of stochastic averaging with the perturbation technique is applied to derive the FPK equation. The operator theory is used to determine the solvability conditions for the stationary probability density. An illustrative example is presented to get the approximate analytical expression of stationary probability density up to order $\varepsilon$. Numerical simulation is carried out to verify the analytical results, and excellent agreement can be easily found. Later the almost-sure stability and the stability in probability of the amplitude process are examined based on stationary probability density function using the FPK equation.

**Acknowledgements** This work was supported by the NSF of China (Grant no. 10502042) and NSF of Shaanxi Province, the Youth and Aoxiang Star Plan for NPU teachers' Scientific and Technological Innovation Foundation.

## References


1. Bountis, T., Mahmoud, G.M.: Synchronized periodic orbits in beam–beam interaction models of one and two spatial dimensions. Part. Accel. **22**, 124–147 (1987)
2. Li, R.: Selfconsistent simulation of the CSR effect. In: Proceedings of the European Particle Accelerator Conference, Stockholm, Sweden, 1998. Institute of Physics, London (1998)
3. Month, M., Herrera, J.C. (eds.): Nonlinear Dynamics and the Beam–Beam Interaction. AIP Conference Proceedings, vol. 57. AIP, New York (1979)
4. Dowell, D.H., O'Shea, P.G.: Coherent synchrotron radiation included emittance growth in a chicane buncher. In: Proceedings of the Particle Accelerator Conference, Vancouver, Canada, 1997. IEEE, Piscataway (1998)
5. Braun, H.H., Corsini, R. et al.: Emmitance growth and energy loss due to coherent synchrotron radiation in the bunch compressor of the CLIC Test Facility (CTFII). In: Proceedings of the European Particle Accelerator Conference, Vienna, Austria, 2000. http://accelconf.web.cern.ch/AccelConf/e00/index.html
6. Mahmoud, G.M.: On periodic orbits of nonlinear dynamical systems with many degrees of freedom. Physica A **181**, 385–395 (1992)
7. Wedig, W.V.: Dynamic stability of beams under axial forces: Lyapunov exponents for general fluctuating loads. In: Kratzig, W.B., et al. (eds.) Structural Dynamics, pp. 141–148. Balkema, Rotterdam (1990)
8. Cveticanin, L., Maretic, R.: A van der Pol absorber for rotor vibrations. J. Sound Vib. **173**(2), 145–155 (1994)
9. Cveticanin, L.: Vibrations of a textile machine rotor. J. Sound Vib. **97**, 181–187 (1984)
10. Cveticanin, L.: Dynamical behavior of a rotor with time dependent parameters. JSME Int. J. Ser. C **37**(1), 41–48 (1994)
11. Muszynska, A.: Whirl and whip-rotor/bearing stability problems. J. Sound Vib. **110**(3), 443–462 (1986)
12. Genin, J., Maybee, J.S.: Stability in the three dimensional whirling problem. Int. J. Non-Linear Mech. **4**, 205–215 (1969)
13. Tondl, A.: Determination of the limit of initiation of self-excited vibration of rotors. Int. J. Non-Linear Mech. **15**, 417–428 (1980)
14. Mahmoud, G.M.: Beam–beam interaction models with a small stochastic perturbation. Physica A **216**, 445–451 (1995)
15. Xu, Y., Xu, W., Mahmoud, G.M.: On a complex beam–beam interaction model with random forcing. Physica A **336**, 347–360 (2004)
16. Xu, Y., Xu, W., Mahmoud, G.M., Lei, Y.M.: Beam–beam interaction models under narrow-band random excitation. Physica A **346**, 372–386 (2005)
17. Xu, Y., Zhang, H.Q., Xu, W.: On stochastic complex beam–beam interaction models with Gaussian colored noise. Physica A **384**, 259–272 (2007)
18. Mahmoud, G.M.: Stability regions for coupled Hill's equations. Physica A **242**, 239–249 (1997)
19. Stratonovich, R.L.: Topics in the Theory of Random Noise. Gordon and Breach, New York (1963)
20. Roberts, J.B.: The energy envelope of a randomly excited non-linear oscillator. J. Sound Vib. **60**, 177–185 (1978)
21. Zhu, W.Q., Lin, Y.K.: On stochastic averaging of energy envelope. ASCE J. Eng. Mech. **117**, 1890–1905 (1991)
22. Lin, Y.K.: Some observations on the stochastic averaging method. Prob. Eng. Mech. **1**, 23–27 (1986)

23. Zhu, W.Q.: Stochastic averaging in random vibration. Appl. Mech. Rev. **41**, 189–199 (1988)
24. Blankenship, G., Papanicolaou, G.C.: Stability and control of stochastic systems with wide-band noise disturbances. SIAM J. Appl. Math. **34**, 437–476 (1978)
25. Roy, R.V.: Stochastic averaging of oscillators excited by colored Gaussian processes. Int. J. Non-Linear Mech. **29**, 463–475 (1994)
26. Klosek-Dygas, M.M., Matkowsky, B.J., Schuss, Z.: Stochastic stability of nonlinear oscillators. SIAM J. Appl. Math. **48**, 1115–1127 (1988)
27. Klosek-Dygas, M.M., Matkowsky, B.J., Schuss, Z.: Colored noise in dynamical systems. SIAM J. Appl. Math. **48**, 425–441 (1988)
28. Liu, X.B., Liew, K.M.: On the stability properties of a Van der Pol-Duffing oscillator that is driven by a real noise. J. Sound Vib. **285**, 27–49 (2005)
29. Gardiner, C.W.: Handbook of Stochastic Methods for Physics, Chemistry and the Natural Sciences. Springer, Berlin (1983)
30. Hänggi, P., Jung, P.: Colored noise in dynamical systems. Adv. Chem. Phys. **89**, 239–326 (1995)
31. Jung, P., Hänggi, P.: Dynamical systems: a unified colored-noise approximation. Phys. Rev. A **35**, 4464–4466 (1987)
32. Haken, H.: The Fokker–Planck Equation. Springer, Berlin (1989)
33. Lin, Y.K., Cai, G.Q.: Probabilistic is Structural Dynamics, Advanced Theory and Applications. McGraw-Hill, New York (1995)
34. Arnold, L.: Random Dynamical Systems. Springer, Berlin (1998)
35. Zhu, W.Q.: Random Vibration. Science, Beijing (1992)
36. Bartosh, L.: Generation of colored noise. Int. J. Mod. Phys. C **12**(6), 851–855 (2001)