Research Article

Quality and Reliability Engineering International

(wileyonlinelibrary.com) DOI: 10.1002/qre.1153

Published online 13 October 2010 in Wiley Online Library

A Control Scheme Integrating the T Chart and TCUSUM Chart

Liang Qu[a], Zhang Wu[a*†] and Tien-I Liu[b]

This article proposes an integrated scheme (*T*&TCUSUM chart) which combines a Shewhart T chart and a TCUSUM chart (a CUSUM-type T chart) to monitor the time interval *T* between the occurrences of an event or the time between events. The performance studies show that the T&TCUSUM chart can effectively improve the overall performance over the entire *T* shift range. On average, it is more effective than the *T* chart by 26.66% and the TCUSUM chart by 14.12%. Moreover, the T&TCUSUM chart performs more consistently than other charts for the detection of either small or large *T* shifts, because it has the strength of both the *T* chart (more sensitive to large shifts) and the TCUSUM chart (more sensitive to small shifts). The implementation of the new chart is almost as easy as the operation of a TCUSUM chart. Copyright © 2010 John Wiley & Sons, Ltd.

Keywords: quality control; statistical process control; control chart; time between events; CUSUM chart; steady-state ATS; average loss

1. Introduction

The control chart is the most powerful tool in Statistical Process Control (SPC). In the recent years, many new charts and their applications have been proposed and studied by SPC practitioners and researchers in engineering, management and statistics[1–6]. One type of control chart was developed to monitor the time interval *T* between the occurrences of an event or TBE (Time Between Events). *Time* and *event* may have different meanings in different situations. For example, in the manufacturing industry, an event may mean the occurrence of a nonconforming product and time means the time between two consecutive nonconforming products. In reliability engineering, an event may mean the failure of a system and time means the time between the failures. The occurrences of many events result in a negative or hazardous consequence. In SPC, one is usually interested in detecting a decrease in the time interval *T* of those events, as a decrease in *T* indicates a move in a loss direction or a transit to a worse status. The users should be warned as soon as possible when a decreasing *T* shift occurs. Otherwise this shift will cause a high rate in damage, cost, or loss.

Many TBE control charts have been developed to monitor the time interval *T* of an event, including the T chart or exponential chart[7–10], the CRL and RL_2 charts[11, 12], the SCRL or Gamma chart[13, 14], the TCUSUM chart[15–18], the geometric CUSUM chart[11, 19], the exponential EWMA chart[20], the synthetic chart[21, 22] and the CQC-r chart[23]. Recently Wu *et al.*[24, 25] presented two charts for simultaneously monitoring the time interval *T* and the magnitude *X* of an event.

The TBE charts are particularly effective for a high yield production line with very low defect rate. Its applications can also be extended to many non-manufacturing sectors, such as the healthcare industry[26–28].

While the Shewhart T chart is simple in design and implementation and has high detection effectiveness against large *T* shifts, the TCUSUM charts (i.e. the CUSUM chart monitoring *T*) have been well recognized recently owing to the fact that online measurement and distributed computing systems have become the norm in today's SPC applications[15–18, 29–31]. As a CUSUM chart incorporates all the information in the sequence of the sample points by monitoring the cumulative sums of the statistics, it is more effective than a Shewhart-type control chart for detecting small and moderate *T* shifts.

In most SPC applications, it is quite difficult to predict the actual magnitudes of process shifts. In order to make the control scheme effective over a wide range of shifts, some researchers[30, 32] recommended using two or three CUSUM charts simultaneously (named as the 2-CUSUM chart or 3-CUSUM chart). Lucas[33] proposed a scheme integrating an $\bar{X}$ (or X) chart and a CUSUM chart. In his scheme, the CUSUM feature will quickly detect small and moderate shifts while the addition of the $\bar{X}$ chart increases the speed of detecting large shifts. According to a recent study[34], Lucas' combined X&CUSUM scheme outperforms the single CUSUM

[a] *School of Mechanical and Aerospace Engineering, Nanyang Technological University, Singapore 639798, Singapore*
[b] *Department of Mechanical Engineering, California State University, Sacramento, CA, U.S.A.*
**Correspondence to: Zhang Wu, School of Mechanical and Aerospace Engineering, Nanyang Technological University, Singapore 639798, Singapore.*
† *E-mail: mzwu@ntu.edu.sg*

chart by 6.3% on average if both charts are designed by an optimization algorithm. Lucas also commented that the X&CUSUM scheme is almost as easy to use as the CUSUM chart.

This article proposes a combined T&TCUSUM chart. As it has both the strength of the T chart for quickly detecting large *T* shifts and the advantage of the TCUSUM chart of being sensitive to small shifts, the T&TCUSUM chart often significantly outperforms an individual T chart or TCUSUM chart from an overall viewpoint. Specifically, the T&TCUSUM chart achieves an improvement over the TCUSUM chart by 14.12%, more than doubling the improvement (6.3%) gained by the X&CUSUM chart over the CUSUM chart[33]. Moreover, like the X&CUSUM chart, the T&TCUSUM chart has more consistent detection effectiveness over the whole shift range.

The performance of a control chart can be measured by the Average Time to Signal (ATS). The out-of-control ATS indicates the average time required to signal an out-of-control case and is commonly used as an indicator of the power (or effectiveness) of the control chart, whereas the in-control ATS_0 means the average time from the commencement of a process to the first false alarm and is used as a measure of the false alarm rate.

As in most reported works[23], the time interval *T* is assumed to follow an exponential distribution. Therefore, the only distribution parameter is the failure rate λ (or the reciprocal of the mean value of *T*). The probability density function f_T and the cumulative probability function F_T of an exponential distribution are given by

$$f_T(t)=\lambda e^{-\lambda t}, \quad F_T(t)=1-e^{-\lambda t}, \quad t\geq 0. \tag{1}$$

Let δ ($\delta\geq 1$) be the shift of λ in terms of its in-control value λ_0, that is,

$$\lambda=\delta\lambda_0. \tag{2}$$

A larger δ or λ leads to a greater reduction in *T*, and vice versa. Or in other words, an increasing (or decreasing) δ or λ represents a decreasing (or increasing) shift in *T*. In this article, it is assumed that the in-control λ_0 is known *a priori*. When a process is in control, $\lambda=\lambda_0$ (or $\delta=1$); and when the process falls out of control, $\delta>1$.

The remainder of the article proceeds as follows: the implementation of the T&TCUSUM chart is described in the next section, followed by the design of this chart. A performance comparison between the T&TCUSUM chart and the individual T and TCUSUM charts is then conducted. An application example is also illustrated. The discussions and conclusions are drawn in the last section.

2. Implementation of the T&TCUSUM chart

A T&TCUSUM chart consists of a T chart component and a TCUSUM chart component. The T chart component has only one charting parameter: the lower control limit LCL, and the TCUSUM chart component has a reference parameter *k* and an upper control limit *H*. The cumulative sum C_t of the TCUSUM chart component is updated by

$$\begin{aligned} C_0 &= 0, \\ C_t &= \max(0, C_{t-1}+k-T_t), \end{aligned} \tag{3}$$

where T_t is the (*t*)th sample value of the time interval *T*. A T&TCUSUM chart is implemented as follows:

1. When the occurrence of an event is detected, find the time interval *T* between the current and the last occurrences.
2. If $T_t<$ LCL, the process is thought to be out of control, then go to step (6) immediately (This is the feature of the T chart component).
3. Otherwise (i.e. $T_t\geq$LCL), update C_t using Equation (3).
4. If $C_t>H$, the process is thought to be out of control and go to step (6) (This is the feature of the TCUSUM chart component).
5. Otherwise (i.e. $C_t\leq H$), the process is thought to be in control, then go back to step (1) and wait for the next occurrence of the event.
6. The T&TCUSUM chart produces an out-of-control signal, and the process is stopped immediately for an investigation.

The implementation of the T&TCUSUM chart is just slightly more difficult than that of the TCUSUM chart with one more action in step (2). This problem becomes even minor nowadays as on-site computers are available in most of today's industrial production lines.

3. Design of the T&TCUSUM chart

3.1. Design objective

Usually, the design objective of a control chart is to minimize the out-of-control ATS for a specified process shift, with the condition that the in-control ATS_0 is no smaller than a specification τ . This is achieved by searching the optimal values of the charting parameters. However, since it is very difficult, if not impossible, to predict the actual sizes of process shifts for most applications[29], there is no guarantee that the resultant control chart will perform well over a wide range of process shifts.

A common practice while comparing the overall performance of two charts is to examine the corresponding out-of-control ATS values of the two charts at some process shift points (e.g. the discrete values of δ in Equation (3)) within an interested range

($1<\delta\leq\delta_{max}$). In most cases, no chart will produce smaller out-of-control ATS than another chart at all shift points[29]. However, if one chart has a smaller average out-of-control ATS, or has smaller ATS values at more points and/or to a larger degree, this chart is thought to be more effective than the other. This comparison scenario can be formulated as the *average ratio* (AR) of ATS values across a shift range of interest,

$$\mathrm{AR}=\frac{\sum_{i=1}^{m}\frac{\mathrm{ATS}(\delta_i)}{\mathrm{ATS}_{\mathrm{bench}}(\delta_i)}}{m},\ 1<\delta_i\leq\delta_{\max}, \tag{4}$$

where δ_i denotes the discrete values of δ at the ith shift point and m is the number of out-of-control points in the shift range. ATS(δ_i) is the ATS value produced by a chart at δ_i and $\mathrm{ATS}_{\mathrm{bench}}(\delta_i)$ is the value generated by a benchmark chart at the same point. Obviously, if the AR value of a chart is larger than one, this chart is generally less effective than the benchmark chart over the shift range, and vice versa.

In this article, the *Average Loss*, AL, is also proposed as an alternative measure of the overall performance of the charts. The average extra loss ℓ per unit time[29] during an out-of-control case is proportional to ($\lambda-\lambda_0$), that is the average number λ of event occurrences per unit time minus the number λ_0 under the in-control situation,

$$\ell=c\cdot(\lambda-\lambda_0)=c\lambda_0(\delta-1), \tag{5}$$

where c is a cost constant and can be set as one. The total loss incurred by an out-of-control case is equal to the product of ℓ and ATS (note ATS is the average time that an out-of-control case sustains without being detected). Both ℓ and ATS are the functions of the T shifts, indicated by δ. If the entire shift range is taken into consideration, the Average Loss AL is obtained similarly to AR in Equation (4)

$$\mathrm{AL}=\frac{\sum_{i=1}^{m}(l(\delta_i)\cdot\mathrm{ATS}(\delta_i))}{m},\quad 1<\delta_i\leq\delta_{\max}. \tag{6}$$

The average loss AL is actually a weighted average ATS across the shift range, using the extra loss ℓ per unit time as the weight. If a chart has a small AL, its out-of-control ATS value at each δpoint is generally small.

The average loss AL has two advantages compared with the average ratio AR of ATS values. First, the calculation of AL does not require a predetermined benchmark chart and is therefore more tractable. Second, while the AR results are more or less influenced by the selection of the benchmark, the results of AL will not. In view of this, AL will be used as the objective function in the optimal design of the control charts, including the T&TCUSUM chart. In fact, if AR (Equation (4)) is taken as the objective function, the optimal design of the benchmark chart is problematic, because the benchmark chart cannot use itself as the benchmark. The minimization of AL will directly reduce the out-of-control ATS as well as the loss, damage, or cost incurred by the out-of-control cases.

3.2. *Design specifications*

To design a T&TCUSUM chart, the following three specifications need to be determined:

1. The allowable minimum value, τ, of the in-control ATS_0. The value of τ is decided with respect to the tolerable false alarm rate. The resultant (or actual) ATS_0 must be no smaller than τ.
2. The in-control value λ_0 of the parameter λ of the exponential probability distribution of the time interval T. The value of λ_0 can be estimated from the in-control sample mean of T, usually obtained from a historical data set.
3. The maximum shift δ_{max} required to calculate the average loss AL in Equation (6). In practice, the value of δ_{max} may be decided based on the knowledge about the process (e.g. the maximum possible shift in a process) or based on the shift range the users are interested to investigate.

3.3. *Design model*

The optimal design of a T&TCUSUM chart will be conducted using the following model:

$$\text{Objective function:}\quad \mathrm{AL}=\text{minimum}, \tag{7}$$

$$\text{Constraint function:}\quad \mathrm{ATS}_0=\tau, \tag{8}$$

$$\text{Independent design variables:}\quad \mathrm{LCL}, k,$$

$$\text{Dependent design variable:}\quad H.$$

The in-control ATS_0 and out-of-control ATS (used for the calculation of AL) are evaluated by a Markov procedure, detailed in Appendix A.

The optimal design will search for the optimal values of the independent design variables LCL (the lower control limit of the T chart component) and k (the reference parameter of the TCUSUM chart component) in order to minimize the objective function

AL (the average loss). The minimization of AL will in turn shorten the ATS values for different values of δ over the entire shift range, or reduce the loss in quality. For a given pair of LCL and k, the dependent design variable H (the upper control limit of the TCUSUM chart component) is adjusted to meet the constraint on ATS_0 (8). This constraint is treated as an equality constraint rather than an inequality one in order to make full use of the available resources and chart capacity.

It is noteworthy that the adjustment of LCL is actually to search for the optimal allocation of the detection power between the T chart component and the TCUSUM chart component, or to balance the detection power between that against small shifts and that against large shifts. If LCL is made larger (tighter), H must be relaxed correspondingly in order to maintain ($ATS_0=\tau$). Consequently, the T chart component becomes relatively more powerful than the TCUSUM chart component and the whole T&TCUSUM combination is more sensitive to larger shifts. On the other hand, if LCL is made smaller, H must be tightened. Then the T&TCUSUM chart becomes more sensitive to smaller shifts.

3.4. *Optimization search*

The optimal design is implemented as a two-level search as outlined below:

1. Specifications: τ, λ_0, and δ_{max}.
2. Initialize a variable AL_{min} as a very large number, say 10^7 (AL_{min} is used to store the minimum value of average loss AL).
3. At the first level, search LCL within the range of ($0 \le LCL \le LCL_{max}$), where the upper bound LCL_{max} is determined by

$$LCL_{max} = -\frac{1}{\lambda_0}\ln\left(1-\frac{1}{\tau\lambda_0}\right). \tag{9}$$

   Constraint (8) on ATS_0 can be satisfied only when LCL $\le LCL_{max}$.
4. At the second level, with the given value of LCL determined in the first level, search k within the range of ($1/\lambda_0 < k < \infty$). Then for a given set of values of LCL and k,

4.1. Adjust H to make the resultant ATS_0 equal to τ (constraint (8)).
4.2. When the values of all three charting parameters, LCL, k, and H, are preliminarily determined, the objective function AL is calculated by Equation (7).
4.3. If the calculated AL is smaller than the current AL_{min}, replace the latter by the former and the current values of LCL, k, and H are stored as a temporary optimal solution.

5. At the end of the entire two-level search, the optimal T&TCUSUM chart that produces the minimum AL and satisfies ATS_0 constraint is identified. The corresponding optimal values of LCL, k, and H are also finalized.

The above search algorithm is fairly reliable. The optimal design of a T&TCUSUM chart can be completed within a few seconds of CPU time on a personal computer. As Duncan[35] mentioned, most design strategies used in SPC are heuristic. They make no attempt to secure the global optimal solution. Instead, they focus on deriving a relatively convenient procedure for approximating the optimum that could be adopted in practice.

4. Comparative studies

The performance of three control charts (the T chart, TCUSUM chart, and T&TCUSUM chart) is compared in this section.

1. *T* chart
   For detecting a decreasing T shift (or an increase in the event frequency), the T chart has only one lower control limit LCL determined by

$$LCL = -\frac{1}{\lambda_0}\ln\left(1-\frac{1}{\tau\lambda_0}\right). \tag{10}$$

   It is also the upper bound for the LCL of the T chart component in a T&TCUSUM combination (see Equation (9)).
2. TCUSUM chart[15]
   The design of a TCUSUM chart is to determine the optimal combination of the reference parameter k and the control limit H so that the average loss AL over the shift range is minimized and the in-control ATS_0 is equal to τ.
3. The T&TCUSUM chart proposed in this article.

4.1. *A study for a general case*

The performance of the three charts is first studied under the following general conditions:

$$\lambda_0 = 0.005, \quad \tau = 10000, \quad \delta_{max} = 60. \tag{11}$$

The specification of ($\lambda_0 = 0.005$) indicates that, on average, the event takes place once for every 200 time units when the process is in control, and ($\tau = 10000 = 50/\lambda_0$) means that a false alarm will be produced for every 50 occurrences of the event, on average. The control charts are studied in the shift range of ($1 \le \delta \le 60$) or ($\lambda_0 \le \lambda \le 60\lambda_0$).

For this case, the charting parameters as well as the ATS values of the three control charts are summarized in Table I. The curves of the normalized ATS (i.e. $ATS/ATS_{T\&TCUSUM}$) of the three charts are displayed in Figure 1. In Table I and Figure 1, there are several interesting findings as follows:

1. First, each of the three charts generates an ATS_0 value very close to τ when the process is in control (i.e. when $\delta=1$). This ensures that the comparison among the charts is fair.
2. As expected, the TCUSUM chart outperforms the T chart for small T shifts (when $\delta\leq 25$), but it is less sensitive to large δ than the latter.
3. The T&TCUSUM chart has shown its high effectiveness for detecting T shifts of different sizes. This chart is more effective than the TCUSUM chart as long as $\delta>9$. When $\delta=60$, the TCUSUM chart produces an ATS value larger than that of the T&TCUSUM chart by 41.25%. On the other hand, the T&TCUSUM chart consistently outperforms the T chart over the whole shift range. When $\delta=5$, the ATS value produced by the T chart is larger than that of the T&TCUSUM chart by 99.51%. However, the T chart becomes nearly as effective as the T&TCUSUM chart when δ gets very large.
4. The performance of the T&TCUSUM chart is robust over the entire shift range. It is effective for detecting both small and large shifts. On the contrary, although the T chart is quite sensitive to large T shifts and the TCUSUM chart is very effective for detecting small T shifts, each of them is ineffective in other circumstances.

The average losses AL of the three charts are enumerated at the bottom of Table I. Also listed are the ratio of $AL/AL_{T\&TCUSUM}$ and AR (*average ratio* of ATS values of two charts using the T&TCUSUM chart as the benchmark). The ratios of ($AL/AL_{T\&TCUSUM}$) reveal that, in this general case, the T&TCUSUM chart reduces the total quality loss by 32.73 and 21.18% compared with the T chart and TCUSUM chart, respectively. The AR values also ascertain that, on average, the T&TCUSUM chart produces smaller out-of-control ATS, namely 28.52% smaller than that of the T chart and 22.54% smaller than that of the TCUSUM chart.

It is interesting to find that the ($AL/AL_{T\&TCUSUM}$) ratio and AR value of a chart are fairly close to each other. Both seem to be reasonable and trustworthy measures of the overall effectiveness of a control chart.

4.2. *A factorial experiment*

Next, the three charts are further studied under different conditions through a 2^3 factorial experiment in which each of the three factors, λ_0, τ, and δ_{max}, varies at two levels as shown below:

λ_0:	0.001,	0.025.
τ:	$20/\lambda_0$,	$50/\lambda_0$.
δ_{max}:	30,	80.

This results in eight different cases or combinations of λ_0, τ, and δ_{max}, as shown in Table II. It is noted that τ is expressed in terms of λ_0 as suggested by Grade and Rattihalli[36].

For each case the three control charts are designed and each of them produces an ATS_0 value equal to τ. The relative operating characteristics of the charts are similar to that revealed in Table I and Figure 1. Generally, the T&TCUSUM chart is more effective than the other two charts across the T shift range, except that it is less sensitive to very small δ than the TCUSUM chart and, occasionally, less sensitive to very large δ than the T chart.

For all the eight cases, the overall performance of the control charts, as reflected by AL and AR, are summarized in Table II together with the charting parameters. The values of $AL/AL_{T\&TCUCUM}$ and AR are uniformly larger than one. This indicates that the T&TCUSUM chart always outperforms the individual T chart and TCUSUM chart in all eight cases. Its superiority over the T chart becomes very high (>60%) when τ is high and δ_{max} is low at the same time. On the other hand, the T&TCUSUM chart outperforms the TCUSUM chart to a more significant degree (close to or more than 20%) when δ_{max} is at high level. The specification λ_0 seems to have no impact on the performance comparison between the charts as τ is expressed in terms of λ_0.

The comparison between the T chart and TCUSUM chart strongly depends on δ_{max}. The TCUSUM chart outperforms the T chart when δ_{max} is low; but it is less effective than the latter when δ_{max} is high. It stands to reason as the TCUSUM chart is more sensitive to smaller T shifts and the T chart to larger ones.

Finally, a grand average $\overline{AL/AL_{T\&TCUSUM}}$ is calculated for each chart. It is the average of the $AL/AL_{T\&TCUSUM}$ ratios encompassing all the eight cases in Table II. The results are $\overline{AL_T/AL_{T\&TCUSUM}}=1.2666$ and $\overline{AL_{TCUSUM}/AL_{T\&TCUSUM}}=1.1412$. This indicates that, from the most comprehensive viewpoint (covering all different values of δ, λ_0, τ, and δ_{max}), the T&TCUSUM chart is more effective than the T chart and TCUSUM chart by 26.66 and 14.12%, respectively. In view of the ratio of ATS, the grand averages $\overline{AR_T}(=1.2364)$ and $\overline{AR_{TCUSUM}}(=1.1507)$ indicate the similar degrees of superiority of the T&TCUSUM chart over the other two charts.

It is noteworthy that neither the T chart nor the TCUSUM chart may outperform the T&TCUSUM chart under any circumstances (for any set of specifications λ_0, τ, and δ_{max}) because each of the T chart and TCUSUM chart is just a special case of the T&TCUSUM chart. If the lower limit LCL of the T&TCUSUM chart is fixed as zero and its reference parameter k and upper limit H are made equal to the k and H of a TCUSUM chart, then this T&TCUSUM chart will perform exactly as that TCUSUM chart. It means that one can always design a T&TCUSUM chart that outperforms the best possible TCUSUM chart, or at least works equally well. Similarly, if the H of a T&TCUSUM chart is set infinitely large and its LCL is made equal to the LCL of the best T chart, then the best T chart becomes a special case of the T&TCUSUM chart. Consequently, one can always design a T&TCUSUM chart that will surely perform better than, or at least equally well as, the best T chart or the best TCUSUM chart.

Table I. Comparison of three control charts ($\lambda_0 = 0.005, \tau = 10\,000, \delta_{max} = 60$)

	ATS		
	T&TCUSUM chart	TCUSUM chart	T chart
δ	LCL = 2.544 $H = 76.814$ $k = 50.667$	$H = 209.215$ $k = 101.333$	LCL = 4.041
1	10002.960	9991.169	10000.000
2	1809.547	1084.686	2624.242
4	356.951	256.996	693.557
5	228.517	181.921	455.915
6	163.928	140.707	324.986
8	102.259	97.033	192.261
9	85.599	84.115	155.658
10	73.461	74.302	129.126
12	57.024	60.380	93.910
14	46.444	50.965	72.123
16	39.077	44.155	57.625
18	33.659	38.985	47.439
20	29.510	34.916	39.978
22	26.235	31.624	34.328
24	23.587	28.903	29.932
25	22.446	27.712	28.088
26	21.405	26.616	26.434
28	19.577	24.664	23.598
30	18.026	22.980	21.261
32	16.694	21.511	19.308
34	15.538	20.219	17.656
36	14.526	19.073	16.243
38	13.634	18.050	15.024
40	12.841	17.131	13.964
42	12.131	16.302	13.033
44	11.493	15.549	12.212
46	10.917	14.862	11.482
48	10.393	14.234	10.830
50	9.914	13.656	10.244
52	9.476	13.124	9.716
54	9.073	12.631	9.237
56	8.701	12.174	8.801
58	8.357	11.749	8.403
60	8.038	11.353	8.039
AL	2.7669	3.3529	3.6726
$AL/AL_{T\&TCUSUM}$	1.0000	1.2118	1.3273
AR	1.0000	1.2254	1.2852

5. Example

An electronic component is produced at a rate of 350 units/h in a production line. The company has 225 records of the time t_i (in seconds) between defective units. They were acquired when the process was in a stable and in-control status. It is found that these t_i data can be fitted very well to an exponential distribution. The in-control λ_0 is estimated as 0.001123/s from the records. This λ_0 value indicates that, on average, there is a defective unit for every 890.47 s (or 0.2474 h). The Quality Assurance (QA) engineer decides to set $\tau = 43\,200$ s (or 12 h). He is most interested to detect the increase of λ up to 60 times of λ_0. The specifications are summarized as follows:

$$\lambda_0 = 0.001123/\text{s}, \quad \tau = 43200\text{s}, \quad \delta_{max} = 60.$$

The three control charts (the T chart, TCSUM chart, and T&TCUSUM chart) are designed for this case and their charting parameters are listed below:

$$\text{T chart:} \quad \text{LCL} = 18.5588.$$

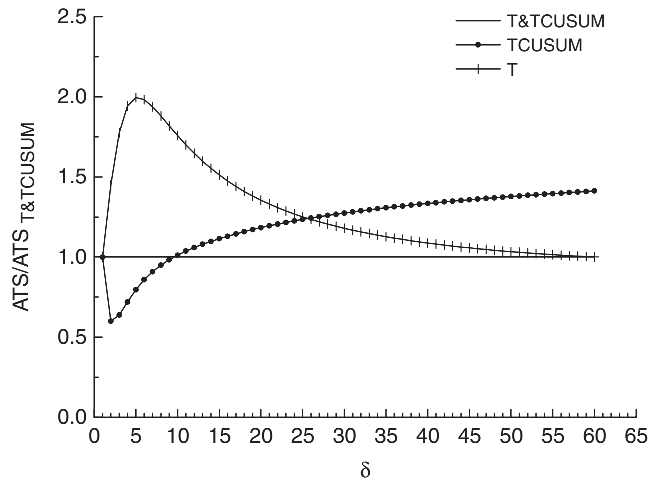

Figure 1. Normalized ATS of three control charts

Table II. Factorial experiment

Case	λ_0	τ	δ_{max}		LCL	H	k	AL	$AL/AL_{T\&TCUSUM}$	AR
1	0.001 (−)	20 000 (−)	30 (−)	T	51.293	—	—	2.960	1.15644	1.13332
				TCUSUM	—	831.411	544.444	2.662	1.04020	1.04762
				T&TCUSUM	28.922	535.370	388.888	2.559	1.00000	1.00000
2	0.001 (−)	20 000 (−)	80 (+)	T	51.293	—	—	2.240	1.03417	1.03345
				TCUSUM	—	763.342	520.833	2.589	1.19508	1.19887
				T&TCUSUM	41.607	508.955	343.750	2.166	1.00000	1.00000
3	0.001 (−)	50 000 (+)	30 (−)	T	20.202	—	—	5.350	1.69230	1.61587
				TCUSUM	—	1113.295	522.222	3.447	1.09034	1.10306
				T&TCUSUM	0.335	553.568	355.555	3.161	1.00000	1.00000
4	0.001 (−)	50 000 (+)	80 (+)	T	20.202	—	—	3.197	1.18364	1.16295
				TCUSUM	—	1062.182	510.416	3.346	1.23900	1.25338
				T&TCUSUM		651.385	312.500	2.701	1.00000	1.00000
5	0.025 (+)	800 (−)	30 (−)	T	2.051	—	—	2.960	1.15644	1.13332
				TCUSUM	—	33.256	21.777	2.662	1.04020	1.04762
				T&TCUSUM	1.156	21.414	15.555	2.559	1.00000	1.00000
6	0.025 (+)	800 (−)	80 (+)	T	2.051	—	—	2.240	1.03417	1.03345
				TCUSUM	—	30.533	20.833	2.589	1.19508	1.19887
				T&TCUSUM	1.664	20.358	13.750	2.166	1.00000	1.00000
7	0.025 (+)	2000 (+)	30 (−)	T	0.808	—	—	5.350	1.69230	1.61587
				TCUSUM	—	44.531	20.888	3.447	1.09034	1.10306
				T&TCUSUM		22.142	14.222	3.161	1.00000	1.00000
8	0.025 (+)	2000 (+)	80 (+)	T	0.808	—	—	3.197	1.18364	1.16295
				TCUSUM	—	42.487	20.416	3.346	1.23900	1.25338
				T&TCUSUM	0.686	26.055	12.500	2.701	1.00000	1.00000
		T		$\overline{AL_T/AL_{T\&TCUSUM}}=1.2666$					$\overline{AR_T}=1.2364$	
		TCUSUM		$\overline{AL_{TCUSUM}/AL_{T\&TCUSUM}}=1.1412$					$\overline{AR_{TCUSUM}}=1.1507$	

TCUSUM chart: $k=470.1217$, $H=1001.0106$.

T&TCUSUM chart: $LCL=14.7304$, $k=319.6827$, $H=693.2005$.

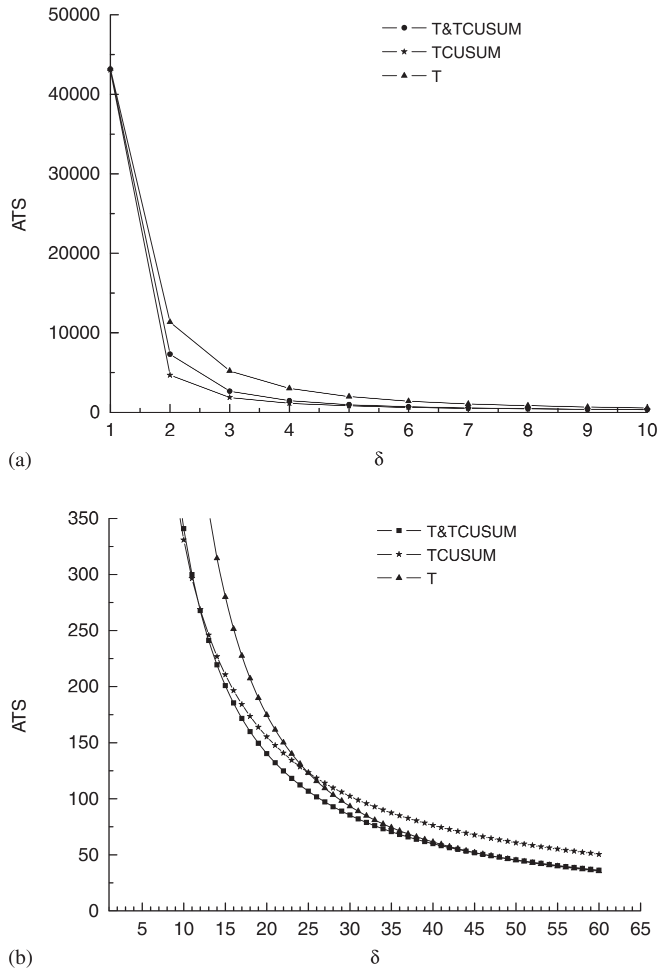

Figure 2. ATS comparison among three charts in the example. (a) Entire shift range. (b) Moderate and large shifts.

Figure 2 displays the ATS curves of the three charts. Figure 2(a) shows the curves over the shift range of ($1 \leq \delta \leq 10$). The in-control ATS_0 values of all charts are equal to $\tau(=43\,200)$. Figure 2(b) zooms in the ATS curves for moderate and large T shifts. The comparison of the effectiveness of the three charts is similar to that summarized in the Section of comparative studies. As expected, the T&TCUSUM chart is more powerful than the other two charts from an overall viewpoint. The ratio of $AL_T/AL_{T\&TCUSUM}$ is equal to 1.2564 and $AL_{TCUSUM}/AL_{T\&TCUSUM}$ is equal to 1.1659.

As the T&TCUSUM chart has excellent detection effectiveness and the SPC operation will be carried out with the help of an on-site computer, this chart is selected for the process monitoring in this application. The operators only have to enter the sample value of T from a keyboard for each detected defective unit, all the computation and plotting will be handled by a computer program.

6. Conclusions and discussions

This article presents the basic ideas, as well as the design, implementation, and performance assessment of the joint T&TCUSUM scheme. The design of this chart is conducted based on an objective function representing the average loss AL incurred by the out-of-control cases. The T&TCUSUM chart always outperforms the individual T and TCUSUM charts in overall effectiveness. It is more effective than the T chart by 26.66% and the TCUSUM chart by 14.12%, considering different specified values of λ_0, τ, and δ_{max} as studied in this article. The improvement achieved by the T&TCUSUM chart over the TCUSUM chart is more significant than the superiority of the X&CUSUM chart over the CUSUM chart[33]. The high detection effectiveness of the T&TCUSUM chart is attributable to its ability of making use of the information about the last time interval T and the information residing in the entire series of data of T. In view of implementation, a T&TCUSUM chart can be operated almost as easily as a TCUSUM chart.

Meanwhile, the T&TCUSUM chart demonstrates the strength in achieving a balanced effectiveness for detecting T shifts of different sizes. Its performance is satisfactory across the entire shift range. Conversely, the TCUSUM chart is only effective for detecting small T shifts but insensitive to large shifts. Similarly, the T chart is only powerful for detecting large T shifts but quite ineffective for small shifts. It is an important issue as both small and large T shifts may arise unpredictably for most of the events.

In this article, it is assumed that the time interval T follows an exponential distribution. It is interesting to carry out further studies on the T&TCUSUM chart for the cases in which T follows other probability distributions. Furthermore, if it is difficult to identify a suitable theoretical distribution for an application, a nonparametric approach can be employed to estimate the probability distribution based on available data. For all these alternatives, the T&TCUSUM chart is expected to work well because the high effectiveness of this chart is not obtained due to the use of a particular probability distribution. Instead, it is attributable to the utilization of the information including the last and all previous sample values of T.

References

1. Costa AFB, De Magalhaes MS. An adaptive chart for monitoring the process mean and variance. *Quality and Reliability Engineering International* 2007; **23**:821–831.
2. Wu Z, Wang QN. A single CUSUM chart using a single observation to monitor a variable. *International Journal of Production Research* 2007; **45**:719–741.
3. Zhang LY, Chen GM, Castagliola P. Ont and EWMAt charts for monitoring changes in the process mean. *Quality and Reliability Engineering International* 2009; **25**:933–945.
4. Shi LH, Zou CL, Wang ZJ, Kapur KC. A new variable sampling control scheme at fixed times for monitoring the process dispersion. *Quality and Reliability Engineering International* 2009; **25**:961–972.
5. Khoo MBC, Xie M. A study of time-between-events control chart for the monitoring of regularly maintained systems. *Quality and Reliability Engineering International* 2009; **25**:805–819.
6. Wang KB, Tsung F. An adaptive dimension reduction scheme for monitoring feedback-controlled processes. *Quality and Reliability Engineering International* 2009; **25**:283–298.
7. Calvin TW. Quality control techniques for 'zero-defects'. *IEEE Transactions on Components, Hybrids, and Manufacturing Technology* 1983; **6**:323–328.
8. Goh TN. A control chart for very high yield processes. *Quality Assurance* 1987; **13**:18–22.
9. Chan LY, Xie M, Goh TN. Cumulative quantity control charts for monitoring production processes. *International Journal of Production Research* 2000; **38**:397–408.
10. Zhang CW, Xie M, Goh TN. Economic design of cumulative count of conforming charts under inspection by samples. *International Journal of Production Economics* 2008; **111**:93–104.
11. Bourke PD. Detecting a shift in fraction nonconforming using run-length control charts with 100% inspection. *Journal of Quality Technology* 1991; **23**:225–238.
12. Bourke PD. The RL_2 chart versus the np chart for detecting upward shifts in fraction defective. *Journal of Applied Statistics* 2006; **33**:1–15.
13. Wu Z, Zhang XL, Yeo SH. Design of the sum-of-conforming-run-length control chart. *European Journal of Operational Research* 2001; **132**:187–196.
14. Zhang CW, Xie M, Liu JY, Goh TN. A control chart for the gamma distribution as a model of time between events. *International Journal of Production Research* 2007; **45**:5649–5666.
15. Lucas JM. Counted data Cusum's. *Technometrics* 1985; **27**:129–144.
16. Vardeman S, Ray D. Average run lengths for CUSUM schemes when observations are exponentially distributed. *Technometrics* 1985; **27**:145–150.
17. Gan FF. Exact run length distributions for one-sided exponential CUSUM schemes. *Statistica Sinica* 1992; **2**:297–312.
18. Borror CM, Keats JB, Montgomery DC. Robustness of the time between events CUSUM. *International Journal of Production Research* 2003; **41**:3435–3444.
19. Bourke PD. The geometric CUSUM chart with sampling inspection for monitoring fraction detective. *Journal of Applied Statistics* 2001; **28**:951–972.
20. Gan FF. Designs of one- and two-sided exponential EWMA charts. *Journal of Quality Technology* 1998; **30**:55–69.
21. Wu Z, Spedding TA. A synthetic control chart for detecting small shifts in the process mean. *Journal of Quality Technology* 2000; **32**:32–38.
22. Wu Z, Spedding TA. Implementing synthetic control charts. *Journal of Quality Technology* 2000; **32**:75–78.
23. Xie M, Goh TN, Lu XS. Some effective control chart procedures for reliability monitoring. *Reliability Engineering and System Safety* 2002; **77**:143–150.
24. Wu Z, Jiao JX, He Z. A single control chart for monitoring the frequency and magnitude of an event. *International Journal of Production Economics* 2009; **119**:24–33.
25. Wu Z, Jiao JX, He Z. A control scheme for monitoring the frequency and magnitude of an event. *International Journal of Production Research* 2009; **47**:2887–2902.
26. Benneyan JC. Number-between g-type statistical quality control charts for monitoring adverse events. *Health Care Management Science* 2001; **4**:305–318.
27. Benneyan JC. Performance of number-between g-type statistical control charts for monitoring adverse events. *Health Care Management Science* 2001; **4**:319–336.
28. Woodall WH. The use of control charts in health-care and public-health surveillance. *Journal of Quality Technology* 2006; **38**:89–134.
29. Reynolds MR Jr, Stoumbos ZG. Should observations be grouped for effective process monitoring. *Journal of Quality Technology* 2004; **36**:343–366.
30. Zhao Y, Tsung F, Wang ZJ. Dual CUSUM control scheme for detecting a range of mean shift. *IIE Transactions* 2005; **37**:1047–1058.
31. Woodall WH, Montgomery DC. Research issues and ideas in statistical process control. *Journal of Quality Technology* 1999; **31**:376–386.
32. Sparks RS. Cusum charts for signalling varying location shifts. *Journal of Quality Technology* 2000; **32**:157–171.
33. Lucas JM. Combined Shewhart-CUSUM quality control schemes. *Journal of Quality Technology* 1982; **14**:51–59.
34. Wu Z, Yang M, Jiang W, Khoo MBC. Optimization designs of the combined Shewhart-CUSUM control charts. *Computational Statistics and Data Analysis* 2008; **53**:496–506.
35. Duncan AJ. The economic design of $\bar{X}$ chart to maintain current control of a process. *Journal of the American Statistical Association* 1956; **51**:228–241.
36. Grade MP, Rattihalli RN. Unit and Group-Runs chart to identify increases in fraction nonconforming. *Journal of Quality Technology* 2005; **37**:199–209.

Appendix A: Calculation of the ATS of the T&TCUSUM chart

Suppose that the statistic C_t in Equation (3) experiences m different transitional states before being absorbed into the out-of-control state. States 0 to $(m-1)$ are in-control states and state m is an out-of-control state (m is set as 100 in this article). The width of the interval of each in-control state is given as

$$d = H/(m-0.5). \tag{A1}$$

The center, O_i, of state i is given by

$$O_i = i \cdot d \quad i = 0, \quad 1, \ldots, m-1. \tag{A2}$$

Let p_{ij} be the transition probability from state i to state j for the T&TCUSUM chart. First, only the TCUSUM chart component is taken into consideration. If j = 0 (referring to Equation (3)),

$$p_{i0} = Pr(O_i + k - T < 0.5d) = Pr(T > L_0), \tag{A3}$$

where

$$L_0 = (i-0.5)d + k. \tag{A4}$$

On the other hand, if $j > 0$,

$$p_{ij} = Pr(O_j - 0.5d < O_i + k - T < O_j + 0.5d) = Pr(L_j < T < U_j), \tag{A5}$$

where

$$L_j = (i-j-0.5)d + k, \quad U_j = (i-j+0.5)d + k. \tag{A6}$$

Then, the T chart component is added. This means that if $T < \mathrm{LCL}$, the T&TCUSUM chart will produce an out-of-control signal immediately, or that the statistic C_t will be transferred to the out-of-control state unconditionally. In view of this, Equations (A3) and (A5) for the transition probability p_{ij} have to be modified as below:

$$p_{i0} = \begin{cases} Pr(T > L_0) & \text{if LCL} < L_0, \\ Pr(T > \mathrm{LCL}) & \text{if LCL} > L_0, \end{cases} \tag{A7}$$

$$p_{ij} = \begin{cases} 0 & \text{if LCL} > U_j, \\ Pr(\mathrm{LCL} < T < U_j) & \text{if } U_j > \mathrm{LCL} > L_j, \\ Pr(L_j < T < U_j) & \text{if LCL} < L_j. \end{cases} \tag{A8}$$

Or

$$p_{i0} = \begin{cases} 1 - Pr(T < L_0) & \text{if LCL} < L_0, \\ 1 - Pr(T < \mathrm{LCL}) & \text{if LCL} > L_0, \end{cases} \tag{A9}$$

$$p_{ij} = \begin{cases} 0 & \text{if LCL} > U_j, \\ Pr(T < U_j) - Pr(T < \mathrm{LCL}) & \text{if } U_j > \mathrm{LCL} > L_j, \\ Pr(T < U_j) - Pr(T < L_j) & \text{if LCL} < L_j. \end{cases} \tag{A10}$$

Based on the transition probability p_{ij}, the transition matrix $\boldsymbol{R}$ can be established. It is a matrix with a size of $(m \times m)$, excluding the elements associated with the absorbing (or out-of-control) state. There are three types of time intervals and transition matrices. Correspondingly, the probabilities of $Pr(T \leq t)$ in Equations (A9) and (A10) will be calculated differently.

1. *Pre-shift interval*: It represents all the time intervals before the T shift takes place. In these time intervals, $\lambda=\lambda_0$. To establish the corresponding transition matrix $\boldsymbol{R}_{\text{pre}}$ for a pre-shift interval, the probabilities of $Pr(T\leq t)$ is determined by Equation (1) using $\lambda=\lambda_0$.

$$P_r(T\leq t)=F_T(t)=1-\mathrm{e}^{-\lambda_0 t}. \tag{A11}$$

2. *Shift interval*: It is the time interval within which the T shift happens in a random time t_s. In this time interval, $\lambda=\lambda_0$ before t_s, but $\lambda=\lambda_1(=\delta\lambda_0)$ after t_s. For the corresponding transition matrix $\boldsymbol{R}_{\text{shift}}$, the probability of $Pr(T\leq t)$ is denoted as $F_T^*(t)$ and is calculated by the following formula derived by Wu *et al*[25]:

$$P_r(T\leq t)=F_T^*(t)=\lambda_1\left[\frac{1-\mathrm{e}^{-\lambda_1 t}}{\lambda_1}-\frac{\mathrm{e}^{-\lambda_0 t}(1-\mathrm{e}^{-(\lambda_1-\lambda_0)t})}{\lambda_1-\lambda_0}\right]. \tag{A12}$$

3. *Post-shift interval*: It represents all the time intervals after the T shift takes place. To establish the corresponding transition matrix $\boldsymbol{R}_{\text{post}}$, the probability of $Pr(T\leq t)$ is calculated by Equation (1) using $\lambda=\lambda_1$.

$$P_r(T\leq t)=F_T(t)=1-\mathrm{e}^{-\lambda_1 t}. \tag{A13}$$

The in-control Average Run Length ARL_0 is equal to the first element of the vector V_0 given by the following formula:

$$\boldsymbol{V_0}=(\boldsymbol{I}-\boldsymbol{R}_{\text{pre}})^{-1}1, \tag{A14}$$

where $\boldsymbol{I}$ is an identity matrix and $\mathbf{1}$ is a vector with all elements equal to one. Then ATS_0 is calculated by

$$\text{ATS}_0=\text{ARL}_0/\lambda_0. \tag{A15}$$

On the other hand, the out-of-control ATS under the steady-state mode is calculated as follows:

$$\text{ATS}=(\boldsymbol{B}^{\mathrm{T}}\boldsymbol{R}_{\text{shift}}\boldsymbol{V_1}+1)/\lambda_1, \tag{A16}$$

where $\boldsymbol{B}$ is the steady-state probability vector obtained by first normalizing $\boldsymbol{R}_{\text{pre}}$ and then solving the following equation:

$$(\boldsymbol{I}-\boldsymbol{R}_{\text{pre}}^{\mathrm{T}})\boldsymbol{B}=\mathbf{0} \tag{A17}$$

subject to

$$\mathbf{1}^{\mathrm{T}}\mathbf{B}=1. \tag{A18}$$

The vector $\boldsymbol{V_1}$ is the out-of-control ARL vector under the zero-state mode and determined by

$$\boldsymbol{V_1}=(\boldsymbol{I}-\boldsymbol{R}_{\text{post}})^{-1}\mathbf{1}. \tag{A19}$$

All the formulae for calculating the in-control ATS_0 and out-of-control ATS have been verified by simulation.

Authors' biographies

Liang Qu is a PhD student in the School of Mechanical and Aerospace Engineering of Nanyang Technological University, Singapore. She obtained her BE and ME in 2006 and 2008 from Harbin Institute Technology, China. Her current research work focuses on Statistical Process Control.

Zhang Wu is an Associate Professor in the School of Mechanical and Aerospace Engineering of Nanyang Technological University, Singapore. He obtained his PhD in 1988 and MEng in 1984 from the McMaster University, Canada, and obtained his BEng in 1982 from the Huazhong University of Science and Technology, China. His current research interests include the development of algorithms in quality control, reliability, nonlinear optimization, and geometrical tolerance. He is a senior member of the American Society for Quality.

Tien-I Liu is an ASME Fellow. He is a Professor of the Department of Mechanical Engineering, California State University, Sacramento. Professor Liu has received both his MS and PhD in Mechanical Engineering from the University of Wisconsin-Madison. He received his BS in Mechanical Engineering from National Taiwan University. His research interests are in quality and manufacturing.