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A Ruin Model with Random Income and Dependence between Claim Sizes and Claim Intervals

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Abstract In this paper, we consider a generalization of the classical ruin model, where the income is random and the distribution of the time between two claim occurrences depends on the previous claim size. This model is more appropriate than the classical ruin model. Explicit expression for the generating function of the Gerber-Shiu expected discounted penalty function are derived. A similar model is discussed. Finally, the result are showed by two examples.

Keywords Ruin model, expected discounted penalty function, dependence, ruin probability
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1 Introduction

The classical Cramer-Lundberg model describing the surplus process of an insurance portfolio relies on the assumption of independence among claim sizes and between claim size and inter-claim time. As the time went by, this assumption turns out to be too restrictive and there is a need for more general models. As discussed in e.g. Albrecher and Boxma[2] and Boudreault et al.[7], there exist many actuarial contexts for which such assumption is inappropriate. Albrecher and Boxma[2] had considered a generalization of the classical ruin model to a dependent setting, where the distribution of the time between two claim occurrences depends on the previous claim size. Boundreault et al[7] had given a risk model with the reverse dependence structure (i.e. the distribution of the next claim size depends on the last interarrival time). Note that, for these risk models, it is explicit that the premium income is a linear function. Bao[4] had discussed a ruin model, in which the premium is no longer a linear function of the time but another Poisson process.

In this paper a generalization of a ruin model is considered in Section 2, where the distribution of the inter-claim time depends on the previous claim size and the income is random. In Section 3, we consider a similar ruin model and obtain explicit expression for the generating function of Gerber-Shiu expected discounted penalty function. Finally, by using the method given in Section 2 and Section 3, two examples are given.

2 Model

Let us consider the following risk model $U(t)$ which is an insurance portfolio:

$$U(t) = u + M(t) - \sum_{i=1}^{N(t)} X_i,$$

where $u = U(0)$ is the initial capital. X_i is the size of the i-th claim and $N(t)$ is the number

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of claims up to time t. We assume the premium income is no longer a linear function of the time bur another Poisson process $\{M(t), t \geqslant 0\}$ with parameter λ , which is independent of $\{N(t), t \geqslant 0\}$ and $\{X_1, X_2, \cdots\}$. For simplicity, the premium rate is supposed to be 1. Let $\{W_i, i \geqslant 1\}$ be a sequence of i.i.d. random variables, which denote the time between claims. $\{X_i, i \geqslant 1\}$ is a sequence of i.i.d. random variables with distribution function $F(x)$, mean μ and probability function $f(x)$. Actually, the model is a discrete ruin model. We assume the claim occurrence process to be of the following Markovian type: If a claim X_i is no less than a threshold T_i, then the time until the next claim is exponentially distributed with rate λ_1, otherwise it is exponentially distributed with rate λ_2. The quantities T_i is assumed to be i.i.d. random variables with distribution function $T(\cdot)$.

Let $\tau = \inf\{t \geqslant 0 : U(t) < 0\}$ be the time of ruin, and the probability of ultimate ruin with initial surplus u is $\psi(u) = P\{\tau < \infty | U(0) = u\}$. Note that if ruin occurs, $|U(\tau)|$ is the deficit at ruin and $U(\tau-)$ is the surplus immediately prior to ruin. Denote by

$$m(u) = E\big[e^{-\delta\tau}\omega\big(U(\tau-), |U(\tau)|\big)I(\tau < \infty)|U(0) = u\big],$$

the Gerber-Shiu expected discounted penalty function. Here $\delta \geqslant 0$ is the discounted factor, $\omega(u_1, u_2)$ is a nonnegative function, and $I(\cdot)$ is the indicator function.

We assume that

$$\mu < \lambda\Big[\frac{P(X \geqslant T)}{\lambda_1} + \frac{P(X < T)}{\lambda_2}\Big], \tag{1}$$

to ensure the positive safety loading condition as the classical case, and $P(X > 0) = P(T > 0) = 1$.

Let $m_i(u)$ $(i = 1, 2)$ denote the discounted penalty function with initial capital u given that the first claim occurs according to the exponential distribution with rate λ_i. By conditioning on the time W_1 and the amount X_1 of the first claim, we get

$$\begin{aligned}
m_1(u) &= \sum_{n=0}^{\infty}\int_0^{\infty} e^{-\delta t}\lambda_1 e^{-\lambda_1 t} P(M(t) = n)\, dt\Big(\sum_{i=1}^{u+n} P(T \leqslant i)\, m_1(u+n-i)\, f(i) \\
&\quad + \sum_{i=1}^{u+n} P(T > i)\, m_2(u+n-i)\, f(i) + \sum_{i=u+n+1}^{\infty}\omega(u+n, i-u-n)\, f(i)\Big) \\
&= \sum_{n=0}^{\infty}\frac{\lambda_1\lambda^n}{n!}\int_0^{\infty} e^{-(\lambda_1+\lambda+\delta)t} t^n dt \gamma(u+n) \\
&= \sum_{n=0}^{\infty}\frac{\lambda_1}{\lambda_1+\lambda+\delta}\Big(\frac{\lambda}{\lambda_1+\lambda+\delta}\Big)^n\gamma(u+n),
\end{aligned} \tag{2}$$

where

$$\begin{aligned}
\gamma(t) &= \sum_{i=1}^{t} P(T \leqslant i) m_1(t-i) f(i) + \sum_{i=1}^{t} P(T > i) m_2(t-i) f(i) + \sum_{i=t+1}^{\infty}\omega(t, i-t) f(i) \\
&= \sum_{i=1}^{t} m_1(t-i) f(i) T(i) + \sum_{i=1}^{t} m_2(t-i) f(i)(1-T(i)) + \omega(t), \qquad t \in N, \\
\omega(j) &= \sum_{i=j+1}^{\infty}\omega(j, i-j) f(i).
\end{aligned}$$

Let $p_i = \frac{\lambda_i}{\lambda_i+\lambda+\delta}$, $q_i = \frac{\lambda}{\lambda_i+\lambda+\delta}$, $i = 1, 2$, then we have by (2)

$$m_1(u) = \sum_{n=0}^{\infty} p_1 q_1^n \gamma(u+n), \tag{3}$$

from (3) with u replaced by $u+1$, it follows, after rearranging, that

$$q_1 m_1(u+1) = m_1(u) - p_1\gamma(u). \tag{4}$$

Then multiplying by z^u and summing over u from 0 to ∞, function (4) satisfy

$$q_1\left[M_1(z) - m_1(0)\right] = z\left[M_1(z) - p_1 M_1(z)A(z) - p_1 M_2(z)B(z) - p_1 W(z)\right], \tag{5}$$

or equivalently

$$\left[q_1 - z + zp_1A(z)\right]M_1(z) = q_1 m_1(0) - zp_1 M_2(z)B(z) - zp_1W(z), \tag{6}$$

where

$$\begin{aligned}
M_i(z) &= \sum_{u=0}^{\infty} z^u m_i(u),\\
A(z) &= \sum_{u=1}^{\infty} z^u f(u)T(u),\\
B(z) &= \sum_{u=1}^{\infty} z^u f(u)\left[1 - T(u)\right],\\
W(z) &= \sum_{u=0}^{\infty} z^u \cdot \omega(u).
\end{aligned}$$

Similarly we obtain

$$\left[q_2 - z + zp_2B(z)\right]M_2(z) = q_2 m_2(0) - zp_2 M_1(z)A(z) - zp_2W(z). \tag{7}$$

From (6) and (7), we obtain

$$M_1(z) = \frac{\left[q_1m_1(0) - zp_1W(z)\right]\left[q_2 - z + zp_2B(z)\right] - \left[q_2m_2(0) - zp_2W(z)\right]zp_1B(z)}{\left[q_1 - z + zp_1A(z)\right]\left[q_2 - z + zp_2B(z)\right] - z^2p_1p_2A(z)B(z)} \tag{8}$$

and

$$M_2(z) = \frac{\left[q_2m_2(0) - zp_2W(z)\right]\left[q_1 - z + zp_1A(z)\right] - \left[q_1m_1(0) - zp_1W(z)\right]zp_2A(z)}{\left[q_1 - z + zp_1A(z)\right]\left[q_2 - z + zp_2B(z)\right] - z^2p_1p_2A(z)B(z)}. \tag{9}$$

Note that the denominators on the right-hand side of (8) and (9) coincide.

Lemma 1. *The denominator of (8) has exactly 2 roots, say, ρ_1, ρ_2 in the unit circle.*

Proof. Set

$$B(z) = \begin{pmatrix} zp_1A(z) & zp_1B(z)\\ zp_2A(z) & zp_2B(z)\end{pmatrix}$$

and $\Lambda(z) = \operatorname{diag}(q_1 - z, q_2 - z)$. Let $A(z,u) = \Lambda(z) + uB(z)$. It is easy to check that the roots of the equation $\det(A(z,1)) = 0$ is equivalent to the zeros of the denominator of (8) and (9).

We first prove that for $0 \leqslant u \leqslant 1$,

$$\det A(z,u) \neq 0 \qquad \text{for} \quad z \in \overline{C},$$

where $\overline{C} = \{z : |z - q_{\min}| \geqslant 1 - q_{\min}, |z| \leqslant 1\}$, $C = \{z : |z - q_{\min}| = 1 - q_{\min}\}$, following the idea in [1]. The matrix $A(z,u)$ is diagonally dominant for $0 \leqslant u \leqslant 1$ in $\overline{C}$.

$$\begin{aligned}
&|q_1 - z + uzp_1A(z)| \geqslant |q_1 - z| - |up_1zA(z)| = |q_1 - q_{\min} + q_{\min} - z| - |up_1zA(z)| \\
&\geqslant |q_{\min} - z| - |q_1 - q_{\min}| - |up_1zA(z)| \geqslant 1 - q_{\min} - q_1 + q_{\min} - |up_1zA(z)| \\
&= 1 - q_1 - up_1|z||A(z)| > p_1u|z|(1 - |A(z)|) \geqslant up_1|z|\left(\sum_{i=1}^{\infty} f(i) - \left|\sum_{i=1}^{\infty} z^iT(i)f(i)\right|\right) \\
&\geqslant up_1|z|\left(\sum_{i=1}^{\infty} f(i)(1 - T(i))\right) \geqslant up_1|z|\left|\sum_{i=1}^{\infty} z^if(i)(1 - T(i))\right| \geqslant up_1|z||B(z)|
\end{aligned}$$

and

$$\begin{aligned}
&|q_2 - z + uzp_2B(z)| \geqslant |q_2 - z| - up_2|z||B(z)| \geqslant |z - q_{\min}| - |q_2 - q_{\min}| - up_2|z|\left|B(z)\right| \\
&\geqslant 1 - q_2 - up_2|z|\left|B(z)\right| \geqslant p_2u|z|(1 - |B(z)|) \geqslant up_2|z|\left|\sum_{i=1}^{\infty} f(i) - \sum_{i=1}^{\infty} f(i)(1 - T(i))\right| \\
&\geqslant up_2|z|\left|\sum_{i=1}^{\infty} z^if(i)T(i)\right| = up_2|z||A(z)|.
\end{aligned}$$

The diagonal dominance implies that $\det(A(z,u)) \neq 0$ for $z \in \overline{C}$, which means there is no root in the set $\overline{C}$ (see [10]).

Then, we prove there are two roots in $C^+ = \{z : |z - q_{\min}| < 1 - q_{\min}\}$. Now let $f(u)$ denote the number of roots of equation $\det(A(z,u)) = 0$ in C^+, then

$$f(u) = \frac{1}{2\pi i}\int_C \frac{\frac{d}{ds}\det(A(z,u))}{\det(A(z,u))}ds.$$

Hence $f(u)$ is a continuous function on $[0,1]$, integer valued, and therefore constant. $f(0) = 2$ since $\det(\Lambda(z)) = 0$ has 2 roots. Thus, $f(1) = 2$. That is to say that the denominator of (8) has exactly 2 roots in the unit circle.

Since $M(z)$ is an analytic function for $|z| \leqslant 1$, ρ_1, ρ_2 must also be zeros of the numerator of (8) and (9), and we have

$$m_1(0) = \frac{p_1\rho_1\rho_2\left[B(\rho_2)W(\rho_1)(q_2 - \rho_1) - B(\rho_1)W(\rho_2)(q_2 - \rho_2)\right]}{q_1\left[\rho_2B(\rho_2)(q_2 - \rho_1) - \rho_1B(\rho_1)(q_2 - \rho_2)\right]} \tag{10}$$

and

$$m_2(0) = \frac{p_2\rho_1\rho_2\left[A(\rho_2)W(\rho_1)(q_1 - \rho_1) - A(\rho_1)W(\rho_2)(q_1 - \rho_2)\right]}{q_2\left[\rho_2A(\rho_2)(q_1 - \rho_1) - \rho_1A(\rho_1)(q_1 - \rho_2)\right]}. \tag{11}$$

Combined with (8), (9), (10), (11), this completes the determination of $M_i(z)$ $i = 1,2$.

Remark. If we set $\lambda_1 = \lambda_2 = \lambda_0$, we obtain $p_1 = p_2, q_1 = q_2, M_1(z) = M_2(z)$. Equation (6) can be turned into[4]

$$[q_1 - z + zp_1A(z)]M_1(z) = q_1m_1(0) - zp_1W(z). \tag{12}$$

3 Another Model

In every exact $t > 0$,the risk process is in one of the two states $i = 1,2$, corresponding to the rate λ_i of the exponential distribution for the time until the first claim occurs. At the time of a claim occurrence the state of the system may change depending on the corresponding claim size. If a claim X_i is smaller than a threshold T_i , then the state of the risk process changes, otherwise it does not. The quantities $\{T_i, i \geqslant 1\}$ are again assumed to be i.i.d. random variables with distribution function $T(\cdot)$. We also assume that the premium income is a Poisson process $\{M(t), t \geqslant 0\}$ with parameter λ, which is independent of $\{N(t), t \geqslant 0\}$ and $\{X_1, X_2, \cdots\}$. Let $\{W_i, i \geqslant 1\}$ be the interclaim times, which is a sequence of i.i.d. random variables. We assume

$$2\mu < \lambda\Big[\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\Big]. \tag{13}$$

The generating function of $m_i(u)$ (which is the Gerber-Shiu expected discounted penalty function with initial capital u, given that the system starts out in state i) is analogous to the previous section. Then we obtain

$$[q_1 - z + zp_1A(z)]\, M_1(z) = q_1m_1(0) - zp_1M_2(z)B(z) - zp_1W(z) \tag{14}$$

and

$$[q_2 - z + zp_2A(z)]\, M_2(z) = q_2m_2(0) - zp_2M_1(z)B(z) - zp_2W(z), \tag{15}$$

from which it follows for $|z| \leqslant 1$,

$$M_1(z) = \frac{[q_1m_1(0) - zp_1W(z)]\,[q_2 - z + zp_2A(z)] - [q_2m_2(0) - zp_2W(z)]\, zp_1B(z)}{[q_1 - z + zp_1A(z)]\,[q_2 - z + zp_2A(z)] - z^2p_1p_2B(z)B(z)}, \tag{16}$$

$$M_2(z) = \frac{[q_2m_2(0) - zp_2W(z)]\,[q_1 - z + zp_1A(z)] - [q_1m_1(0) - zp_1W(z)]\, zp_2B(z)}{[q_1 - z + zp_1A(z)]\,[q_2 - z + zp_2A(z)] - z^2p_1p_2B(z)B(z)}, \tag{17}$$

where $M_i(z)$ is again the generating function. Note that the denominators on the right-hand side of (16) and (17) coincide too.

As for Model 1, we could prove that the denominator of (16) or (17) has exactly 2 roots in the unit circle, say, ρ_1, ρ_2. We have

$$m_1(0) = \frac{p_1\rho_1\rho_2\,[W(\rho_1)\,B(\rho_2)\,(\eta_2(\rho_1) + \rho_1p_2B(\rho_1)) - W(\rho_2)\,B(\rho_1)\,(\eta_2(\rho_2) + \rho_2p_2B(\rho_2))]}{q_1\,[\rho_2B(\rho_2)\,\eta_2(\rho_1) - \rho_1B(\rho_1)\,\eta_2(\rho_2)]}, \tag{18}$$

$$m_2(0) = \frac{p_2\rho_1\rho_2\,[W(\rho_1)\,B(\rho_2)\,(\eta_1(\rho_1) + \rho_1p_1B(\rho_1)) - W(\rho_2)\,B(\rho_1)\,(\eta_1(\rho_2) + \rho_2p_1B(\rho_2))]}{q_2\,[\rho_2B(\rho_2)\,\eta_1(\rho_1) - \rho_1B(\rho_1)\,\eta_1(\rho_2)]}, \tag{19}$$

where

$$\eta_j(\rho_i) = q_j - \rho_i + \rho_ip_jA(\rho_i), \qquad i, j = 1, 2. \tag{20}$$

Remark. For the special case $\lambda_1 = \lambda_2 = \lambda_0$ we obtain $p_1 = p_2, q_1 = q_2, M_1(z) = M_2(z)$. Equation (14) can be turned into[4]

$$[q_1 - z + zp_1A(z)]\, M_1(z) = q_1m_1(0) - zp_1W(z).$$

If alternatively, the state of the risk process changes at the time of a claim occurrence, given that X_j is larger than a threshold T_j and remains in its state otherwise, we get instead of (16) and (17),

$$M_1(z) = \frac{[q_1m_1(0) - zp_1W(z)]\,[q_2 - z + zp_2B(z)] - [q_2m_2(0) - zp_2W(z)]\, zp_1A(z)}{[q_1 - z + zp_1A(z)]\,[q_2 - z + zp_2A(z)] - z^2p_1p_2B(z)B(z)}, \tag{21}$$

$$M_2(z) = \frac{[q_2 m_2(0) - zp_2 W(z)]\,[q_1 - z + zp_1 B(z)] - [q_1 m_1(0) - zp_1 W(z)]\, zp_2 A(z)}{[q_1 - z + zp_1 A(z)]\,[q_2 - z + zp_2 A(z)] - z^2 p_1 p_2 B(z) B(z)}, \tag{22}$$

where $m_i\,(0)\,, i = 1, 2$ could also be found in this case.

Example 1. Let $T \sim Exp(1)$, $B \sim G(0.8)$, $\lambda = 2, \lambda_1 = 1, \lambda_2 = 2, \delta = 0.5$ and the positive safety loading condition (1) is obviously fulfilled. If we put $\omega\,(U\,(\tau-)\,, |U\,(\tau)|) = 1$, the Laplace transform of the ruin time can be obtained. That is to say, using (10) and (11), we obtain

$$m_1\,(0) = 0.4313245351,$$

$$m_2\,(0) = 0.6030962228.$$

Then the inversion of the Z transforms (8) and (9) yields

$$m_1\,(u) = -\frac{0.1105768549}{(11.28308554)^{u+1}} + \frac{0.7610532798}{(1.725256292)^{u+1}}, \tag{23}$$

$$m_2\,(u) = -\frac{0.2505598154}{(11.28308554)^{u+1}} + \frac{1.147055211}{(1.725256292)^{u+1}}. \tag{24}$$

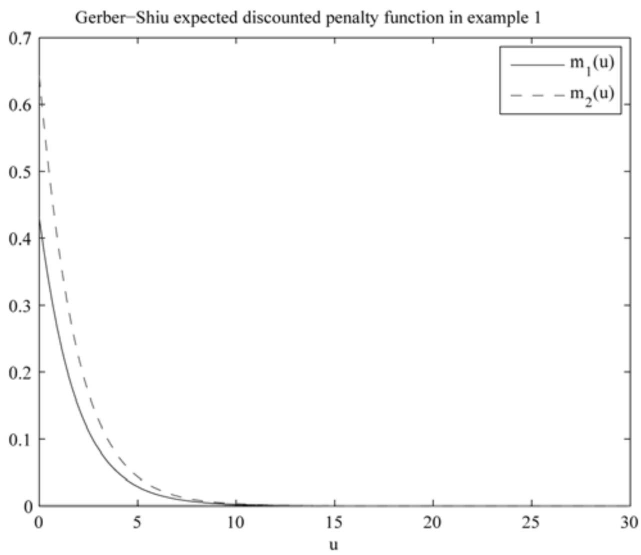

Figure 1. Gerber-Shiu expected discounted penalty function in Example 1.

In Figure 1, we see that the Laplace transform of the ruin time is the decreasing function of u, which is consistent with the real situations.

Example 2. Let $T \sim Exp(1)$, $B \sim G(0.8)$, $\lambda = 2$, $\lambda_1 = 1$, $\lambda_2 = 2$, $\delta = 0.5$ the positive safety loading condition (13) is obviously fulfilled too. As in example 1, we put $\omega\,(U\,(\tau-)\,, |U\,(\tau)|) = 1$ in the equation (16) and (17). The Laplace transform of the ruin time can also be obtained. Using (18) and (19), we find

$$m_1\,(0) = 0.3202969035,$$

$$m_2\,(0) = 0.5433933648.$$

Then the inversion of the Z transforms (16) and (17) yields

$$m_1(u) = \frac{0.4388791850}{(-60.57536918)^{(u+1)}} + \frac{0.7281654172}{(1.574703752)^{(u+1)}} \tag{25}$$

and

$$m_2(u) = \frac{0.5283105388}{(-60.57536918)^{(u+1)}} + \frac{0.9172949623}{(1.574703752)^{(u+1)}}. \tag{26}$$

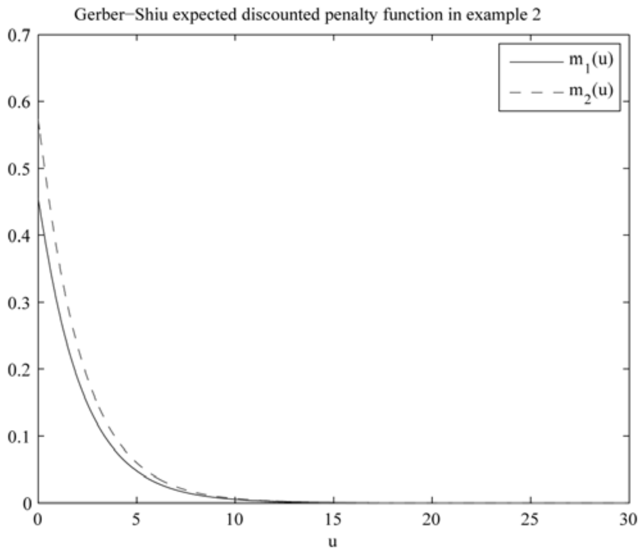

Figure 2. Gerber-Shiu expected discounted penalty function in Example 2.

In Figure 2, we find that the ruin time is the decreasing function of u.

4 Concluding Remarks

This paper studies the generating function of the expected discounted penalty function of two general ruin models. Because the independence assumption of the classical ruin model is too restrictive, two more general ruin models are shown in Sections 2 and 3. These two ruin models have many similarities such as the random income, the independence between premiums and the claims process, the dependence between claim size and inter-claim time. Although there are minor differences concerning the dependence, the methods used for solving the expected discounted penalty function are the same.

The results given in this paper can be generalized to similar dependence ruin models. For example, the dependence between claim size and inter-claim time could be the distribution of the next claim size depending on the last inter-claim time.

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