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A new fractal algorithm to model discrete sequences*

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Employing the properties of the affine mappings, a very novel fractal model scheme based on the iterative function system is proposed. We obtain the vertical scaling factors by a set of the middle points in each affine transform, solving the difficulty in determining the vertical scaling factors, one of the most difficult challenges faced by the fractal interpolation. The proposed method is carried out by interpolating the known attractor and the real discrete sequences from seismic data. The results show that a great accuracy in reconstruction of the known attractor and seismic profile is found, leading to a significant improvement over other fractal interpolation schemes.

Keywords: fractal interpolation, the vertical scaling factors, iterative function system, seismic data

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1. Introduction

There are many objects in nature which are so complicated and irregular that they cannot be modeled properly with the classical geometry. When classical geometry fails to serve as a tool to analyse the complexity of such object, fractal geometry begins. The concept of fractal geometry was first introduced as an extension of classical geometry by Mandelbrot, which can be used to make accurate models of physical structures from ferns to galaxies.[1]

A fractal object is self similar in that subsections of the object are similar, in some sense, to the whole object. No matter how small a subdivision is taken, the subsection contains details no less than the whole.[2] The fractal dimension, thus, is introduced as a measure of the scaling property of the features. Therefore, any irregular shaped body, including signals and discrete sequences, whose Housdorff–Besicovitch dimension strictly exceeds the topological dimension, can be measured by the fractal dimension:[3,4] many natural shapes such as coastlines, mountains and clouds are easily described by fractal models.

Many signals or discrete sequences, such as stock price, seismic data, etc, are scale invariance, thus, the best way to model such signals or discrete sequences is to use a fractal model.[5] There are two popular ways in which a fractal object can be constructed. One is in terms of fractional Brownian motion, which is statistically self similar. The other involves the iterated function system (IFS) theory,[6] which is a more general approach than fractional Brownian motion.[5]

The IFS theory has received a great deal of attention.[7–9] Since IFSs are capable of generating complicated and varied functions even if maps, as few as two ($m = 2$), are involved, we can find their applications in many fields, such as signal processing, computer graphics, metallurgy, geology, chemistry, medical sciences and other areas.[10–17] However, one of the challenges of using IFS for modeling the discrete sequences is the difficulty in determining the vertical scaling factors. In Ref. [17], an algorithm of the computation of the vertical scaling factor was introduced based on the local information. This method, however, needs to employ a great many of known points to achieve a desired data model accuracy because it actually is a linear interpolation. The rest of the present paper is organised as follows. In Section 2, after analysing the IFS based on the affine transformation, we propose a very novel and simple method to compute the vertical scaling factors, in which used are only the uniformly distributed known points in the discrete sequences. The underlying fundamental behind the algorithm is mentioned in Section 3. The proposed method is used to model two discrete sequences

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in Section 4. Finally, the conclusions are presented in Section 5.

2. Background of IFS theory

Consider a one-dimensional signal, also known as a time serial or a discrete sequence, in which the value of real-valued function $F(x)$ is measured as a function of a real variable x in the following form:[6]

$$\{(x_n, F_n) \in \mathbb{R} \times \mathbb{R} : n \in [0, 1, 2, \ldots, N]\}, \tag{1}$$

where N is a positive integer, $F_n = f(x_n)$, and the x_n's are real numbers such that

$$x_0 < x_1 < x_2 < \cdots < x_N. \tag{2}$$

In the fractal modeling, the data set $\{(x_n, F_n) \in \mathbb{R}^2 : n \in [0, 1, 2, \ldots, N]\}$ are called interpolation points. An IFS, also known as fractal interpolation function, corresponding to this set of data, is a continuous function $f : [x_0, x_N] \to \mathbb{R}$ such that[6]

$$f(x_n) = F_n \quad \text{for} \quad n \in [0, 1, 2, \ldots, N]. \tag{3}$$

Affine mapping or affine transform is a simple and effective way to construct an IFS. Generally the affine transform is defined as

$$\omega_n \begin{bmatrix} x \\ f(x) \end{bmatrix} = \begin{bmatrix} L_n(x) \\ F_n(x, y) \end{bmatrix} \quad \text{for all} \quad n \in [1, 2, \ldots, N]. \tag{4}$$

According to the IFS theory,[6] for an IFS defined in expression 4, there is a unique nonempty compact set $\mathbb{G} \subset \mathbb{R}^2$, also known as attractor of the IFS, which is the graph of the desired function $f(x)$:

$$\mathbb{G} = \bigcup_{n=1}^{N} \omega_n(\mathbb{G}). \tag{5}$$

For simplicity, we can assume that $L_n(x)$ and $F_n(x, y)$ are linear functions. Thus, the affine mappings defined in expression 4 can be expressed as

$$\omega_n \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a_n & 0 \\ c_n & d_n \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} e_n \\ f_n \end{bmatrix}, \; n \in [1, 2, \cdots, N]. \tag{6}$$

In order to determine coefficients a_n, c_n, d_n, e_n and f_n, the following conditions are applied:

$$\begin{cases} \omega_n \begin{bmatrix} x_0 \; y_0 \end{bmatrix} = \begin{bmatrix} x_{n-1} \; y_{n-1} \end{bmatrix}, \\ \omega_n \begin{bmatrix} x_N \; y_N \end{bmatrix} = \begin{bmatrix} x_n \; y_n \end{bmatrix}, \end{cases} \quad n \in [1, 2, \ldots, N]. \tag{7}$$

Thus, we can obtain the other four coefficients while the coefficient d_n are considered as a parameter for each affine transform ω_n:

$$\begin{aligned} a_n &= \frac{x_n - x_{n-1}}{x_N - x_0}, \\ e_n &= \frac{x_N x_{n-1} - x_0 x_n}{x_N - x_0}, \\ c_n &= \frac{F_n - F_{n-1}}{x_N - x_0} - d_n \frac{F_N - F_0}{x_N - x_0}, \\ f_n &= \frac{x_N F_{n-1} - x_0 F_n}{x_N - x_0} - d_n \frac{x_N F_0 - x_0 F_N}{x_N - x_0}. \end{aligned} \tag{8}$$

3. Determination of vertical scaling factor

Because the vertical scaling factor must satisfy $|d_n| < 1$, it is also known as a contraction factor for map ω_n. By choosing the vertical scaling factor d_n in Eq. (8) for each affine transform ω_n, we are able to specify the vertical scaling produced by the transformation. Therefore, the vertical scaling factor plays a critical role in implementing a fractal interpolation. In order to determine d_n, $n \in [1, 2, \ldots, N]$, we consider the properties of some special points under the affine mapping ω_n, see Fig. 1.

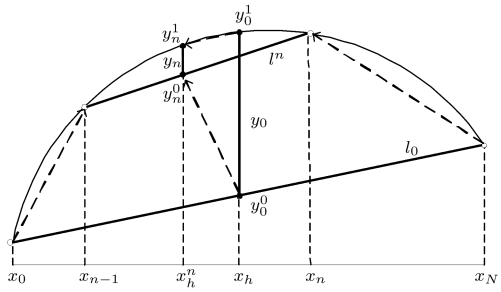

Fig. 1. Relationship among some special points under affine mapping.

Let x_h be a mid-point of the interpolation area $[x_0, x_N]$ and l_0 denote the line connecting the two interpolation points (x_0, y_0) and (x_N, y_N). Let l_n, $n \in [1, 2, \ldots, N]$, be the line connecting points of the nth affine mapping (x_{n-1}, y_{n-1}) and (x_n, y_n).

Because the affine mapping is linear, line l_0 is transformed into line l_n under affine mapping ω_n. Therefore, we discuss point $x = x_h$ on the x-coordination. The two points corresponding to point $x = x_h$ are (x_h, y_0^1) on the attractor and (x_h, y_0^0) on the line l_n respectively. By the affine mapping ω_n, these two points will become points (x_n^h, y_n^0) and (x_n^h, y_n^1) respectively.

The vertical length between (x_h, y_0^1) and (x_h, y_0^0) is Δy_0, however it becomes Δy_n by the affine mapping ω_n. According to fundamental of the affine mapping, the relationship can be constructed by

$$\Delta y_n = d_n \times \Delta y_0. \tag{9}$$

Therefore, the vertical scaling factor corresponding to ω_n can be defined as

$$d_n = \frac{\Delta y_0}{\Delta y_n}, \quad n \in [1, 2, \ldots, N]. \tag{10}$$

4. Simulations

To testify the reliability and the validity of the proposed fractal modeling method, we give two numerical examples: one is to reconstruct a known attractor and the other is to try to interpolate the discrete sequence-the recorded field seismic data.

4.1. Reconstruction of known attractor

The graph of the interpolation function is $\mathbb{G} = \{(x, 2x - x^2) : x \in [0, 2]\}$. According to the proposed method in the paper, we can claim that $\mathbb{G}$ is the attractor of the hyperbolic IFS$\{\mathbb{R}^2; \omega_1, \omega_2\}$ if we choose the data set of $\{(0, 0), (1, 1), (2, 0)\}$ as the interpolation points. Therefore, the two corresponding affine mappings can be expressed as

$$\omega_1 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0.5 & 0 \\ 0.5 & 0.25 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$
$$\omega_2 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0.5 & 0 \\ -0.5 & 0.25 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{11}$$

As x varies over $[0, 2]$, the term on the right-hand side of the first equation in expression (11) yields a part of the graph of $f(x)$ lying in self-affine region $[0, 1]$, while the term on the right-hand side of the second equation yields a part of the graph of $f(x)$ lying in interval $[l, 2]$. Hence $\mathbb{G} = \omega_1(\mathbb{G}) \cup \omega_2(\mathbb{G})$. Since $\mathbb{G} \in \mathbb{H}(\mathbb{R}^2)$ we conclude that it is the attractor of the IFS by noticing that the IFS is just-touching. The reconstructed known attractor $\mathbb{G}$ by the proposed method is shown in Fig. 2. For comparison, the same attractor reconstructed by the method in Ref. [17] is shown in Fig. 3. From the interpolation errors for these two methods shown in Fig. 4, we can see that the error for the proposed method is much less than that for the compared one.

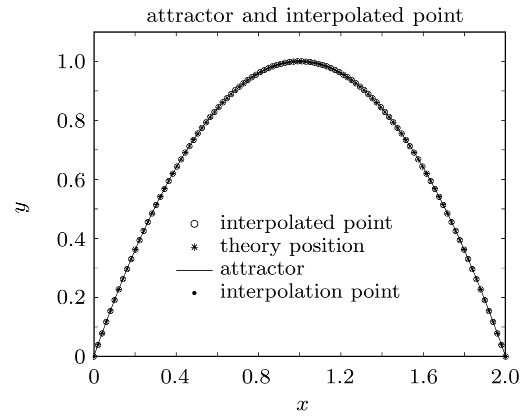

Fig. 2. Reconstructed known attractor by proposed method.

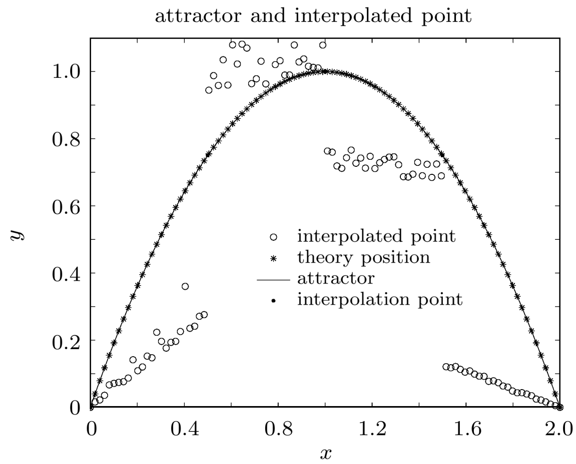

Fig. 3. Reconstructed known attractor by compared method.

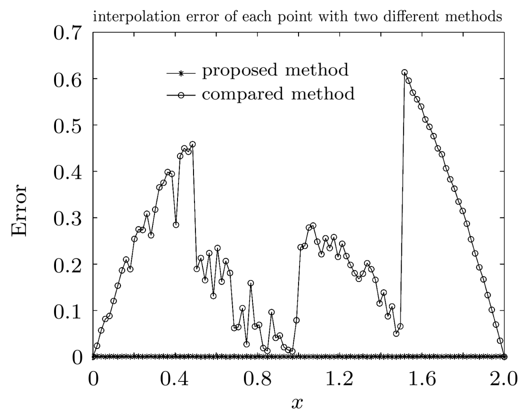

Fig. 4. Reconstruction errors induced by proposed and compared methods.

4.2. Reconstruction of the real discrete sequence

In the second simulation, the recorded field seismic data are considered as the graph of the IFS, and we try to obtain this graph by the proposed method with the following procedure.

(1) Select a set of points $(x_n, y_n), n \in [1, 2, \ldots, N]$ as the interpolation point grouped into the set $\mathbb{G}$ that has been used in the Eq. (5): $\mathbb{G} = \{(x_n, y_n), n \in [1, 2, \ldots, N]\}$. The points selected as the interpolation points can be distributed uniformly or not in the sequence. In our test, the interpolation points are uniformly distributed.

(2) Compute the affine mappings $\omega_n, n \in [1, 2, \ldots, N]$ according to Eqs. (8) and (10).

(3) Substitute the set $\mathbb{G} = \{(x_n, y_n), n \in [1, 2, \ldots, N]\}$ and the affine mappings $\omega_n, n \in [1, 2, \ldots, N]$ into Eq. (5). A new set of interpolation points will be obtained and the procedure can be continued until the interpolation points are dense enough to obtain the expected interpolation resolution. Note that once the affine mappings are obtained from the second step, they will not change with the interpolation points obtained from later circulations.

The reconstruction result can be seen in Fig. 5. In this figure, the reconstructed traces numbers are 10, 20, 40, 65, 80, and they match the whole profile quite well. Figure 6 shows the original and the reconstructed waveforms obtained by the two methods. In general, these two methods can recover the main features of traces. However, if the traces are magnified, the difference between the proposed and the compared method becomes quite obvious, which can be seen in Fig. 7. The trace recovered by the compared method is not very smooth, while the one by the proposed method is as smooth as the original one. This ability is quite important in the modern seismic processing because some other methods need dense and regular sampling traces, such as surface-related multiple elimination and wave-equation migration. The improvement by the proposed method in terms of interpolation error can also be seen in Fig. 8. The error for each point, induced by the proposed method, is much smaller than that induced by the compared method, while the interpolation errors for each trace are 2 or 3 times lower than that induced by the method in Ref. [17], see Tabel 1.

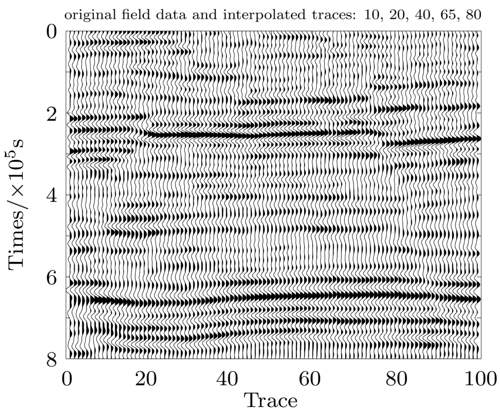

Fig. 5. Reconstructed seismic data profiles obtained by proposed method. Reconstructed trace numbers are 10, 20, 40, 65 and 80.

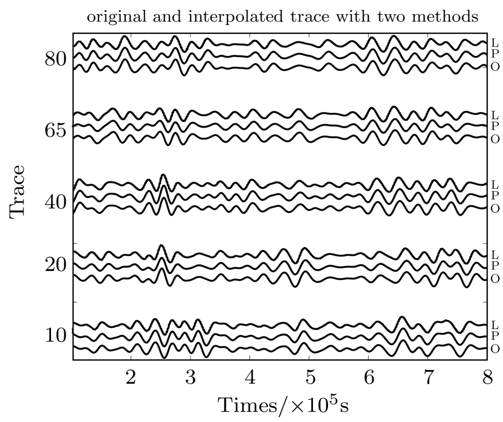

Fig. 6. Reconstructed traces obtained by proposed and compared methods. The corresponding trace numbers are also 10, 20, 40, 65 and 80. L means reconstructed sequence obtained by compared method, P represents that by proposed method, and O refers to original sequence.

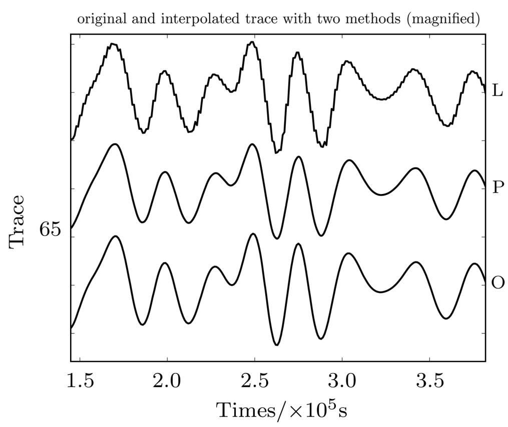

Fig. 7. Magnified part of reconstructed traces obtained by proposed and compared methods. The corresponding trace numbers are also 10, 20, 40, 65 and 80. L means reconstructed sequence obtained by compared method, P represents that by proposed method, and O refers to original sequence.

Table 1. Errors for each reconstructed trace by two methods.

trace number	10	20	40	65	80
the proposed method	24.3636	179.7889	70.6344	209.4774	44.6208
the compared method	292.6753	382.815	331.5845	296.1738	278.1165

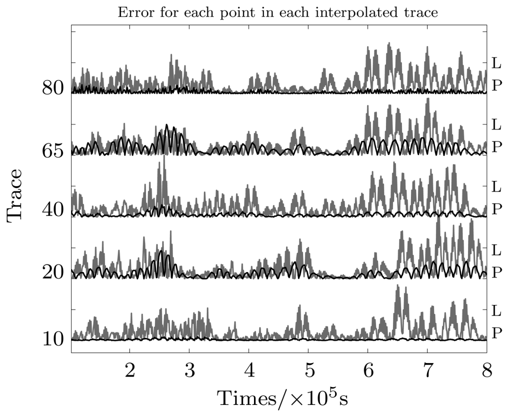

Fig. 8. Reconstruction errors induced by proposed and compared methods for reconstructed trace numbers 10, 20, 40, 65 and 80. L means reconstructed sequence obtained by compared method, P represents that by proposed method, and O refers to original sequence.

5. Conclusions

In this paper we derive an efficient algorithm of performing IFS interpolation and parameter estimation, say the vertical scaling factor in each affine mapping, for a given self-affine signal. We emphasise goodness-of-fit to the given signal. The algorithms are applied to two examples of self-affine signals. The simulation results show that the robust algorithm is strong enough to converge to the true result when the alternative method cannot do so. Real field seismic data are also used to test the algorithm, and the results show that the proposed algorithm achieves better fidelity to data than the alternative algorithm. The results also illustrate that the proposed interpolation algorithm yields more significant improvement over methods suggested in the previous literature.

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