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Changes in the natural frequency of a ferromagnetic rod in a magnetic field due to magnetoelastic interaction *

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Abstract Based on the magnetoelastic generalized variational principle and Hamilton's principle, a dynamic theoretical model characterizing the magnetoelastic interaction of a soft ferromagnetic medium in an applied magnetic field is developed in this paper. From the variational manipulation of magnetic scale potential and elastic displacement, all the fundamental equations for the magnetic field and mechanical deformation, as well as the magnetic body force and magnetic traction for describing magnetoelastic interaction are derived. The theoretical model is applied to a ferromagnetic rod vibrating in an applied magnetic field using a perturbation technique and the Galerkin method. The results show that the magnetic field will change the natural frequencies of the ferromagnetic rod by causing a decrease with the bending motion along the applied magnetic field where the magnetoelastic buckling will take place, and by causing an increase when the bending motion of the rod is perpendicular to the field. The prediction by the mode presented in this paper qualitatively agrees with the natural frequency changes of the ferromagnetic rod observed in the experiment.

Key words ferromagnetic rod, magnetoelastic interaction, generalized variational principle, vibration frequency, perturbation method

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Introduction

Ferromagnetic structures, such as rods, beams, plates and shells, are widely employed in modern electromagnetic equipments. Research on the behavior of magnetoelasticity of ferromagnetic structures can be traced back to the 1960s. Moon and Pao[1] were the first to conduct a magnetoelastic experiment on a ferromagnetic beam-plate in a transverse magnetic field. They found that the natural frequency of the plate decreased with magnetic-field intensity and became near zero as the field attained a critical value. This is the so-called magnetoelastic buckling, and the critical magnetic field decreases with a $-\frac{3}{2}$ power of the ratio of length to thickness. In subsequent studies, Moon and Pao[2] proposed a theoretical model named a magnetic body couple model for predicting this experimental phenomenon. However, since there

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may sometimes be a big discrepancy between theoretical predictions and experimental data on critical magnetic fields, many researchers[3−5] devoted their attention to finding an explanation for this. On the other hand, there are some monographs[6−8] on the general magnetoelastic theory of ferromagnetic solids based on different theories, such as the axiomatic method and the variational method where the magnetic forces exerted on the magnetic body are pre-chosen by Maxwell stress tensor. It has been found that these theoretical models can simulate the experimental phenomenon of a ferromagnetic plate buckling in an uniform transverse magnetic field, but fail in predicting the increase in the natural frequency of a low susceptibility ferromagnetic plate vibrating in an in-plane magnetic field as observed by Tagaki et al[9]. Based on a generalized variational principle of magnetoelasticity, Zhou and Zheng[10−11] developed a new theoretical model for the magnetoelastic buckling and bending of ferromagnetic plates in a transverse and/or oblique magnetic field. They showed that the predictions from their model were much more close to the experimental data for thin plates buckling in a transverse field[12], and at the same time the model could theoretically simulate and explain the experimental result of natural frequency increasing with magnetic-field intensity for a cantilever beam-plate in a in-plane field[9].

Besides research on the buckling of ferromagnetic plates in the transverse applied magnetic fields, Moon[13] conducted an experiment on ferromagnetic elastic rod vibration in a magnetic field. It showed that the rod had two directions of bending: one along the field and the other transverse to the field. Without the applied magnetic field, these two bending modes were degenerate, i.e., the frequencies were the same. However, experimental results showed that in the presence of a magnetic field transverse to the rod, "the modes split with the mode along the field decreasing in frequency and the mode normal to the field increasing slightly in frequency"[13]. For these experimental phenomena, up to now, there have been no reports on further investigations. In this paper, a dynamic model based on Hamilton's principle and the magnetoelastic generalized variational principle for a soft ferromagnetic system is suggested to predict natural frequency changes in a ferromagnetic rod in an applied field. In the model, the magnetic forces arise from the interaction between the ferromagnetic structure and the magnetic field, which is described by the magnetic body force and magnetic traction acting on the ferromagnetic structure. A perturbation technique and the Galerkin method are adopted to analyze the characteristic of vibration frequency of a ferromagnetic rod in an applied transverse magnetic field. The results show that the theoretical model suggested by this paper can theoretically explain the frequency changes of the ferromagnetic elastic rod in applied fields, i.e., the decrease or increase in frequency for the vibration modes along or normal to the magnetic field.

1 Dynamic model of magnetoelastic interaction

Here we consider an isotropic, homogeneous, linear soft ferromagnetic medium in an applied magnetic field $\boldsymbol{B}_0$. For simplicity, we restrict ourselves to low frequency in the magnetoelastic system, and the effect of an electric field and eddy currents on the natural frequency are neglected. For a magnetic field without charge distribution and with conduction current on/in the magnetoelastic medium, the quasi-magnetostatic magnetic energy for the dynamic ferromagnetic system can be expressed as follows[10,11,14]:

$$\Pi_{\mathrm{em}}\{\phi, \boldsymbol{u}\} = \frac{1}{2}\mu_0 \int_t \left[\int_{\Omega^+(\boldsymbol{u})} \mu_{\mathrm{r}} (\nabla\phi^+)^2 dv + \int_{\Omega^-(\boldsymbol{u})} (\nabla\phi^-)^2 dv\right] dt + \int_t \int_{S_0} \boldsymbol{n}\cdot\boldsymbol{B}_0 \phi^- ds dt, \quad (1)$$

where the last term is the external work given by the applied magnetic field $\boldsymbol{B}_0$ in the interval dt; ϕ is a magnetic scalar potential function which satisfies $-\nabla\phi = \boldsymbol{H}$, and ∇ is a gradient operator in the space; S_0 is a closed surface which encloses but is far away from the ferromagnetic medium; $\boldsymbol{u}$ denotes the displacement of the ferromagnetic medium; and Ω^+ and Ω^- represent the inside and outside regions of a deformed ferromagnetic medium, respectively.

For the ferromagnetic medium, one can write the magnetic constitutive relationships between the magnetic field $\boldsymbol{H}$ and magnetization $\boldsymbol{M}$ or magnetic induction $\boldsymbol{B}$ as follows:

$$\boldsymbol{M}^+ = \chi \boldsymbol{H}^+ \quad \text{in } \Omega^+, \tag{2}$$

$$\boldsymbol{B}^+ = \mu_0 \mu_\mathrm{r} \boldsymbol{H}^+ \quad \text{in } \Omega^+, \tag{3}$$

$$\boldsymbol{B}^- = \mu_0 \boldsymbol{H}^- \quad \text{in } \Omega^-, \tag{4}$$

in which the superscripts "+" and "−" are used to identify the quantities in $\Omega^+$and Ω^-, respectively; μ_0 and μ_r denote the magnetic permeability of vacuum and the relative permeability of the ferromagnetic medium, respectively; χ is the susceptibility of the ferromagnetic medium, and the formula $\chi = \mu_\mathrm{r} - 1$ holds for the linear ferromagnetic material.

For the deformation of the ferromagnetic medium without the external mechanical loading, the Lagrangian functional can be expressed as

$$\Pi_\mathrm{me}\{\phi, \boldsymbol{u}\} = \frac{1}{2} \int_t \int_{\Omega^+} (\rho \dot{\boldsymbol{u}} \cdot \dot{\boldsymbol{u}} - \boldsymbol{t} : \boldsymbol{\varepsilon}) dv dt, \tag{5}$$

where ρ is the mass density; $\boldsymbol{t}$ and $\boldsymbol{\varepsilon}$ represent the stress and strain tensors, respectively; $\dot{\boldsymbol{u}}$ denotes the velocity of the ferromagnetic material. The assumption of the infinitesimal deformation is introduced. The following elasticity constitutive and geometrical relationships are given by

$$\boldsymbol{t} = \lambda (\mathrm{tr}\, \boldsymbol{\varepsilon}) \boldsymbol{I} + 2G\boldsymbol{\varepsilon}, \quad \boldsymbol{\varepsilon} = [\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^\mathrm{T}]/2, \tag{6}$$

in which λ and G are Lame's constants for the ferromagnetic material; $\boldsymbol{I}$ is the identity tensor and the superscript "T" represents the transpose of a tensor.

By adding the magnetic energy functional Π_em and Lagrangian functional Π_me, a generalized magnetoelastic energy functional for a soft ferromagnetic dynamic system can be gotten by

$$\begin{aligned} \Pi\{\phi, \boldsymbol{u}\} &= \Pi_\mathrm{em}\{\phi, \boldsymbol{u}\} + \Pi_\mathrm{me}\{\phi, \boldsymbol{u}\} \\ &= \frac{1}{2}\mu_0 \int_t \left[\int_{\Omega^+(\boldsymbol{u})} \mu_\mathrm{r} (\nabla \phi^+)^2 dv + \int_{\Omega^-(\boldsymbol{u})} (\nabla \phi^-)^2 dv \right] dt \\ &\quad + \int_t \int_{S_0} \boldsymbol{n} \cdot \boldsymbol{B}_0 \phi^- ds dt + \frac{1}{2} \int_t \int_{\Omega^+} (\rho \dot{\boldsymbol{u}} \cdot \dot{\boldsymbol{u}} - \boldsymbol{t} : \boldsymbol{\varepsilon}) dv dt. \end{aligned} \tag{7}$$

It should be noted that the deformation $\boldsymbol{u}$ may influence the magnetic energy of the system by changing the inside and outside regions of the ferromagnetic material, although it does not explicitly appear in the magnetic scale potential ϕ. The effect of deformation on region $\Omega^+$of the integral term for kinetic and strain energy in Eq. (7) is neglected due to the infinitesimal deformation of the ferromagnetic body.

Let $\delta \boldsymbol{u}$ and $\delta \phi$ be the admissible variations of displacement and the magnetic potential of the magnetoelastic system, respectively. We have

$$\delta \boldsymbol{u} = \boldsymbol{0} \quad \text{on} \quad S_\mathrm{u}, \tag{8}$$

and

$$\delta \phi = \delta \phi^+ = \delta \phi^- \quad \text{on} \quad S, \tag{9}$$

where $S = \Omega^+ \cap \Omega^-$ denotes the surface of the ferromagnetic medium; and S_u represents the displacement boundary surface of S. Considering that variations of variables $\boldsymbol{u}$ and ϕ are independent of each other, by means of the variation operation[10–11] we can get

$$\begin{aligned}\delta\Pi\{\phi,\boldsymbol{u}\} =& -\int_t \mu_0\left[\int_{\Omega^+(\boldsymbol{u})}\mu_{\rm r}(\nabla^2\phi^+)\delta\phi^+ dv+\int_{\Omega^-(\boldsymbol{u})}(\nabla^2\phi^-)\delta\phi^+ dv\right]dt\\ &+\int_t\left[\oint_{S_0}\mu_0\left(\mu_{\rm r}\frac{\partial\phi^+}{\partial n}-\frac{\partial\phi^-}{\partial n}\right)\delta\phi^+ ds+\int_{S_0}\boldsymbol{n}\cdot(\mu_0\nabla\phi^-+\boldsymbol{B}_0)\delta\phi^- ds\right]dt\\ &+\int_t\left[-\int_{\Omega^+}\rho\ddot{\boldsymbol{u}}\delta\boldsymbol{u}dv+\int_{\Omega^+}(\nabla\cdot\boldsymbol{t}+\boldsymbol{f}^{\rm em})\cdot\delta\boldsymbol{u}dv\right.\\ &\left.+\int_{S_{\rm t}}(\boldsymbol{n}\cdot\boldsymbol{t}+\boldsymbol{F}^{\rm em})\cdot\delta\boldsymbol{u}dv+\int_{S_{\rm u}}\boldsymbol{n}\cdot\boldsymbol{t}\cdot\delta\boldsymbol{u}dv\right]dt.\end{aligned}\tag{10}$$

Here, $\boldsymbol{f}^{\rm em}$ and $\boldsymbol{F}^{\rm em}$ are, respectively, the magnetic body force in the ferromagnetic material and the magnetic traction on the surface of the ferromagnetic material, which are similar to the formulas in the Refs. [10–11].

By $\delta\Pi\{\phi,\boldsymbol{u}\}=0$ and arbitrariness, and by the independence of $\delta\boldsymbol{u}$ and $\delta\phi$, all governing equations and boundary conditions for the ferromagnetic system can be expressed as follows.

Governing equations for quasi-static magnetic fields:

$$\nabla^2\phi^+ = 0 \quad \text{in} \quad \Omega^+(\boldsymbol{u});\tag{11}$$

$$\nabla^2\phi^- = 0 \quad \text{in} \quad \Omega^-(\boldsymbol{u}),\tag{12}$$

with the connected conditions

$$\phi^+=\phi^-,\quad \mu_{\rm r}\frac{\partial\phi^+}{\partial n}=\frac{\partial\phi^-}{\partial n}\quad \text{on} \quad S,\tag{13}$$

and the boundary condition

$$-\nabla\phi^- = \frac{1}{\mu_0}\boldsymbol{B}_0 \quad \text{at} \quad \infty \quad \text{or on} \quad S_0.\tag{14}$$

The motion equation of the elastic field:

$$\nabla\cdot\boldsymbol{t}+\boldsymbol{f}^{\rm em}=\rho\ddot{\boldsymbol{u}} \quad \text{in} \quad \Omega^+,\tag{15}$$

with the boundary conditions

$$\boldsymbol{u}=\boldsymbol{u}^* \quad \text{on} \quad S_{\rm u},\tag{16}$$

$$\boldsymbol{n}\cdot\boldsymbol{t}=\boldsymbol{f}^{\rm em} \quad \text{on} \quad S_{\rm t},\tag{17}$$

in which $\boldsymbol{u}^*$ is a specific displacement vector on the boundary $S_{\rm u}$; and the magnetic body force $\boldsymbol{f}^{\rm em}$ and magnetic traction $\boldsymbol{F}^{\rm em}$ are respectively formulated by

$$\boldsymbol{f}^{\rm em}=\frac{\mu_0\mu_{\rm r}\chi}{2}\nabla(\boldsymbol{H}^+)^2 \quad \text{in} \quad \Omega^+,\tag{18}$$

$$\boldsymbol{F}^{\rm em}=-\frac{\mu_0(\mu_{\rm r}^2-1)}{2}(H_\tau^+)^2\boldsymbol{n} \quad \text{on} \quad S_{\rm t},\tag{19}$$

where H_τ^+ is the tangent component of magnetic field $\boldsymbol{H}^+$ on the surface of the ferromagnetic medium; $\boldsymbol{n}$ denotes the unit vector outward normal to the surface of the rod. From the governing equations (11)–(12), one can find that the distributions of magnetic fields inside and outside the ferromagnetic medium are dependent upon the current configuration of the ferromagnetic medium. Therefore, the magnetic body and traction forces, Eqs. (18)–(19), are the functions of displacement, and Eq. (15) is a nonlinear partial differential equation.

2 Vibration of a ferromagnetic rod

In this section, we will study the experimental phenomena of the natural frequency change by using the dynamic magnetoelastic model developed in the last section. Consider a soft ferromagnetic elastic rod with a circular section in a transverse uniform magnetic field $\boldsymbol{B}_0$, as shown in Fig. 1. The transect radius and the length of the rod are denoted by a and L. Two sets of coordinate systems, the Cartesian system $\{oxyz\}$ and the cylindrical system $\{or\theta z\}$, are adopted for convenience in the following descriptions on the quantities and deformation of the ferromagnetic system.

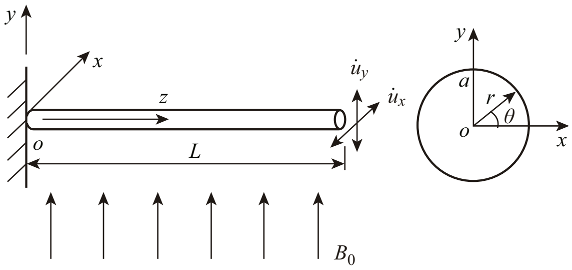

Fig. 1 Ferromagnetic rod with a circular section in a transverse magnetic field

Firstly, we try to find the solution of the magnetic field distributions before the ferromagnetic rod deforms. When the effect of two ends of the rod on the magnetic field is neglected by reasons of assumption of a large ratio of rod length to radius (i.e., $L \gg a$), one can write the fundamental equations of the magnetic field for the "rigid rod", from Eqs. (11)–(14), in the following forms:

$$\frac{\partial^2\Phi^+}{\partial r^2} + \frac{\partial^2\Phi^+}{r\partial r} + \frac{\partial^2\Phi^+}{r^2\partial\theta^2} = 0 \quad \text{in} \quad \Omega^+(\boldsymbol{u}=\boldsymbol{0}), \tag{20}$$

$$\frac{\partial^2\Phi^-}{\partial r^2} + \frac{\partial^2\Phi^-}{r\partial r} + \frac{\partial^2\Phi^-}{r^2\partial\theta^2} = 0 \quad \text{in} \quad \Omega^-(\boldsymbol{u}=\boldsymbol{0}), \tag{21}$$

$$\Phi^+ = \Phi^-, \quad \mu_{\mathrm{r}}\frac{\partial\Phi^+}{\partial r} = \frac{\partial\Phi^-}{\partial r} \quad \text{on} \quad r=a, \tag{22}$$

$$-\nabla\Phi^- = \frac{\boldsymbol{B}_0}{\mu_0} \quad \text{on} \quad S_0 \quad \text{or at} \quad \infty, \tag{23}$$

where Φ^+ or Φ^- is the magnetic scale potential for the ferromagnetic rod without deformation. It is not difficult to get the solutions for the magnetic fields,

$$\boldsymbol{H}_0^+ = -\left(\frac{\partial}{\partial r}\boldsymbol{e}_r + \frac{\partial}{r\partial\theta}\boldsymbol{e}_\theta\right)\Phi^+ = \frac{2B_0}{\mu_0(\mu_{\mathrm{r}}+1)}(\sin\theta\boldsymbol{e}_r + \cos\theta\boldsymbol{e}_\theta) \qquad \text{in} \quad \Omega^+(\boldsymbol{u}=\boldsymbol{0}), \tag{24}$$

$$\boldsymbol{H}_0^- = -\left(\frac{\partial}{\partial r}\boldsymbol{e}_r + \frac{\partial}{r\partial\theta}\boldsymbol{e}_\theta\right)\Phi^- = \frac{B_0}{\mu_0}\left[\left(1+\frac{a^2}{r^2}\frac{\mu_{\mathrm{r}}-1}{\mu_{\mathrm{r}}+1}\right)\sin\theta\boldsymbol{e}_r + \left(1-\frac{a^2}{r^2}\frac{\mu_{\mathrm{r}}-1}{\mu_{\mathrm{r}}+1}\right)\cos\theta\boldsymbol{e}_\theta\right] \quad \text{in} \quad \Omega^-(\boldsymbol{u}=\boldsymbol{0}), \tag{25}$$

in which $\boldsymbol{e}_r$ and $\boldsymbol{e}_\theta$ are, respectively, the unit vectors along the r and θ-axes in the cylindrical coordinate system. There are two kinds of bending modes for the ferromagnetic rod in a transverse field $\boldsymbol{B}_0$, one mode along the magnetic field (i.e., along the y-axis) and the other perpendicular to the field (i.e., along the x-axis), as shown in Fig. 1.

2.1 Vibration along the field

With the classical Bernouli-Euler beam theory, the vibration equation and boundary conditions for the ferromagnetic cantilevered rod give

$$z \in (0, L): \quad EJ\frac{\partial^4 u_y}{\partial z^4} + \rho A\frac{\partial^2 u_y}{\partial t^2} = q_y^{\mathrm{em}}(z, t); \tag{26}$$

$$z = 0: \quad u_y = 0, \quad \frac{\partial u_y}{\partial z} = 0; \tag{27}$$

$$z = L: \quad \frac{\partial^2 u_y}{\partial z^2} = 0, \quad \frac{\partial^3 u_y}{\partial z^3} = 0. \tag{28}$$

Here E is Young's modulus and $J = \pi a^4/4$ is the moment of inertia; A is the transect area of the rod; q_y^{em} is the equivalent magnetic force exerted on the midline of the rod along the vibration direction (i.e., y-axis with the unit vector $\boldsymbol{e}_y$, that is,

$$q_y^{\mathrm{em}}(z, t) = \int_0^{2\pi}\int_0^a \boldsymbol{f}^{\mathrm{em}} \cdot \boldsymbol{e}_y r dr d\theta + \int_0^{2\pi} [\boldsymbol{f}^{\mathrm{em}} \cdot \boldsymbol{e}_y]_{r=a} a d\theta. \tag{29}$$

A perturbation method is employed to analyze the vibration of the ferromagnetic rod. Let $\bar{u}_y(z)$ be a normalized eigenfunction of the dynamic problem and

$$u_y(z, t) = \alpha_1 \bar{u}_y(z)\mathrm{e}^{\mathrm{i}\omega_1 t} \tag{30}$$

where α_1 is a small positive number, and ω_1 is referred to a frequency of the loaded rod. When the deflection is small we can expand the magnetic field vector $\boldsymbol{H}^+(x, \bar{y}, z, t)$ in a deformed state of the ferromagnetic rod by the magnetic field $\boldsymbol{H}_0^+(x, y, z)$ of the undeformed state of the rod. Taking the coordinate $\bar{y}$ as the deformed state and y as the undeformed state of the rod, we have

$$\bar{y} = y + u_y(z, t) = y + \alpha_1 \bar{u}_y(z)\mathrm{e}^{\mathrm{i}\omega_1 t} \tag{31}$$

and

$$\boldsymbol{H}^+(x, \bar{y}, z, t) = \boldsymbol{H}_0^+(x, y) + \alpha_1 \bar{u}_y(z)\frac{\partial \boldsymbol{H}_0^+(x, y)}{\partial y}\mathrm{e}^{\mathrm{i}\omega_1 t} + \cdots. \tag{32}$$

With the relationship between the Cartesian coordinate system and the cylindrical one, the following formulas hold:

$$\boldsymbol{e}_y = \sin\theta \boldsymbol{e}_r + \cos\theta \boldsymbol{e}_\theta, \quad \frac{\partial(\cdot)}{\partial y} = \sin\theta\frac{\partial(\cdot)}{\partial r} + \frac{\cos\theta}{r}\frac{\partial(\cdot)}{\partial \theta}. \tag{33}$$

Substituting the magnetic field distribution of Eqs. (24) and (33) into Eq. (32), we can obtain the magnetic field distribution for the deformed state by

$$\begin{aligned}
&\boldsymbol{H}^+(r, \theta, z, t) \\
&= \boldsymbol{H}_0^+(r, \theta) + \alpha_1 \bar{u}_y(z)\left[\sin\theta\frac{\partial \boldsymbol{H}_0^+(r, \theta)}{\partial r} + \frac{\cos\theta}{r}\frac{\partial \boldsymbol{H}_0^+(r, \theta)}{\partial \theta}\right]\mathrm{e}^{\mathrm{i}\omega_1 t} + \cdots \\
&= \frac{2B_0}{\mu_0(\mu_{\mathrm{r}} + 1)}(\sin\theta \boldsymbol{e}_r + \cos\theta \boldsymbol{e}_\theta) + \alpha_1 \bar{u}_y(z)\frac{2B_0}{\mu_0(\mu_{\mathrm{r}} + 1)}\left(\frac{\cos^2\theta}{r}\boldsymbol{e}_r - \frac{\sin(2\theta)}{2r}\boldsymbol{e}_\theta\right)\mathrm{e}^{\mathrm{i}\omega_1 t} + \cdots.
\end{aligned} \tag{34}$$

From Eqs. (18)–(19), we can further get the magnetic body force and the magnetic traction by neglecting the terms of order 2 and those higher than order 2 of α_1 in the forms

$$\boldsymbol{f}^{\mathrm{em}} = \frac{\mu_0 \mu_{\mathrm{r}} \chi}{2}\nabla(\boldsymbol{H}^+)^2 = \boldsymbol{0}, \tag{35}$$

$$\boldsymbol{F}^{\mathrm{em}} = -\frac{\mu_0(\mu_{\mathrm{r}}^2 - 1)}{2}(H_\tau^+)^2\boldsymbol{n} = -\frac{2B_0^2\chi}{\mu_0(\mu_{\mathrm{r}} + 1)}\left[1 - 2\alpha_1 \bar{u}_y\frac{\sin\theta}{r}\mathrm{e}^{\mathrm{i}\omega_1 t}\right]\cos^2\theta \boldsymbol{e}_r, \tag{36}$$

and the equivalent magnetic force

$$q_y^{\mathrm{em}}(z,t)=\frac{B_0^2\pi\chi}{\mu_0(\mu_{\mathrm{r}}+1)}\alpha_1\bar{u}_y(z)\mathrm{e}^{\mathrm{i}\omega_1 t}. \tag{37}$$

Substituting Eq. (37) into Eqs. (26)–(28), we get the eigenvalue equations of $\bar{u}_y(z)$ as follows:

$$z\in(0,L):\quad EJ\frac{d^4\bar{u}_y(z)}{dz^4}-\frac{B_0^2\pi\chi}{\mu_0(\mu_{\mathrm{r}}+1)}\bar{u}_y(z)=\rho A\omega_1^2\bar{u}_y(z); \tag{38}$$

$$z=0:\quad \bar{u}_y=0,\quad \frac{d\bar{u}_y}{dz}=0; \tag{39}$$

$$z=L:\quad \frac{d^2\bar{u}_y}{dz^2}=0,\quad \frac{d^3\bar{u}_y}{dz^3}=0. \tag{40}$$

In order to get the relation between the vibration frequency of the ferromagnetic rod in the presence of a magnetic field and one in its absence, here we choose the eigenfunction of a cantilevered beam without loading[15],

$$\bar{u}_y(z)=\sin(\beta z)-\sinh(\beta z)+\frac{\cos(\beta L)+\cosh(\beta L)}{\sin(\beta L)-\sinh(\beta L)}[\cos(\beta z)-\cosh(\beta z)]. \tag{41}$$

It is taken as an admissible function for the Galerkin method to approximately calculate the frequency of the structure. It is obvious that Eq. (41) can satisfy all boundary conditions of Eqs. (39)–(40). Taking Eq. (41) into Eq. (38) and integrating the two sides of Eq. (38) on variable z from 0 to L, one can get the vibration frequency as

$$\omega_1^2=\omega_0^2-\frac{1}{\rho A}\frac{B_0^2\pi\chi}{\mu_0(\mu_{\mathrm{r}}+1)}, \tag{42}$$

in which

$$\omega_0^2=\frac{\int_0^L EJ\frac{d^4\bar{u}_y}{dz^4}\bar{u}_y dz}{\int_0^L \rho A\bar{u}_y^2 dz}=\frac{EJ\beta^4}{\rho A} \tag{43}$$

is the natural frequency of a cantilevered rod without loading. From Eq. (42), it can be found that the frequency of the ferromagnetic rod decreases with the applied field B_0^2. When the field B_0^2 reaches a critical point, the frequency becomes zero ($\omega_1^2=0$), at which the ferromagnetic rod will lose stability like a ferromagnetic thin plate in a transverse field[1−2]. The corresponding critical value of the magnetic field is reduced to

$$\frac{B_{\mathrm{c}}^2}{\mu_0 E}=\frac{J\beta^4(\mu_{\mathrm{r}}+1)}{\pi\chi}. \tag{44}$$

For the first vibration mode, the approximate eigenvalue is $\beta=1.875/L$ (e.g., Meirovitch[15]). Substituting the moment of inertia of the circular transect $J=\pi a^4/4$ and β into Eq. (44), we obtain

$$\frac{B_{\mathrm{c}}}{\mu_0 E}=(1.875)^2\sqrt{\frac{\mu_{\mathrm{r}}+1}{\chi}}\left(\frac{L}{a}\right)^{-2}. \tag{45}$$

The above equation shows that the critical field of the ferromagnetic rod buckling varies with −2 power of the ratio of length to radius L/a when the rod vibrates along the magnetic field. This result is qualitatively coincident with the measurements in the experiment conducted by Moon[13]. However, here we do not show the predicted values from the model in this paper in comparison to the experimental data because there are no details on the material and magnetic parameters of the ferromagnetic rod given by Moon[13]. Reasonable comparisons may be conducted from new experimental investigations which need to consider the edge effect for approximation of the magnetic field distributions and the finite long rod in the theoretical analysis.

2.2 Vibration perpendicular to the field

In this case, we denote u_x as the displacement of the ferromagnetic cantilevered rod perpendicular to the field. It is not difficult to get the vibration equation and the corresponding boundary conditions as follows:

$$z \in (0, L): \quad EJ\frac{\partial^4 u_x}{\partial z^4} + \rho A\frac{\partial^2 u_x}{\partial t^2} = q_x^{\mathrm{em}}(z, t); \tag{46}$$

$$z = 0: \quad u_x = 0, \quad \frac{\partial u_x}{\partial z} = 0; \tag{47}$$

$$z = L: \quad \frac{\partial^2 u_x}{\partial z^2} = 0, \quad \frac{\partial^3 u_x}{\partial z^3} = 0, \tag{48}$$

where q_x^{em} is the equivalent magnetic force exerted on the midline of the rod along the vibration direction (i.e., x-axis with the unit vector $\boldsymbol{e}_x$), that is,

$$q_x^{\mathrm{em}}(z, t) = \int_0^{2\pi}\int_0^a \boldsymbol{f}^{\mathrm{em}} \cdot \boldsymbol{e}_x r dr d\theta + \int_0^{2\pi} [\boldsymbol{F}^{\mathrm{em}} \cdot \boldsymbol{e}_x]_{r=a} a d\theta. \tag{49}$$

Let $\bar{u}_x(z)$ be a normalized eigenfunction and $\bar{x}$ be the coordinate of a point in the deformed state; x is the coordinate of a point in the undeformed state of the rod. We have

$$u_x(z, t) = \alpha_2 \bar{u}_x(z)\mathrm{e}^{\mathrm{i}\omega_2 t}, \tag{50}$$

$$\bar{x} = x + \alpha_2 \bar{u}_x(z)\mathrm{e}^{\mathrm{i}\omega_2 t}, \tag{51}$$

$$\boldsymbol{H}^+(\bar{x}, y, z, t) = \boldsymbol{H}_0^+(x, y) + \alpha_2 \bar{u}_x(z)\frac{\partial \boldsymbol{H}_0^+(x, y)}{\partial x}\mathrm{e}^{\mathrm{i}\omega_2 t} + \cdots, \tag{52}$$

where α_2 is a small positive number and ω_2 is referred to a natural frequency of the rod in a magnetic field. With the relationships between the Cartesian and cylindrical coordinate systems, the following formulas hold:

$$\boldsymbol{e}_x = \cos\theta\boldsymbol{e}_r - \sin\theta\boldsymbol{e}_\theta, \quad \frac{\partial(\cdot)}{\partial x} = \cos\theta\frac{\partial(\cdot)}{\partial r} - \frac{\sin\theta}{r}\frac{\partial(\cdot)}{\partial\theta}. \tag{53}$$

Substituting the magnetic field distribution of Eqs. (24) and (53) into Eq. (52), we can obtain the magnetic field distribution for the deformed state as

$$\begin{aligned}\boldsymbol{H}^+(r, \theta, z, t) =& \boldsymbol{H}_0^+(r, \theta) + \alpha_2 \bar{u}_x(z)\left[\cos\theta\frac{\partial \boldsymbol{H}_0^+(r, \theta)}{\partial r} - \frac{\sin\theta}{r}\frac{\partial \boldsymbol{H}_0^+(r, \theta)}{\partial\theta}\right]\mathrm{e}^{\mathrm{i}\omega_2 t} + \cdots \\ =& \frac{2B_0}{\mu_0(\mu_{\mathrm{r}} + 1)}(\sin\theta\boldsymbol{e}_r + \cos\theta\boldsymbol{e}_\theta) + \alpha_2 \bar{u}_x(z) \\ & \cdot \frac{2B_0}{\mu_0(\mu_{\mathrm{r}} + 1)}\left(-\frac{\sin(2\theta)}{2r}\boldsymbol{e}_r + \frac{\sin^2\theta}{r}\boldsymbol{e}_\theta\right)\mathrm{e}^{\mathrm{i}\omega_2 t} + \cdots.\end{aligned} \tag{54}$$

From Eqs. (18)–(19), (49) and (54), the equivalent magnetic force gives

$$q_x^{\mathrm{em}}(z, t) = -\frac{B_0^2 \pi\chi}{\mu_0(\mu_{\mathrm{r}} + 1)}\alpha_2 \bar{u}_x(z)\mathrm{e}^{\mathrm{i}\omega_2 t}. \tag{55}$$

We choose the eigenfunction of a cantilevered beam without loading as Eq. (41), and use the Galerkin method to approximately calculate the frequency by

$$\omega_2^2 = \omega_0^2 + \frac{1}{\rho A}\frac{B_0^2 \pi\chi}{\mu_0(\mu_{\mathrm{r}} + 1)}, \tag{56}$$

in which,

$$\omega_0^2 = \frac{\int_0^L EJ\frac{d^4\bar{u}_x}{dz^4}\bar{u}_x dz}{\int_0^L \rho A\bar{u}_x^2 dz} = \frac{EJ\beta^4}{\rho A} \tag{57}$$

is a natural frequency of the cantilevered rod. Contrary to the ferromagnetic rod vibration along the applied magnetic field, Eq. (56) shows that the frequency of the rod increases with the applied field B_0^2. The results from the proposed model in this paper are the same as the experimental observations of Moon[13].

2.3 Vibration frequency change predicted by magnetic body couple

Moon and Pao[1−2] proposed the magnetic body couple model based on the dipole model of a micro-current for predicting the magnetoelastic buckling of a ferromagnetic plate in a transverse magnetic field. In the model, the magnetic couple body in the ferromagnetic medium gives

$$\boldsymbol{c} = \boldsymbol{M}^+ \times \boldsymbol{B}_0 = \chi\boldsymbol{H}^+ \times \boldsymbol{B}_0. \tag{58}$$

The equivalent magnetic couple acting on the midline of the rod is expressed by

$$\boldsymbol{C} = \chi\int_0^{2\pi}\int_0^a \boldsymbol{H}^+ \times \boldsymbol{B}_0 r dr d\theta. \tag{59}$$

By substituting the approximations of the magnetic fields in Eq. (34) for a rod vibrating along the magnetic field, or Eq. (54) for a rod vibrating perpendicular to the field into the above equation, there will be a reduction $\boldsymbol{C} = \boldsymbol{0}$ in the two cases. That is, the equivalent magnetic forces will always be zero. It means that the natural frequency changes of a ferromagnetic rod vibrating along or perpendicular to the applied magnetic field will not be predicted by the magnetic body couple model.

3 Conclusions

A dynamic theoretical model characterizing the magnetoelastic interaction of a soft ferromagnetic medium is developed based on the magnetoelastic generalized variational principle and Hamilton's principle. All the fundamental equations of the quasi-static magnetic field and mechanical deformation field, together with an expression for the magnetic body force and magnetic traction exerted on the ferromagnetic medium are obtained. A theoretical model is used further to analyze the frequency characteristic of a ferromagnetic rod vibration in an applied magnetic field. With a perturbation technique and the Galerkin method, we find that the frequency of magnetoelastic vibration of a ferromagnetic rod increases with the applied field bending along the field, but decreases with the applied field bending transverse to the field. Predictions on the changes in the natural frequency of a ferromagnetic rod in the presence of a magnetic field are consistent with the experimental ones.

References

[1] Moon F C, Pao Y H. Magnetoelastic buckling of a thin plate[J]. *ASME J Appl Mech*, 1968, **35**(1):53–58.

[2] Moon F C, Pao Y H. Vibration and dynamic instability of a beam-plate in a transverse magnetic field[J]. *ASME J Appl Mech*, 1969, **36**(1):1–9.

[3] Dalrymple J M, Peach M O, Viegelaha G L. Magnetoelastic buckling of thin magnetically soft plate in cylinder model[J]. *ASME J Appl Mech*, 1974, **41**(1):145–150.

[4] Miya K, Hara K, Someya K. Experimental and theoretical study on magnetoelastic buckling of a ferromagnetic cantilevered beam-plate[J]. *ASME J Appl Mech*, 1978, **45**(3):355–360.

[5] Peach M O, Christopherson N S, Dalrymple J M, Viegelahn G L. Magnetoelastic buckling: why theory and experiment disagree[J]. *Exp Mech*, 1988, **28**(1):65–69.

[6] Brown W F. Magnetoelastic interaction[M]. In: *Springer Tracts in Natural Philosophy*, No. 9. Berlin: Springer-Verlag, 1966.

[7] Pao Y H, Yeh C S. A linear theory for soft ferromagnetic elastic solid[J]. *Int J Engng Sci*, 1973, **11**(4):415–436.

[8] Van Lieshout P H, Rongen, P M J, Van de Ven, A A F. A variational principle for magnetoelastic buckling[J]. *J Eng Math*, 1987, **21**(2):227–252.

[9] Tagaki T, Tani J, Matsubara Y, Mogi I. Electromagneto-mechanical coupling effects for non-ferromagnetic and ferromagnetic structures[C]. In: Miya K (ed). *Proc 2nd Int Workshop on Electromagnetic Forces and Related Effects on Blankets and Other Structures Surrounding the Fission Plasma Torus*, Tokai, Japan, Sept 1993, 81–90.

[10] Zhou Youhe, Zheng Xiaojing. A general expression of magnetic force for soft ferromagnetic plates in complex magnetic fields[J]. *Int J Engng Sci*, 1997, **35**(15):1405–1417.

[11] Zhou Youhe, Zheng Xiaojing. A generalized variational principle and theoretical model for magnetoelastic interaction of ferromagnetic bodies[J]. *Science in China* (*Series A*), 1999, **42**(6):618–626.

[12] Zheng Xiaojing, Zhou Youhe, Wang Xingzhe, Lee J S. Bending and buckling of ferroelastic plates[J]. *ASCE J Eng Mech*, 1999, **125**(2):180–185.

[13] Moon F C. Problems in magneto-solid mechanics[M]. In: *Mechanics Today*, Vol 4 (A78-35706 14-70), New York: Pergamon Press, Inc., 1978, 307–390.

[14] Wang Xingzhe, Lee J S, Zheng Xiaojing. Magneto-thermo-elastic instability of ferromagnetic plates in thermal and magnetic fields[J]. *Int J Solids and Structures*, 2003, **40**(22):6125–6142.

[15] Meirovitch L. Analytical methods in vibrations[M]. New York: Macmillan, 1967.