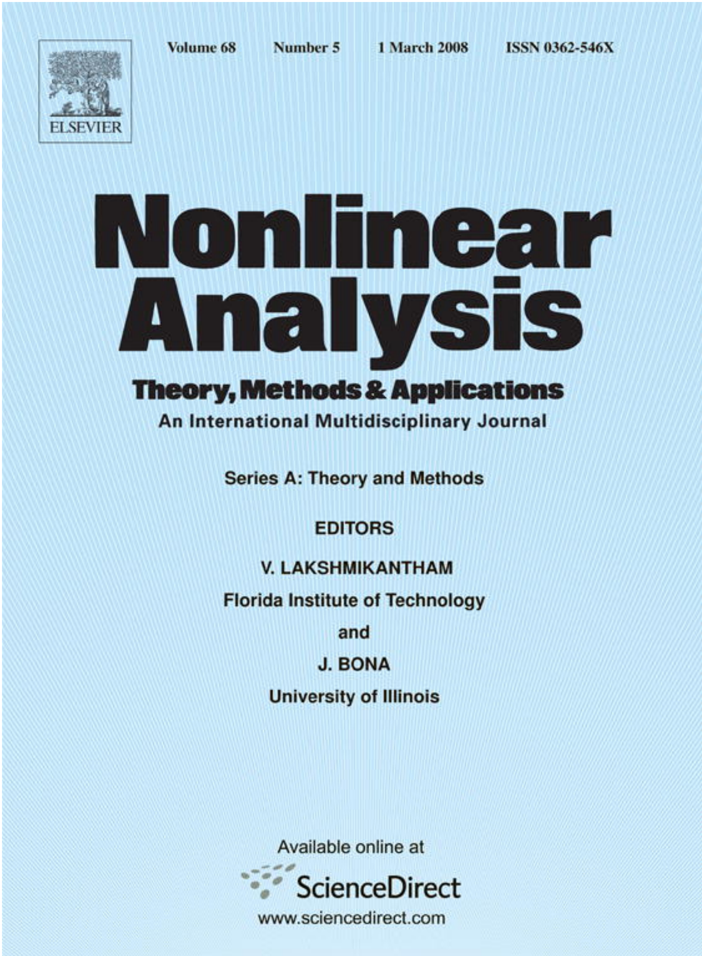
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# Delay-dependent $H_\infty$ control for uncertain fuzzy systems with time-varying delays

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**Abstract**

This paper addresses the robust stabilization and robust $H_\infty$ control problems of uncertain fuzzy delayed systems via a parallel distributed compensation (PDC) scheme. The uncertainties are norm bounded, and the delay may be either constant or time varying, and either differentiable or non-differentiable. Presented in this paper are some robust stabilization and robust $H_\infty$ control criteria, which are delay dependent in general and delay derivative dependent when the delay is differentiable. These results enjoy much less conservatism and more computational simplicity compared with those in the existing literature, and permit, if the delay is differentiable, the derivative to be larger than one. Some examples are given to illustrate the results in this paper.


## 1. Introduction

Stability is one of the most important problems in the synthesis of control systems. With the development of fuzzy systems, various kinds of fuzzy control methods have appeared. A quite popular one is performing control design based on the Takagi–Sugeno (T–S) fuzzy model via the PDC design technique [1–7]. The underlying idea is that for each local linear model, a linear feedback control is designed and the resulting overall controller, which is nonlinear in general, is a fuzzy blending of all the local linear controllers. Therefore, it provides a good opportunity to employ the well-established linear systems theory for theoretical analysis and design of the overall closed-loop controlled systems. The most conspicuous among these systems are those with uncertainties and/or time delays, since they often appear in various engineering systems and are frequently a source of instability [8–16]. Related literature usually adopts a Lyapunov functional for system analysis and design, and sometimes uses the Razumikhin method in conjunction with the Riccati inequality [14].

In recent years, much attention has been paid to studying the robust $H_\infty$ control of fuzzy systems so as to stabilize systems robustly or reject disturbance [10,11,13,17]. As far as the author knows, for time delay systems, most reported results are delay independent [11,13,17], and for the time-varying delay systems, few results are delay dependent or delay derivative dependent. It is widely believed that criteria which are both delay derivative dependent and delay dependent have less conservatism than those that are delay independent or delay derivative independent. Ref. [10] discussed the delay-dependent robust $H_\infty$ control condition for uncertain systems with constant delay, but that method,

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if applied to time-varying delay systems, only results in a highly conservative delay-derivative-dependent $H_\infty$ control condition that is subject to following restrictions: the delay derivative exists, and is less than 1.

On these grounds, the problem mainly addressed in this paper is the robust $H_\infty$ control of a fuzzy system with uncertainties and time-varying delay, especially as regards weakening the restrictions and hence improving the conservatism of the results in [10]. Since robust $H_\infty$ control augments robust stabilization with guaranteed disturbance attenuation, the robust stabilization problem is first explored to pave the way to our goal.

In this paper, the uncertainties are norm bounded, and delay may be either constant or time varying and either differentiable or non-differentiable. The main contribution of this paper lies in the following aspects. Firstly, on the basis of a novel Lyapunov–Krasovskii functional and PDC design technique, some robust stabilization criteria are established via state feedback. Secondly, some robust $H_\infty$ control criteria are proposed. All the criteria are less conservative than many existing results and, if the delay is differentiable, allow its derivative to be arbitrarily large. Note that these advantages are not obtained at the cost of high computational complexity.

The rest of this paper is organized as follows. In Section 2, the main problem is formulated. Some robust stabilization criteria are derived in Section 3 via state feedback. Section 4 discusses the robust $H_\infty$ control condition for such a system. In Section 5, two numerical examples are given to illustrate the correctness and advantages of our theoretical results. And Section 6 concludes this paper.

*Notation*: $L_2$ denotes the space of square integrable functions on $[0,\infty)$ and $\|.\|_2$ the $L_2$ norm, $A > 0$ ($<0$) means $A$ is a symmetrical positive (negative) definite matrix, $A^{-1}$ denotes the inverse of matrix $A$ and $A'$ the transpose, $\overrightarrow{A}$ represents the sum of $A$ and its transpose, $C^+$ means the set of all positive real constants, and $\mathrm{diag}(a_1, a_2, \ldots, a_n)$ a diagonal matrix with diagonal entries $a_1, a_2, \ldots, a_n$.

## 2. Problem formulation

Consider a uncertain fuzzy system with arbitrary delay, which is represented by T–S fuzzy model composed of a set of fuzzy implications each of which is expressed by a linear system model. The $i$th rule of this T–S fuzzy model is represented as: *Rule* $i$: IF $z_1(t)$ is $M_{i1}$ and $\cdots$ and $z_p(t)$ is $M_{ip}$, THEN

$$\begin{aligned}
&\dot{\boldsymbol{x}}(t) = (A_i + \Delta A_i)\boldsymbol{x}(t) + (\mathcal{A}_i + \Delta\mathcal{A}_i)\boldsymbol{x}(t-\tau(t)) + (B_i + \Delta B_i)\boldsymbol{u}(t) + \mathcal{B}_i\boldsymbol{w}(t),\\
&\boldsymbol{y}(t) = D_i\boldsymbol{x}(t) + \mathcal{D}_i\boldsymbol{u}(t), \quad t \ge 0, \qquad \boldsymbol{x}(t) = \boldsymbol{\varphi}(t), \quad t \in [-h, 0],
\end{aligned}$$

in which $z_1(t), \ldots, z_p(t)$ are the premise variables and each $M_{ij}$ $(j = 1, 2, \ldots, p)$ is a fuzzy set. $A_i, \mathcal{A}_i \in R^{n\times n}$, $B_i \in R^{n\times m}$, $\mathcal{B}_i \in R^{n\times r}$, $D_i \in R^{p\times n}$, and $\mathcal{D}_i \in R^{p\times m}$ are constant matrices, $\boldsymbol{x}(t) \in R^n$ is a state variable, $\boldsymbol{u}(t) \in R^m$ is a control input vector, $\boldsymbol{w}(t) \in R^r$ is the disturbance input which is assumed to belong to $L_2[0,\infty)$, and $\boldsymbol{y}(t) \in R^p$ is the controlled output vector. $\Delta A_i, \Delta\mathcal{A}_i, \Delta B_i$ are called *admissible uncertainties* satisfying $[\Delta A_i, \Delta\mathcal{A}_i, \Delta B_i] = NF(t)[E_i, E_{1i}, E_{2i}]$, where $N, E_i, E_{1i}, E_{2i}$ are known constant matrices and $F(t)$ is an unknown matrix with the property $F(t)F'(t) \le I$. $\boldsymbol{\varphi}(t)$: $[-h, 0] \to R^n$ is an initial vector-valued smooth function, and $0 \le \tau(t) \le h \in C^+$. $\tau(t)$ may be non-differentiable, and when it is differentiable, suppose that $\dot{\tau}(t) \le d \in C^+$.

Using a center-average defuzzifier, the final output of the fuzzy system is

$$\begin{aligned}
\dot{\boldsymbol{x}}(t) &= \frac{\sum_{i=1}^{r} \omega_i(\boldsymbol{z}(t))\{(A_i + \Delta A_i)\boldsymbol{x}(t) + (\mathcal{A}_i + \Delta\mathcal{A}_i)\boldsymbol{x}(t-\tau(t)) + (B_i + \Delta B_i)\boldsymbol{u}(t) + \mathcal{B}_i\boldsymbol{w}(t)\}}{\sum_{i=1}^{r} \omega_i(\boldsymbol{z}(t))}\\
&= \sum_{i=1}^{r} h_i(t)\left\{\bar{A}_i\boldsymbol{x}(t) + \bar{\mathcal{A}}_i\boldsymbol{x}(t-\tau(t)) + \bar{B}_i\boldsymbol{u}(t) + \mathcal{B}_i\boldsymbol{w}(t)\right\}\\
&= A\boldsymbol{x}(t) + \mathcal{A}\boldsymbol{x}(t-\tau(t)) + B\boldsymbol{u}(t) + \mathcal{B}\boldsymbol{w}(t),\\
\boldsymbol{y}(t) &= \sum_{i=1}^{r} h_i(t)\{D_i\boldsymbol{x}(t) + \mathcal{D}_i\boldsymbol{u}(t)\} = D\boldsymbol{x}(t) + \mathcal{D}\boldsymbol{u}(t), \quad t \ge 0, \qquad \boldsymbol{x}(t) = \boldsymbol{\varphi}(t), \quad t \in [-h, 0],
\end{aligned} \tag{1}$$

where $r$ is the number of fuzzy implications, $\omega_i(\boldsymbol{z}(t)) = \prod_{l=1}^{p} M_{il}(z_l(t))$ with $\boldsymbol{z}(t) = [z_1(t), \ldots, z_p(t)]'$. $M_i(z_l(t))$ is the grade of membership of $z_l(t)$ in $M_{il}$, and $h_i(t) = \omega_i(\boldsymbol{z}(t))/\sum_{i=1}^{r}\omega_i(\boldsymbol{z}(t))$. It is assumed that $\omega_i(\boldsymbol{z}(t)) \ge 0$;

therefore $h_i(t) \geq 0$, $\sum_{i=1}^{r} h_i(t) = 1$, $\forall t \geq 0$, and

$$\bar{A}_i = A_i + \Delta A_i, \qquad \bar{\mathcal{A}}_i = \mathcal{A}_i + \Delta \mathcal{A}_i, \qquad \bar{B}_i = B_i + \Delta B_i, \qquad A = \sum_{i=1}^{r} h_i(t)\bar{A}_i, \qquad \mathcal{A} = \sum_{i=1}^{r} h_i(t)\bar{\mathcal{A}}_i,$$

$$B = \sum_{i=1}^{r} h_i(t)\bar{B}_i, \qquad \mathcal{B} = \sum_{i=1}^{r} h_i(t)\mathcal{B}_i, \qquad D = \sum_{i=1}^{r} h_i(t)D_i, \qquad \mathcal{D} = \sum_{i=1}^{r} h_i(t)\mathcal{D}_i.$$

For notational simplicity, let $h_i(t) = h_i, \boldsymbol{x}(t) = \boldsymbol{x}, \boldsymbol{x}(t - \tau(t)) = \boldsymbol{x}_\tau, \boldsymbol{w}(t) = \boldsymbol{w}, \boldsymbol{y}(t) = \boldsymbol{y}, \boldsymbol{u}(t) = \boldsymbol{u}$.

**Definition.** Given scalars $\bar{\theta} > 0$ and $\gamma > 0$, if there is a fuzzy control law for (1) such that (i) for any time delay $0 \leq \tau(t) \leq \theta$ satisfying $0 \leq \theta \leq \bar{\theta}$ and all admissible uncertainties, the resulting closed-loop system is asymptotically stable when $\boldsymbol{w} \equiv \boldsymbol{0}$, (ii) under the zero initial conditions, $\|z\|_2 \leq \gamma \|\boldsymbol{w}\|_2$ holds for any nonzero $\boldsymbol{w} \in L_2$, then system (1) is said to be *robustly asymptotically stabilizable with disturbance attenuation* $\gamma$.

## 3. Robust stabilization

Initially, we consider the problem of stabilization of system (1) with $\boldsymbol{w} = \boldsymbol{0}$, i.e.

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + \mathcal{A}\boldsymbol{x}_\tau + B\boldsymbol{u} \quad \text{or} \quad \dot{\boldsymbol{x}} = \boldsymbol{f}, \quad \boldsymbol{f} = A\boldsymbol{x} + \mathcal{A}\boldsymbol{x}_\tau + B\boldsymbol{u}. \tag{2}$$

On the basis of the PDC design technique, we consider the following fuzzy control rules:
*Control Rule i*: IF $z_1(t)$ is $M_{i1}$ and $\cdots$ and $z_p(t)$ is $M_{ip}$, THEN $\boldsymbol{u} = -\mathcal{K}_i\boldsymbol{x}$, where $\mathcal{K}_i \in R^{m\times n}$ is a constant gain matrix to be determined later. The overall controller is represented as

$$\boldsymbol{u} = -\sum_{i=1}^{r} h_i\mathcal{K}_i\boldsymbol{x}. \tag{3}$$

Applying (3) to (2), the resulting closed-loop system can be recast as

$$\dot{\boldsymbol{x}} = \boldsymbol{f}, \quad \boldsymbol{f} = \left(A - B\sum_{j=1}^{r} h_j\mathcal{K}_j\right)\boldsymbol{x} + \mathcal{A}\boldsymbol{x}_\tau = W\boldsymbol{x} + \mathcal{A}\boldsymbol{x}_\tau, \tag{4}$$

where $W = A - B\sum_{j=1}^{r} h_j\mathcal{K}_j = \sum_{i=1}^{r}\sum_{j=1}^{r} h_ih_j(\bar{A}_i - \bar{B}_i\mathcal{K}_j)$.

Before presenting the main results, we define some notation which will be used throughout this paper.

$$\mathcal{H}_1 = hX_{22} - \overrightarrow{X_{23}} + (d-1)Q, \qquad \mathcal{H}_{2ij} = \frac{A_iR - B_iK_j + A_jR - B_jK_i}{2},$$
$$\mathcal{H}_{3ij} = hX_{33} - 2R + e_{ij}NN', \qquad \mathcal{H}_5 = h\mathcal{X}_{22} - \overrightarrow{\mathcal{X}_{23}} + (d-1)\mathcal{Q}.$$

First, assume that $0 \leq \tau(t) \leq h$ and $\dot{\tau}(t) \leq d$, and we have the following theorem.

**Theorem 1.** *Closed-loop system* (4) *is asymptotically stabilizable if there exist matrices* $P > 0$, $Q > 0$, $R > 0$, $K_j$, $X_{lk}$ $(l, k = 1, 2, 3)$, *and positive scalars* $e_{ij}$ $(1 \leq i \leq j \leq r)$, *such that* (5)–(7) *hold, and the control gain matrix is given by* $\mathcal{K}_j = K_jR^{-1}$.

$$\Theta_{ii} = \begin{bmatrix} \overrightarrow{A_iR - B_iK_i} + hX_{11} + \overrightarrow{X_{13}} + Q + e_{ii}NN' & * & * & * \\ P - R + A_iR - B_iK_i + e_{ii}NN' & \mathcal{H}_{3ii} & * & * \\ R\mathcal{A}_i' + hX_{21} - X_{31} + X_{23} & R\mathcal{A}_i' & \mathcal{H}_1 & * \\ E_iR - E_{2i}K_i & \boldsymbol{0} & E_{1i}R & -e_{ii} \end{bmatrix} < 0, \quad i = 1, 2, \ldots, r, \tag{5}$$

$$\Theta_{ij} = \begin{bmatrix} \overrightarrow{\mathcal{H}_{2ij}} + hX_{11} + \overrightarrow{X_{13}} + Q + e_{ij}NN' & * & * & * \\ P - R + \mathcal{H}_{2ij} + e_{ij}NN' & \mathcal{H}_{3ij} & * & * \\ R\dfrac{\mathcal{A}_i' + \mathcal{A}_j'}{2} + hX_{21} - X_{31} + X_{23} & R\dfrac{\mathcal{A}_i' + \mathcal{A}_j'}{2} & \mathcal{H}_1 & * \\ \dfrac{E_iR - E_{2i}K_j + E_jR - E_{2j}K_i}{2} & \boldsymbol{0} & \dfrac{E_{1i} + E_{1j}}{2}R & -e_{ij} \end{bmatrix} \leq 0, \quad 1 \leq i < j \leq r, \tag{6}$$

$$\begin{bmatrix} X_{11} & * & * \\ X_{21} & X_{22} & * \\ X_{31} & X_{32} & X_{33} \end{bmatrix} > 0. \tag{7}$$

**Proof.** Let $\mathcal{R} = R^{-1}$, $\mathcal{X}_{lk} = \mathcal{R}X_{lk}\mathcal{R}$, $\mathcal{Q} = \mathcal{R}Q\mathcal{R}$, $\mathcal{P} = \mathcal{R}P\mathcal{R}$, where $R, P, Q, X_{lk}$ $(l, k = 1, 2, 3)$ are solutions to (5)–(7). Chose the Lyapunov functional candidate

$$V = \int_{-h}^{0}\int_{t+\beta}^{t} \dot{\boldsymbol{x}}'(s)\mathcal{X}_{33}\dot{\boldsymbol{x}}(s)\,\mathrm{d}s\,\mathrm{d}\beta + \int_{t-\tau(t)}^{t} \boldsymbol{x}'(s)\mathcal{Q}\boldsymbol{x}(s)\,\mathrm{d}s + V_1 + V_2, \tag{8}$$

in which

$$V_1 = \boldsymbol{x}'\mathcal{P}\boldsymbol{x}, \quad V_2 = \int_0^t \int_{\theta-\tau(\theta)}^{\theta} \begin{bmatrix} \boldsymbol{x}(\theta) \\ \boldsymbol{x}(\theta-\tau(\theta)) \\ \dot{\boldsymbol{x}}(s) \end{bmatrix}' \begin{bmatrix} \mathcal{X}_{11} & * & * \\ \mathcal{X}_{21} & \mathcal{X}_{22} & * \\ \mathcal{X}_{31} & \mathcal{X}_{32} & \mathcal{X}_{33} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}(\theta) \\ \boldsymbol{x}(\theta-\tau(\theta)) \\ \dot{\boldsymbol{x}}(s) \end{bmatrix} \mathrm{d}s\,\mathrm{d}\theta.$$

Taking the time derivative of $V$ along trajectories of (4), we have

$$\begin{aligned} \dot{V}_1 &= 2\boldsymbol{x}'\mathcal{P}\dot{\boldsymbol{x}} = 2\boldsymbol{x}'\mathcal{P}\boldsymbol{f} = 2\boldsymbol{x}'\mathcal{P}\boldsymbol{f} + 2\left(\boldsymbol{x}' + \boldsymbol{f}'\right)\mathcal{R}\left(W\boldsymbol{x} + \mathcal{A}\boldsymbol{x}_\tau - \boldsymbol{f}\right) \\ &= 2\boldsymbol{x}'\mathcal{P}\boldsymbol{f} - 2\boldsymbol{x}'\mathcal{R}\boldsymbol{f} - 2\boldsymbol{f}'\mathcal{R}\boldsymbol{f} + 2\boldsymbol{x}'\mathcal{R}W\boldsymbol{x} + 2\boldsymbol{x}'\mathcal{R}\mathcal{A}\boldsymbol{x}_\tau + 2\boldsymbol{f}'\mathcal{R}W\boldsymbol{x} + 2\boldsymbol{f}'\mathcal{R}\mathcal{A}\boldsymbol{x}_\tau. \end{aligned}$$

The derivative of $V_2$ is

$$\begin{aligned} \dot{V}_2 &= \int_{t-\tau(t)}^{t} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{x}_\tau \\ \dot{\boldsymbol{x}}(s) \end{bmatrix}' \begin{bmatrix} \mathcal{X}_{11} & * & * \\ \mathcal{X}_{21} & \mathcal{X}_{22} & * \\ \mathcal{X}_{31} & \mathcal{X}_{32} & \mathcal{X}_{33} \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{x}_\tau \\ \dot{\boldsymbol{x}}(s) \end{bmatrix} \mathrm{d}s \\ &\le \int_{t-h}^{t} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{x}_\tau \end{bmatrix}' \begin{bmatrix} \mathcal{X}_{11} & * \\ \mathcal{X}_{21} & \mathcal{X}_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{x}_\tau \end{bmatrix} \mathrm{d}s + \int_{t-\tau(t)}^{t} \dot{\boldsymbol{x}}(s)\left\{2\mathcal{X}_{31}\boldsymbol{x} + 2\mathcal{X}_{32}\boldsymbol{x}_\tau + \mathcal{X}_{33}\dot{\boldsymbol{x}}(s)\right\}\mathrm{d}s \\ &\le \boldsymbol{x}'\left\{h\mathcal{X}_{11} + \overrightarrow{\mathcal{X}_{13}}\right\}\boldsymbol{x} + 2\boldsymbol{x}'\left\{h\mathcal{X}_{12} - \mathcal{X}_{13} + \mathcal{X}_{32}\right\}\boldsymbol{x}_\tau + \boldsymbol{x}_\tau'\left\{h\mathcal{X}_{22} - \overrightarrow{\mathcal{X}_{23}}\right\}\boldsymbol{x}_\tau + \int_{t-h}^{t} \boldsymbol{f}'(s)\mathcal{X}_{33}\boldsymbol{f}(s)\,\mathrm{d}s. \end{aligned}$$

Then it follows that

$$\begin{aligned} \dot{V} &\le 2\boldsymbol{f}'\mathcal{R}\mathcal{A}\boldsymbol{x}_\tau + \boldsymbol{x}'\left\{\overrightarrow{\mathcal{R}W} + h\mathcal{X}_{11} + \overrightarrow{\mathcal{X}_{13}} + \mathcal{Q}\right\}\boldsymbol{x} + 2\boldsymbol{x}'\left\{\mathcal{R}\mathcal{A} + h\mathcal{X}_{12} - \mathcal{X}_{13} + \mathcal{X}_{32}\right\}\boldsymbol{x}_\tau \\ &\quad + 2\boldsymbol{x}'\left\{\mathcal{P} - \mathcal{R} + W'\mathcal{R}\right\}\boldsymbol{f} + \boldsymbol{f}'\left\{h\mathcal{X}_{33} - 2\mathcal{R}\right\}\boldsymbol{f} + \boldsymbol{x}_\tau'\left\{h\mathcal{X}_{22} - \overrightarrow{\mathcal{X}_{23}} + (d-1)\mathcal{Q}\right\}\boldsymbol{x}_\tau \\ &= \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{f} \\ \boldsymbol{x}_\tau \end{bmatrix}' \begin{bmatrix} \overrightarrow{\mathcal{R}W} + h\mathcal{X}_{11} + \overrightarrow{\mathcal{X}_{13}} + \mathcal{Q} & * & * \\ \mathcal{P} - \mathcal{R} + \mathcal{R}W & h\mathcal{X}_{33} - 2\mathcal{R} & * \\ \mathcal{A}'\mathcal{R} + h\mathcal{X}_{21} - \mathcal{X}_{31} + \mathcal{X}_{23} & \mathcal{A}'\mathcal{R} & \mathcal{H}_5 \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{f} \\ \boldsymbol{x}_\tau \end{bmatrix}. \end{aligned} \tag{9}$$

Apply $W = \sum_{i=1}^{r}\sum_{j=1}^{r} h_i h_j(\bar{A}_i - \bar{B}_i\mathcal{K}_j)$ to (9), and note that $\sum_{i=1}^{r} h_i = 1$. Then it is yielded that

$$\dot{V} \le \sum_{i=1}^{r}\sum_{j=1}^{r} h_i h_j \boldsymbol{\rho}' M_{ij}\boldsymbol{\rho} = \sum_{i=1}^{r} h_i^2 \boldsymbol{\rho}' M_{ii}\boldsymbol{\rho} + 2\sum_{1\le i<j\le r} h_i h_j \boldsymbol{\rho}'\frac{M_{ij} + M_{ji}}{2}\boldsymbol{\rho}, \tag{10}$$

where $\boldsymbol{\rho} = \left[\boldsymbol{x}', \boldsymbol{f}', \boldsymbol{x}_\tau'\right]'$, and

$$M_{ij} = \begin{bmatrix} \overrightarrow{\mathcal{R}(\bar{A}_i - \bar{B}_i\mathcal{K}_j)} + h\mathcal{X}_{11} + \overrightarrow{\mathcal{X}_{13}} + \mathcal{Q} & * & * \\ \mathcal{P} - \mathcal{R} + \mathcal{R}(\bar{A}_i - \bar{B}_i\mathcal{K}_j) & h\mathcal{X}_{33} - 2\mathcal{R} & * \\ \bar{\mathcal{A}}_i'\mathcal{R} + h\mathcal{X}_{21} - \mathcal{X}_{31} + \mathcal{X}_{23} & \bar{\mathcal{A}}_i'\mathcal{R} & \mathcal{H}_5 \end{bmatrix}.$$

Define $\boldsymbol{\sigma} = \left[\boldsymbol{a}', \boldsymbol{b}', \boldsymbol{c}', (\boldsymbol{a}' + \boldsymbol{b}')NF(t)\right]'$ with arbitrary vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ of appropriate dimension. Since $\Theta_{ii} < 0, i = 1, \ldots, r$ and $\Theta_{ij} \le 0, 1 \le i < j \le r$,

$$\boldsymbol{\sigma}'\Theta_{ii}\boldsymbol{\sigma} < 0, \quad i = 1, \ldots, r, \qquad \boldsymbol{\sigma}'\Theta_{ij}\boldsymbol{\sigma} \le 0, \quad 1 \le i < j \le r. \tag{11}$$

Recalling that $F(t)F'(t) \le I$, we can easily obtain the following inequalities:

$$\begin{aligned}
&e_{ij}(\boldsymbol{a}'+\boldsymbol{b}')NF(t)F'(t)N'(\boldsymbol{a}+\boldsymbol{b}) \le e_{ij}\left(\boldsymbol{a}'NN'\boldsymbol{a}+2\boldsymbol{b}'NN'\boldsymbol{a}+\boldsymbol{b}'NN'\boldsymbol{b}\right),\\
&(\boldsymbol{a}'+\boldsymbol{b}')NF(t)E_iR\boldsymbol{a} = (\boldsymbol{a}'+\boldsymbol{b}')\Delta A_iR\boldsymbol{a},\ (\boldsymbol{a}'+\boldsymbol{b}')NF(t)E_{1i}R\boldsymbol{c} = (\boldsymbol{a}'+\boldsymbol{b}')\Delta\mathcal{A}_iR\boldsymbol{c},\\
&(\boldsymbol{a}'+\boldsymbol{b}')NF(t)E_{2i}K_j\boldsymbol{a} = (\boldsymbol{a}'+\boldsymbol{b}')\Delta B_iK_j\boldsymbol{a}.
\end{aligned} \tag{12}$$

It follows from (11) and (12) that

$$\bar{\boldsymbol{\sigma}}'\bar{M}_{ii}\bar{\boldsymbol{\sigma}} \le \boldsymbol{\sigma}'\Theta_{ii}\boldsymbol{\sigma} < 0, \quad i=1,\dots,r, \qquad \bar{\boldsymbol{\sigma}}'\bar{M}_{ij}\bar{\boldsymbol{\sigma}} \le \boldsymbol{\sigma}'\Theta_{ij}\boldsymbol{\sigma} \le 0, \quad 1\le i<j\le r, \tag{13}$$

where $\bar{\boldsymbol{\sigma}} = \left[\boldsymbol{a}', \boldsymbol{b}', \boldsymbol{c}'\right]'$, and the symmetrical matrix

$$\bar{M}_{ii} = \begin{bmatrix} \overrightarrow{\bar{A}_iR - \bar{B}_iK_i} + hX_{11} + \overrightarrow{X_{13}} + Q & * & * \\ P - R + \bar{A}_iR - \bar{B}_iK_i & hX_{33} - 2R & * \\ R\bar{\mathcal{A}}_i' + hX_{21} - X_{31} + X_{23} & R\bar{\mathcal{A}}_i' & \mathcal{H}_1 \end{bmatrix}, \quad i=1,\dots,r,$$

$$\bar{M}_{ij} = \begin{bmatrix} \dfrac{\overrightarrow{\bar{A}_iR - \bar{B}_iK_j + \bar{A}_jR - \bar{B}_jK_i}}{2} + hX_{11} + \overrightarrow{X_{13}} + Q & * & * \\ P - R + \dfrac{\bar{A}_iR - \bar{B}_iK_j + \bar{A}_jR - \bar{B}_jK_i}{2} & hX_{33} - 2R & * \\ R\dfrac{\bar{\mathcal{A}}_i' + \bar{\mathcal{A}}_j'}{2} + hX_{21} - X_{31} + X_{23} & R\dfrac{\bar{\mathcal{A}}_i' + \bar{\mathcal{A}}_j'}{2} & \mathcal{H}_1 \end{bmatrix}, \quad 1\le i<j\le r.$$

Since $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ are arbitrary vectors, we have $\bar{M}_{ii} < 0, i=1,\dots,r, \bar{M}_{ij} \le 0,\ 1\le i<j\le r$.

Let $\Sigma_1 = \mathrm{diag}[\mathcal{R}, \mathcal{R}, \mathcal{R}]$; it follows that

$$M_{ii} = \Sigma_1\bar{M}_{ii}\Sigma_1 < 0, \quad i=1,\dots,r, \qquad \frac{M_{ij}+M_{ji}}{2} = \Sigma_1\bar{M}_{ij}\Sigma_1 \le 0, \quad 1\le i<j\le r.$$

On the basis of (10), we have $\dot{V} < 0$. The proof ends. ■

**Remark 1.** Compared with [10], the present paper generalizes its Lyapunov–Krasovskii functional with the summand $V_2$. This key generalization, together with the additional free weighting matrices that it results in, generally leads to less conservatism than for the results in [10]. The performance improvement is demonstrated in Section 5.

**Remark 2.** When $\tau(t)$ is a constant, i.e., $d \equiv 0$, this theorem still holds.

Consider a special case of system (1) where $\boldsymbol{w} = 0$, $\boldsymbol{u} = 0$, and there are no uncertainties, i.e.

$$\dot{\boldsymbol{x}} = \sum_{i=1}^{r} h_i(A_i\boldsymbol{x} + \mathcal{A}_i\boldsymbol{x}_\tau). \tag{14}$$

We can construct a Lyapunov functional which is the same as that in Theorem 1 but with $\mathcal{P}$, $\mathcal{Q}$, and $\mathcal{X}_{lk}$ replaced by $P$, $Q$, $X_{lk}$ $(l,k=1,2,3)$, respectively. According to the proof of Theorem 1, one has

**Corollary 1.** *System (14) is asymptotically stable if there exist matrices $P>0, Q>0, R>0$, and $X_{lk}$ $(l,k=1,2,3)$ such that (7) and (15) hold:*

$$\begin{bmatrix} \overrightarrow{RA_i} + hX_{11} + \overrightarrow{X_{13}} + Q & * & * \\ P - R + RA_i & hX_{33} - 2R & * \\ \mathcal{A}_i'R + hX_{21} - X_{31} + X_{23} & \mathcal{A}_i'R & hX_{22} - \overrightarrow{X_{23}} + (d-1)Q \end{bmatrix} < 0, \quad i=1,2,\dots,r. \tag{15}$$

If $\tau(t)$ in (1) is *non-differentiable*, one can construct a Lyapunov–Krasovskii functional similar to that in Theorem 1 but without the summand $\int_{t-\tau(t)}^{t}\boldsymbol{x}'(s)\mathcal{Q}\boldsymbol{x}(s)\,\mathrm{d}s$. Following the proof process of Theorem 1, we have

**Corollary 2.** *Closed-loop system (4) is asymptotically stabilizable if there exist matrices $P>0, R>0$, $K_j$, $X_{lk}$ $(l,k=1,2,3)$, and positive scalars $e_{ij}$ $(1\le i\le j\le r)$, such that (5), (6) with $Q=\mathbf{0}$ and (7) hold, and the control gain matrix is given by $\mathcal{K}_j = K_jR^{-1}$.*

## 4. Robust $H_\infty$ control

System (1) can be recast as

$$\dot{\boldsymbol{x}} = \boldsymbol{f}, \quad \boldsymbol{f} = A\boldsymbol{x} + \mathcal{A}\boldsymbol{x}_\tau + B\boldsymbol{u} + \mathcal{B}\boldsymbol{w}, \qquad \boldsymbol{y} = D\boldsymbol{x} + \mathcal{D}\boldsymbol{u}. \tag{16}$$

Applying (3) to (16), we have

$$\dot{\boldsymbol{x}} = \boldsymbol{f}, \quad \boldsymbol{f} = W\boldsymbol{x} + \mathcal{A}\boldsymbol{x}_\tau + \mathcal{B}\boldsymbol{w}, \qquad \boldsymbol{y} = \left(D - \mathcal{D}\sum_{j=1}^{r} h_j \mathcal{K}_j\right)\boldsymbol{x}. \tag{17}$$

Under the condition that $\dot{\tau}(t) \le d$, we have the following theorem.

**Theorem 2.** *Given scalar $\gamma > 0$, if there exist matrices $P > 0$, $Q > 0$, $R > 0$, $X_{lk}$ $(l, k = 1, 2, 3)$, $K_j \in R^{m\times n}$, and positive scalars $e_{ij}$ $(1 \le i \le j \le r)$, such that (18), (19) and (7) hold, then (1) is robustly asymptotically stabilizable with disturbance attenuation $\gamma$, and moreover, the control gain matrix $\mathcal{K}_j = K_j R^{-1}$.*

$$\Gamma_{ii} = \begin{bmatrix} \overrightarrow{A_i R - B_i K_i} + hX_{11} + \overrightarrow{X_{13}} + Q + e_{ii}NN' & * & * & * & * & * \\ P - R + A_i R - B_i K_i + e_{ii}NN' & \mathcal{H}_{3ii} & * & * & * & * \\ R\mathcal{A}_i' + hX_{21} - X_{31} + X_{23} & R\mathcal{A}_i' & \mathcal{H}_1 & * & * & * \\ \mathcal{B}_i' & \mathcal{B}_i' & \mathbf{0} & -\gamma^2 & * & * \\ D_i R - \mathcal{D}_i K_i & \mathbf{0} & \mathbf{0} & \mathbf{0} & -I & * \\ E_i R - E_{2i} K_i & \mathbf{0} & E_{1i} R & \mathbf{0} & \mathbf{0} & -e_{ii} \end{bmatrix} < 0,$$
$$i = 1, 2, \ldots, r, \tag{18}$$

$$\Gamma_{ij} = \begin{bmatrix} \overrightarrow{\mathcal{H}_{2ij}} + hX_{11} + \overrightarrow{X_{13}} + Q + e_{ij}NN' & * & * & * & * & * & * \\ P - R + \mathcal{H}_{2ij} + e_{ij}NN' & \mathcal{H}_{3ij} & * & * & * & * & * \\ R\dfrac{\mathcal{A}_i' + \mathcal{A}_j'}{2} + hX_{21} - X_{31} + X_{23} & R\dfrac{\mathcal{A}_i' + \mathcal{A}_j'}{2} & \mathcal{H}_1 & * & * & * & * \\ \mathcal{B}_i' & \mathcal{B}_i' & \mathbf{0} & -\gamma^2 & * & * & * \\ D_i R - \mathcal{D}_i K_j & \mathbf{0} & \mathbf{0} & \mathbf{0} & -2I & * & * \\ D_j R - \mathcal{D}_j K_i & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -2I & * \\ \dfrac{E_i R - E_{2i} K_j + E_j R - E_{2j} K_i}{2} & \mathbf{0} & \dfrac{E_{1i} + E_{1j}}{2} R & \mathbf{0} & \mathbf{0} & \mathbf{0} & -e_{ij} \end{bmatrix} \le 0,$$
$$1 \le i < j \le r. \tag{19}$$

**Proof.** Chose the Lyapunov functional candidate defined by (8). The derivative of $V_2$ is the same as that in Theorem 1 too. On the basis of (17), one gets

$$\dot{V}_1 = 2\boldsymbol{x}'\mathcal{P}\boldsymbol{f} + 2\left(\boldsymbol{x}' + \boldsymbol{f}'\right)\mathcal{R}\left(W\boldsymbol{x} + \mathcal{A}\boldsymbol{x}_\tau + \mathcal{B}\boldsymbol{w} - \boldsymbol{f}\right). \tag{20}$$

By a process like that going from (8) to (9), the following inequality can be obtained:

$$\dot{V} \le \sum_{i=1}^{r}\sum_{j=1}^{r} h_i h_j \boldsymbol{\theta}' \begin{bmatrix} \overrightarrow{\mathcal{R}(\bar{A}_i - \bar{B}_i\mathcal{K}_j)} + h\mathcal{X}_{11} + \overrightarrow{\mathcal{X}_{13}} + \mathcal{Q} & * & * & * \\ \mathcal{P} - \mathcal{R} + \mathcal{R}(\bar{A}_i - \bar{B}_i\mathcal{K}_j) & h\mathcal{X}_{33} - 2\mathcal{R} & * & * \\ \bar{\mathcal{A}}_i'\mathcal{R} + h\mathcal{X}_{21} - \mathcal{X}_{31} + \mathcal{X}_{23} & \bar{\mathcal{A}}_i'\mathcal{R} & \mathcal{H}_5 & * \\ \mathcal{B}_i'\mathcal{R} & \mathcal{B}_i'\mathcal{R} & \mathbf{0} & \mathbf{0} \end{bmatrix} \boldsymbol{\theta} \tag{21}$$

with $\boldsymbol{\theta} = \left[\boldsymbol{x}', \boldsymbol{f}', \boldsymbol{x}_\tau', \boldsymbol{w}'\right]'$.

On the other hand, we have

$$\begin{aligned} \boldsymbol{y}'\boldsymbol{y} &= \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{k=1}^{r}\sum_{l=1}^{r} h_i h_j h_k h_l \boldsymbol{x}'(D_i - \mathcal{D}_i\mathcal{K}_j)'(D_k - \mathcal{D}_k\mathcal{K}_l)\boldsymbol{x} \\ &\le \sum_{i=1}^{r}\sum_{j=1}^{r} h_i h_j \boldsymbol{x}'(D_i - \mathcal{D}_i\mathcal{K}_j)'(D_i - \mathcal{D}_i\mathcal{K}_j)\boldsymbol{x}. \end{aligned}$$

Consequently

$$\begin{aligned}\boldsymbol{y}'\boldsymbol{y} - \gamma^2\boldsymbol{w}'\boldsymbol{w} &\leq \sum_{i=1}^{r}\sum_{j=1}^{r} h_i h_j \boldsymbol{x}'(D_i - \mathcal{D}_i\mathcal{K}_j)'(D_i - \mathcal{D}_i\mathcal{K}_j)\boldsymbol{x} - \gamma^2\boldsymbol{w}'\boldsymbol{w} + \dot{V} - \dot{V} \\ &= \sum_{i=1}^{r}\sum_{j=1}^{r} h_i h_j \boldsymbol{\theta}'\mathcal{M}_{ij}\boldsymbol{\theta} - \dot{V} = \sum_{i=1}^{r} h_i^2\boldsymbol{\theta}'\mathcal{M}_{ii}\boldsymbol{\theta} + 2\sum_{1\leq i<j\leq r} h_i h_j \boldsymbol{\theta}'\frac{\mathcal{M}_{ij}+\mathcal{M}_{ji}}{2}\boldsymbol{\theta} - \dot{V},\end{aligned} \tag{22}$$

where

$$\mathcal{M}_{ij} = \begin{bmatrix} \mathcal{H}_4 & * & * & * \\ \mathcal{P} - \mathcal{R} + \mathcal{R}(\bar{A}_i - \bar{B}_i\mathcal{K}_j) & h\mathcal{X}_{33} - 2\mathcal{R} & * & * \\ \bar{\mathcal{A}}_i'\mathcal{R} + h\mathcal{X}_{21} - \mathcal{X}_{31} + \mathcal{X}_{23} & \bar{\mathcal{A}}_i'\mathcal{R} & \mathcal{H}_5 & * \\ \mathcal{B}_i'\mathcal{R} & \mathcal{B}_i'\mathcal{R} & \mathbf{0} & -\gamma^2 \end{bmatrix}$$

with $\mathcal{H}_4 = \overrightarrow{\mathcal{R}(\bar{A}_i - \bar{B}_i\mathcal{K}_j)} + h\mathcal{X}_{11} + \overrightarrow{\mathcal{X}_{13}} + \mathcal{Q} + (D_i - \mathcal{D}_i\mathcal{K}_j)'(D_i - \mathcal{D}_i\mathcal{K}_j)$. It is obvious that if

$$\mathcal{M}_{ii} < 0, \quad i = 1, 2, \ldots, r, \quad \text{and} \quad \frac{\mathcal{M}_{ij} + \mathcal{M}_{ji}}{2} \leq 0, \quad 1 \leq i < j \leq r, \tag{23}$$

then $\boldsymbol{y}'\boldsymbol{y} - \gamma^2\boldsymbol{w}'\boldsymbol{w} + \dot{V} < 0$. Under the zero initial conditions, we have $\int_0^\infty \left(\boldsymbol{y}'\boldsymbol{y} - \gamma^2\boldsymbol{w}'\boldsymbol{w}\right)\mathrm{d}s \leq 0$, namely, $\|\boldsymbol{y}\|_2 \leq \gamma\|\boldsymbol{w}\|_2$.

By the Schur complement, $\mathcal{M}_{ii} < 0$ $(i = 1, \ldots, r)$ is equivalent to

$$\bar{\mathcal{M}}_{ii} = \begin{bmatrix} \overrightarrow{\mathcal{R}(\bar{A}_i - \bar{B}_i\mathcal{K}_i)} + h\mathcal{X}_{11} + \overrightarrow{\mathcal{X}_{13}} + \mathcal{Q} & * & * & * & * \\ \mathcal{P} - \mathcal{R} + \mathcal{R}(\bar{A}_i - \bar{B}_i\mathcal{K}_i) & h\mathcal{X}_{33} - 2\mathcal{R} & * & * & * \\ \bar{\mathcal{A}}_i'\mathcal{R} + h\mathcal{X}_{21} - \mathcal{X}_{31} + \mathcal{X}_{23} & \bar{\mathcal{A}}_i'\mathcal{R} & \mathcal{H}_5 & * & * \\ \mathcal{B}_i'\mathcal{R} & \mathcal{B}_i'\mathcal{R} & \mathbf{0} & -\gamma^2 & * \\ D_i - \mathcal{D}_i\mathcal{K}_i & \mathbf{0} & \mathbf{0} & \mathbf{0} & -I \end{bmatrix} < 0, \tag{24}$$

and $\frac{\bar{\mathcal{M}}_{ij}+\bar{\mathcal{M}}_{ji}}{2} \leq 0$ $(1 \leq i < j \leq r)$ is equivalent to

$$\bar{\mathcal{M}}_{ij} = \begin{bmatrix} \overrightarrow{\mathcal{R}\frac{\bar{A}_i - \bar{B}_i\mathcal{K}_j + \bar{A}_j - \bar{B}_j\mathcal{K}_i}{2}} + h\mathcal{X}_{11} + \overrightarrow{\mathcal{X}_{13}} + \mathcal{Q} & * & * & * & * & * \\ \mathcal{P} - \mathcal{R} + \mathcal{R}\frac{\bar{A}_i - \bar{B}_i\mathcal{K}_j + \bar{A}_j - \bar{B}_j\mathcal{K}_i}{2} & h\mathcal{X}_{33} - 2\mathcal{R} & * & * & * & * \\ \frac{\bar{\mathcal{A}}_i' + \bar{\mathcal{A}}_j'}{2}\mathcal{R} + h\mathcal{X}_{21} - \mathcal{X}_{31} + \mathcal{X}_{23} & \frac{\bar{\mathcal{A}}_i' + \bar{\mathcal{A}}_j'}{2}\mathcal{R} & \mathcal{H}_5 & * & * & * \\ \mathcal{B}_i'\mathcal{R} & \mathcal{B}_i'\mathcal{R} & \mathbf{0} & -\gamma^2 & * & * \\ D_i - \mathcal{D}_i\mathcal{K}_j & \mathbf{0} & \mathbf{0} & \mathbf{0} & -2I & * \\ D_j - \mathcal{D}_j\mathcal{K}_i & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -2I \end{bmatrix} \leq 0. \tag{25}$$

By a process like that from (11) to (13), on the basis of (18) and (19), we know that

$$\hat{\mathcal{M}}_{ii} = \begin{bmatrix} \overrightarrow{\bar{A}_i R - \bar{B}_i K_i} + hX_{11} + \overrightarrow{X_{13}} + Q & * & * & * & * \\ P - R + \bar{A}_i R - \bar{B}_i K_i & hX_{33} - 2R & * & * & * \\ R\bar{\mathcal{A}}_i' + hX_{21} - X_{31} + X_{23} & R\bar{\mathcal{A}}_i' & \mathcal{H}_1 & * & * \\ \mathcal{B}_i' & \mathcal{B}_i' & \mathbf{0} & -\gamma^2 & * \\ D_i R - \mathcal{D}_i K_i & \mathbf{0} & \mathbf{0} & \mathbf{0} & -I \end{bmatrix} < 0, \quad i = 1, 2, \ldots, r, \tag{26}$$

$$\hat{\mathcal{M}}_{ij} = \begin{bmatrix} \overrightarrow{\frac{\bar{A}_i R - \bar{B}_i K_j + \bar{A}_j R - \bar{B}_j K_i}{2}} + hX_{11} + \overrightarrow{X_{13}} + Q & * & * & * & * & * \\ P - R + \frac{\bar{A}_i R - \bar{B}_i K_j + \bar{A}_j R - \bar{B}_j K_i}{2} & hX_{33} - 2R & * & * & * & * \\ R\frac{\bar{\mathcal{A}}_i' + \bar{\mathcal{A}}_j'}{2} + hX_{21} - X_{31} + X_{23} & R\frac{\bar{\mathcal{A}}_i' + \bar{\mathcal{A}}_j'}{2} & \mathcal{H}_1 & * & * & * \\ \mathcal{B}_i' & \mathcal{B}_i' & \mathbf{0} & -\gamma^2 & * & * \\ D_i R - \mathcal{D}_i K_j & \mathbf{0} & \mathbf{0} & \mathbf{0} & -2I & * \\ D_j R - \mathcal{D}_j K_i & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -2I \end{bmatrix} \le 0,$$

$$1 \le i < j \le r. \tag{27}$$

Let $\Sigma_2 = \text{diag}[\mathcal{R}, \mathcal{R}, \mathcal{R}, I, I]$, $\Sigma_3 = \text{diag}[\mathcal{R}, \mathcal{R}, \mathcal{R}, I, I, I]$; then it is straightforward to get

$$\bar{\mathcal{M}}_{ii} = \Sigma_2 \hat{\mathcal{M}}_{ii} \Sigma_2 < 0, \quad i = 1, 2, \ldots, r, \qquad \bar{\mathcal{M}}_{ij} = \Sigma_3 \hat{\mathcal{M}}_{ij} \Sigma_3 \le 0, \quad 1 \le i < j \le r. \tag{28}$$

i.e., (23) holds. Simultaneously, inequalities (18) and (19) imply (5) and (6), respectively. According to Theorem 1, (1) is asymptotically stabilizable when $\boldsymbol{w} = \mathbf{0}$. Thus, the proof is completed. ■

**Remark 3.** Now, we compare the computational complexity of the main results in this paper with that in [10]. Let $r$ be the number of fuzzy rules. Theorem 4 in Ref. [10] needs to compute $3 + r + 2r^2$ parameters including matrices and scalars, while Theorem 2 of this paper needs $\frac{r(r+1)}{2} + 9 + r$ parameters, which is far fewer when $r \ge 3$. The dimension of LMIs in this paper is much lower, too.

If $\tau$ is non-differentiable or $\dot{\tau}$ is unknown, we can easily get:

**Corollary 3.** *Given scalar $\gamma > 0$, if there exist matrices $P > 0$, $R > 0$, $X_{lk}$ $(l, k = 1, 2, 3)$, $K_j \in R^{m\times n}$, and positive scalars $e_{ij}$ $(1 \le i \le j \le r)$, such that (18), (19) and (7) hold with $Q = 0$, then (1) is robustly asymptotically stabilizable with disturbance attenuation $\gamma$, and moreover, the control gain matrix $\mathcal{K}_j = K_j R^{-1}$.*

**Remark 4.** It is conceivable that generally, Corollary 3 is more conservative than Theorem 2, as demonstrated in Section 5. The lack of rate information on the time delay is responsible for the relative conservatism of Corollary 3, which indicates the important role of rate information in robust control.

## 5. Examples

**Example 1.** First we consider a fuzzy system $\dot{\boldsymbol{x}} = \sum_{i=1}^{2} h_i(A_i\boldsymbol{x} + \mathcal{A}_i\boldsymbol{x}(t-h))$ with $\boldsymbol{x}(t) \in R^2$,

$$A_1 = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix} \quad A_2 = \begin{bmatrix} -1.5 & 1 \\ 0 & -0.75 \end{bmatrix} \quad \mathcal{A}_1 = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} \quad \mathcal{A}_2 = \begin{bmatrix} -1 & 0 \\ 1 & -0.85 \end{bmatrix}.$$

According to Theorem 1 in [10], the upper boundary of $h$ is 0.8005. However, on the basis of Corollary 1 of this paper, the upper boundary of $h$ is 1.2163.

**Example 2.** Now we apply Theorem 2 to design the controllers of the following fuzzy system:

$$\begin{aligned} \dot{\boldsymbol{x}} &= \sum_{i=1}^{2} h_i\{(A_i + NF(t)E)\boldsymbol{x} + (\mathcal{A}_i + NF(t)E)\boldsymbol{x}_\tau + (B_i + NF(t)E)\boldsymbol{u} + \mathcal{B}_i\boldsymbol{w}\}, \\ \boldsymbol{y} &= \sum_{i=1}^{2} h_i\,(D_i\boldsymbol{x} + \mathcal{D}_i\boldsymbol{u})\,, \quad t \ge 0, \qquad \boldsymbol{x} = \boldsymbol{\varphi}(t), \quad t \in [-h, 0], \end{aligned} \tag{29}$$

where $\boldsymbol{x} \in R^2$, $\boldsymbol{u} \in R^2$, $\boldsymbol{w} \in R^2$, $0 \le \tau(t) \le \tau$, and

$$A_1 = \begin{bmatrix} 0.3 & 0.1 \\ 0 & 0.2 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0.1 & 0.3 \\ 0.7 & 0.1 \end{bmatrix} \quad \mathcal{A}_1 = \begin{bmatrix} -1 & 0.4 \\ 0.2 & 0.1 \end{bmatrix} \quad \mathcal{A}_2 = \begin{bmatrix} 0.1 & 0 \\ 0.5 & 0.4 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 0.1 & 0.4 \\ 0 & -1 \end{bmatrix} \quad B_2 = \begin{bmatrix} 0.2 & 1 \\ 0.4 & -0.3 \end{bmatrix} \quad \mathcal{B}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \mathcal{B}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$D_1 = \begin{bmatrix} 0.1 & 1 \\ 0 & 1 \end{bmatrix} \quad D_2 = \begin{bmatrix} 0.1 & 0.4 \\ 0 & 0.1 \end{bmatrix} \quad \mathcal{D}_1 = \begin{bmatrix} 0.3 & -0.3 \\ 0 & 0.2 \end{bmatrix} \quad \mathcal{D}_2 = \begin{bmatrix} 0.2 & 0 \\ 0 & -0.1 \end{bmatrix}$$

$$N = \begin{bmatrix} -0.1 & 0 \\ 0.1 & -0.1 \end{bmatrix} \quad E = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix} \quad F(t) = \begin{bmatrix} \cos t & 0 \\ 0 & \sin t \end{bmatrix}.$$

To begin, assume that $\tau(t)$ is differentiable and $\dot{\tau} \le d$. Fixing $\gamma = 2$ and applying Theorem 2, we can get

| $d$ | 0.1 | 0.5 | 0.6 | 0.9 | 1.1 | $5 \times 10^5$ |
|---|---|---|---|---|---|---|
| $\tau$ (the upper bound of $\tau(t)$) | 0.9103 | 0.8763 | 0.8626 | 0.7832 | 0.7454 | 0.7454 |

and then we can design controllers for (29) accordingly. For example, when $d = 0.5$ and $\tau = 0.8763$, the gain matrices are as follows:

$$\mathcal{K}_1 = \begin{bmatrix} 2.6779 & 5.9123 \\ 0.2088 & -3.4727 \end{bmatrix}, \qquad \mathcal{K}_2 = \begin{bmatrix} 3.0965 & 6.5227 \\ 0.0689 & -3.1462 \end{bmatrix}.$$

According to Theorem 2, controller $\boldsymbol{u} = -\sum_{i=1}^{2} h_i \mathcal{K}_i \boldsymbol{x}$ can robustly asymptotically stabilize the above-mentioned system with disturbance attenuation $\gamma = 2$.

In contrast, to show the conservatism caused by the lack of rate information, now assume that $\tau$ is non-differentiable or $\dot{\tau}$ is unknown. Applying Corollary 3 we have that, corresponding to the fixed $\gamma = 2$, the upper bound of $\tau(t)$ is $\tau = 0.7454$. It can be seen from the above table that Theorem 2 is less conservative than Corollary 3.

## 6. Conclusions

Using a novel Lyapunov–Krasovskii functional, we have explored the robust stabilization and robust $H_\infty$ control problems of delayed fuzzy systems with uncertainties and perturbation. All results have less conservatism than the previous ones. Most importantly, the derivatives of the delays are allowed to be arbitrarily large. Two numerical examples exemplify our theoretical results, indicating the advantage of our criteria. In the future, we hope to further explore the robust $H_\infty$ control problems of uncertain delayed fuzzy systems via observer-based output feedback using a similar approach.


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