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Abstract

Purpose – To study the multiplicative perturbation of local C-regularized cosine functions associated with the following incomplete second order abstract differential equations in a Banach space X $u''(t) = A(I + B)u(t), \quad u(0) = x, \; u'(0) = y, (*)$ where A is a closed linear operator on X and B is a bounded linear operator on X.

Design/methodology/approach – The multiplicative perturbation of exponentially bounded regularized C-cosine functions is generally studied by the Laplace transformation. However, C-cosine functions might not be exponentially bounded, so that the new method for the multiplicative perturbation of the nonexponentially bounded regularized C-cosine functions should be applied. In this paper, the property of regularized C-cosine functions is directly used to obtain the desired results.

Findings – The new results of the multiplicative perturbations of the nonexponentially bounded C-cosine functions are obtained.

Originality/value – The new techniques differing from those given previously in the literature are employed to deduce the desired conclusions. The results can be applied to deal with incomplete second order abstract differential equations which stem from cybernetics, engineering, physics, etc.

Keywords Differential equations, Cybernetics

Paper type Research paper

Introduction

Many initial value or initial-boundary value problems for partial differential equations, stemmed from biology, physics, engineering, control theory, etc. (Zhou *et al.*, 2002; Qin and Ren, 2007) can be translated into the incomplete second order abstract differential equations. The theory of the cosine and sine functions is an important tool to study the well-posedness of the incomplete second order abstract differential equations. For recent critical discussions on initial value (boundary) problems and their well-posedness, please refer to OuYang *et al.* (2000, 2001) and Lin (2001).

In the spirit of the works (Song *et al.*, 2006; Huang and Huang, 1995; Kuo and Shaw, 1997; Li *et al.*, 2003; Liang *et al.*, 2006; Xiao and Liang, 1998, 2001, 2002, 2003; Xiao *et al.*, 2006), we study further multiplicative perturbations of a local C-regularized cosine function associated with an incomplete second order abstract differential equation in a Banach space X, which is one of important mathematical systems coming from the real world, in the case where:

The work was supported partly by the National Natural Science Foundation of China 10571165, the NCET-04-0572 and Research Fund for the Key Program of the Chinese Academy of Sciences and the Youth Foundation of USTC.

Kybernetes
Vol. 37 No. 9/10, 2008
pp. 1431-1437

0368-492X
DOI 10.1108/03684920810907742

- the range of the regularizing operator C is not dense in X; and
- the operator C may not commute with the perturbation operator.

Throughout this paper, all operators are linear; $L(X, X)$ denotes the space of all continuous linear operators from X to X (in short $L(X)$); C is an injective operator in $L(X)$; $C([0, t], L(X))$ denotes all continuous $L(X)$-valued functions, equipped with the norm $\|F\|_\infty = \sup_{r\in[0,t]}\|F(r)\|$. Moreover, we write $D(A)$, $R(A)$, respectively, for the domain, the range of an operator A. We abbreviate C-regularized cosine function to C-cosine function.

Definition 1.1. Assume $\tau > 0$. A one-parameter family $\{C(t); |t| \leq \tau\} \subset L(X)$ is called a local C-cosine function on X if:

- $C(0) = C$ and $C(t+s)C + C(t-s)C = 2C(t)C(s) \quad (\forall |s|, |t|, |s+t| \leq \tau)$; and
- $C(\cdot)x : [-\tau, \tau] \to X$ is continuous for every $x \in X$.

The associated sine operator function $S(\cdot)$ is defined by:

$$S(t) := \int_0^t C(s)ds(|t| \leq \tau).$$

The operator A defined by:

$$D(A) = \left\{ x \in X;\ \lim_{t\to 0^+} \frac{2}{t^2}(C(t)x - Cx) \text{ exists and is in } R(C) \right\},$$

$$Ax = C^{-1}\lim_{t\to 0^+}\frac{2}{t^2}(C(t)x - Cx), \quad \forall x \in D(A),$$

is called the generator of $\{C(t); |t| \leq \tau\}$. It is also called that A generates $\{C(t); |t| \leq \tau\}$.

Lemma 1.2. (Huang and Huang (1995)) Let A generates a local C-cosine function $\{C(t); |t| \leq \tau\}$ on X.

Then:

- For $x \in D(A)$, $t \in [-\tau, \tau]$, $C(t)x$, $S(t)x \in D(A)$, $AC(t)x = C(t)Ax$, $AS(t)x = S(t)Ax$.
- For $x \in X$, $t \in [0, \tau]$, $\int_0^t S(s)xds \in D(A)$ and $A\int_0^t S(s)xds = C(t)x - Cx$.
- For $x \in D(A)$, $t \in [0, \tau]$, $\int_0^t S(s)Axds = A\int_0^t S(s)xds = C(t)x - Cx$.

It is easy to prove the following lemma.

Lemma 1.3. Suppose an extension of A, $^*A$ generates a local C-cosine function. Then $C(D(^*A)) \subset D(A)$ is equivalent to $C^{-1}AC = {}^*A$.

Definition 1.4. Assume A is a closed operator on X and $\tau > 0$. A one-parameter family $\{C(t)\}_{t\in[-\tau,\tau]} \subset L(X)$ is called a local C-existence family for A if:

- $C(\cdot)x : [-\tau, \tau] \to X$ is continuous for every $x \in X$; and
- $\int_0^t S(s)xds \in D(A)$ and $A\int_0^t S(s)xds = C(t)x - Cx$ for every $x \in X, t \in [-\tau, \tau]$.

Clearly, a local C-cosine function generated by A is a local C-existence family for A.

By the arguments similar to those in the proof of (Xiao and Liang, 2002, Theorem 2.4), we obtain:

Proposition 1.5. Assume that $\{C(t); |t| \leq \tau\}$ is a local C-existence family on X for A. Then for each $x \in D(A)$ with $Ax \in R(C)$ and $y \in X$, the Cauchy problem, $u''(t) = Au(t), \quad u(0) = x, \quad u'(0) = y$, has a unique solution $u \in C^2([-\tau, \tau], X)$.

Results and proofs

Theorem 2.1. Assume A generates a local C-cosine function $\{C(t); |t| \leq \tau\}$ on X. If $B \in L(X)$ satisfies:

$$H1. \ R(B) \subset R(C),$$

$$H2. \forall \Phi \in C([0,\tau], X), \quad \int_0^t S(t-s)C^{-1}B\Phi(s)\mathrm{d}s \in D(A)$$

and:

$$\left\| A\int_0^t S(t-s)C^{-1}B\Phi(s)\mathrm{d}s \right\| \leq M\int_0^t \sup_{0\leq s\leq\sigma}\Phi(s)\mathrm{d}\sigma, \quad t \in [0,\tau],$$

where $M > 0$ is a constant, $S(t) = \int_0^t C(s)\mathrm{d}s$:

H3. There exists an injective operator $C_1 \in L(X)$ satisfying $R(C_1) \subset R(C)$ and $C_1A(I+B) \subset A(I+B)C_1$, then $C_1^{-1}A(I+B)C_1$ generates a local C_1-cosine function.

Proof. For every $x \in X$, we define the operator functions $\{\bar{C}_n(t)\}_{t\in[0,\tau]}$ on X as follows:

$$\bar{C}_0(t)x = C(t)x; \quad \bar{C}_n(t)x = A\int_0^t S(t-s)C^{-1}B\bar{C}_{n-1}(s)x\mathrm{d}s, \quad t \in [0,\tau], \quad n = 1, 2, \ldots.$$

By induction, we obtain:

- $\forall x \in X, \bar{C}_n(t)x \in C([0,\tau], X)$,
- $\|\bar{C}_n(t)\| \leq \frac{M^n t^n}{n!}\sup_{0\leq s\leq\tau} PC(s)P, \quad t \in [0,\tau], \quad \forall n \geq 0.$

Thus, ${}^*C(t) := \sum_{n=0}^{\infty}\bar{C}_n(t)C^{-1}C_1 \in C([0,\tau], X)$ and satisfies:

$${}^*C(t) = C(t)C^{-1}C_1x + A\int_0^t S(t-s)C^{-1}B\,{}^*C(s)x\mathrm{d}s, x \in X, t \in X, t \in [0,\tau].$$

This means that ${}^*C(t)$x is a solution of the equation:

$$u(t) = C(t)C^{-1}C_1x + A\int_0^t S(t-s)C^{-1}Bu(s)\mathrm{d}s, \quad t \in [0,\tau]. \tag{1}$$

Let $v(\cdot) \in C([0,\tau], X)$ satisfy equation (1). Then:

$${}^*C(t) - v(t) = A\int_0^t S(t-s)C^{-1}B[{}^*C(s)x - v(s)]\mathrm{d}s, \quad t \in [0,\tau].$$

Using *H2* and Gronwall-Bellman's inequality, we get $^*C(t) = v(t)$ on $[0,\tau]$. It shows the uniqueness of solutions of equation (1). Clearly:

$$\int_0^s {}^*C(\sigma)x\mathrm{d}\sigma = \int_0^s C(\sigma)C^{-1}C_1 x\mathrm{d}\sigma + A\int_0^s \int_0^\sigma S(\sigma - w)C^{-1}B^*C(w)x\mathrm{d}w\mathrm{d}\sigma$$
$$= S(s)C^{-1}C_1x + \frac{d}{\mathrm{d}s}\int_0^s S(s-\sigma)C^{-1}B^*C(\sigma)x\mathrm{d}\sigma - C\int_0^s C^{-1}B^*C(\sigma)x\mathrm{d}\sigma$$

So:

$$(I+B)\int_0^s {}^*C(\sigma)x\mathrm{d}\sigma = S(s)C^{-1}C_1x + \frac{d}{\mathrm{d}s}\int_0^s S(s-\sigma)C^{-1}B^*C(\sigma)x\mathrm{d}\sigma,$$

integrating this from O to t, and putting:

$$\tilde{S}(t)x := \int_0^t {}^*C(s)x\mathrm{d}s, \quad t \in [0,\tau], \quad x \in X,$$

we have:

$$(I+B)\int_0^t \tilde{S}(s)x\mathrm{d}s = \int_0^t S(s)C^{-1}C_1x\mathrm{d}s + \int_0^t S(t-\sigma)C^{-1}B\,{}^*C(\sigma)x\mathrm{d}\sigma \in D(A)$$

and:

$$A(I+B)\int_0^t \tilde{S}(s)x\mathrm{d}s = S(t)C^{-1}C_1x - C_1x + A\int_0^t S(t-\sigma)C^{-1}B\,{}^*C(\sigma)x\mathrm{d}\sigma$$
$$= {}^*C(t)x - C_1x, \quad \forall x \in X, t \in [0,\tau], \tag{2}$$

that is, $^*C(t)$ is a solution of the integral equation:

$$g(t) = C_1x + A(I+B)\int_0^t \int_0^s g(\sigma)\mathrm{d}\sigma\mathrm{d}s, \quad x \in X, \ t \in [0,\tau]. \tag{3}$$

On the other hand, if $g(t)$ is a solution of equation (3), then:

$$\int_0^t S(t-s)A(I+B)\int_0^s g(\sigma)\mathrm{d}\sigma\mathrm{d}s = \int_0^t C(t-s)g(s)\mathrm{d}s - \int_0^t C(s)C_1x\mathrm{d}s.$$

Therefore:

$$A\int_0^t \int_0^s S(s-\sigma)Bg(\sigma)\mathrm{d}\sigma\mathrm{d}s = C\int_0^t g(s)\mathrm{d}s - \int_0^t C(s)C_1x\mathrm{d}s \in R(C).$$

Thus:

$$g(t) = C(t)C^{-1}C_1x + A\int_0^t S(t-s)C^{-1}Bg(s)\mathrm{d}s,$$

i.e. $g(\cdot)$ is the solution of equation (1). This shows that equation (3) has a unique solution. Hence:

$$^*C(-t)x = ^*C(t)x; \quad C_1{}^*C(t)x = ^*C(t)C_1x, \quad x \in X, \ t \in [-\tau, \tau]$$

By equation (3), we get:

$$^*C(-t)x = C_1x + A(I+B)\int_0^t \int_0^s {}^*C(\sigma)x\mathrm{d}\sigma\mathrm{d}s, \quad x \in X, \ t \in [0, \tau].$$

Therefore, for each $t, h \in [0, \tau]$, $x \in X$:

$$2^*C(t)^*C(h)x = A(I+B)\int_0^t \int_0^s 2^*C(\sigma)^*C(h)x\mathrm{d}\sigma\mathrm{d}s + 2C_1^2x + 2A(I+B) \times \int_0^h (h-s)^*C(s)C_1x\mathrm{d}s$$

On the other hand, for any $t, h, t \pm h \in [0, \tau]$, $x \in X$, we get:

$$\begin{aligned} {}^*C(t+h)C_1x - C_1^2x &= A(I+B)\int_0^{t+h}\int_0^s {}^*C(\sigma)C_1x\mathrm{d}\sigma\mathrm{d}s \\ &= A(I+B)\left\{\int_{-h}^0\int_{-h}^w {}^*C(\mu+h)C_1x\mathrm{d}\mu\mathrm{d}w \right. \\ &\quad \left. + \int_0^t\left[\int_{-h}^0 + \int_0^w\right] {}^*C(\mu+h)C_1x\mathrm{d}\mu\mathrm{d}w\right\} \\ &= A(I+B)\left\{\int_0^t\int_0^w {}^*C(\mu+h)C_1x\mathrm{d}\mu\mathrm{d}w \right. \\ &\quad + \int_0^t\int_0^w {}^*C(-\sigma+h)C_1x\mathrm{d}\sigma\mathrm{d}w \\ &\quad \left. - \int_0^h\int_0^{s-h} {}^*C(-\mu)C_1x\mathrm{d}\mu\mathrm{d}s\right\} \end{aligned}$$

Similarly:

$$\begin{aligned} {}^*C(t-h)C_1x - C_1^2x = A(I+B)&\left\{\int_0^t\int_0^s {}^*C(\sigma-h)C_1x\mathrm{d}\sigma\mathrm{d}s \right. \\ &\left. - \int_0^t\int_0^h {}^*C(\sigma-h)C_1x\mathrm{d}\sigma\mathrm{d}s - \int_0^h\int_0^{s-h} {}^*C(\mu)C_1x\mathrm{d}\mu\mathrm{d}s\right\} \end{aligned}$$

Therefore:

$$\begin{aligned} &{}^*C(t+h)C_1x + {}^*C(t-h)C_1x - 2C_1^2x \\ &= A(I+B)\left[\int_0^t\int_0^s ({}^*C(\sigma+h)C_1x + {}^*C(\sigma-h)C_1x)\mathrm{d}\sigma\mathrm{d}s + 2\int_0^h (h-s)^*C(s)C_1x\mathrm{d}s\right] \end{aligned}$$

Thus, we get:

$$2^*C(t)^*C(h)x - [^*C(t+h)C_1x + {}^*C(t-h)C_1x]$$

$$= A(I+B)\int_0^t \int_0^s \{2^*C(\sigma)^*C(h) - [^*C(\sigma+h)C_1x + {}^*C(\sigma-h)C_1x]\}\mathrm{d}\sigma\mathrm{d}s.$$

Again by the uniqueness of the solution of equation (3), we obtain:

$$2^*C(t)^*C(h)x = {}^*C(t+h)C_1x + {}^*C(t-h)C_1x, x \in, X, t, h, t \pm h \in [0, \tau].$$

So, $\{^*C(t) : |t| \leq \tau\}$ is a local C_1-cosine function on X. Assume that $x \in D(A(I+B))$ and:

$$p(t)x = C_1x, \int_0^t \int_0^s {}^*C(\sigma)A(I+B)x\mathrm{d}\sigma\mathrm{d}s, \quad t \in [0, \tau].$$

Since:

$$^*C(t)A(I+B)x = A(I+B)p(t)x,$$

i.e. $p(t)x$ is a solution of equation (2) for $x \in D(A(I+B))$, $^*C(t)x = p(t)x$. So:

$$\frac{2}{t^2}(^*C(t)x - C_1x) = \frac{2}{t^2}\int_0^t \int_0^s {}^*C(\sigma)A(I+B)x\mathrm{d}\sigma\mathrm{d}s \rightarrow C_1A(I+B)x$$

$$\in R(C_1), \quad (t \rightarrow 0^+).$$

Therefore, if $\mathcal{G}$ is the generator of $\{^*C(t) : |t| \leq \tau\}$, then $x \in D(\mathcal{G})$ and $A(I+B)x = \mathcal{G}x$, that is, $A(I+B) \subset \mathcal{G}$.

Moreover, for any $x \in D(\mathcal{G})$, $\lim_{n\to\infty} 2n^2 \int_0^{1/n} \int_0^s {}^*C(\sigma)x\mathrm{d}\sigma\mathrm{d}s = C_1x$. By equation (2):

$$\lim_{n\to\infty} A(I+C)\left(2n^2 \int_0^{1/n} \int_0^s {}^*C(\sigma)x\mathrm{d}\sigma\mathrm{d}s\right) = \lim_{n\to\infty} 2n^2\left({}^*C\left(\frac{1}{n}\right)x - C_1x\right) = C_1Gx.$$

Thus, the closedness of $A(1 + B)$ implies that $C_1x \in D(A(I+B))$ and $C_1\mathcal{G}x = A(I+B)C_1x$. Therefore, $C_1(D(\mathcal{G})) \subset D(A(I+B))$. The conclusion now follows from Lemma 1.3. End of the proof. □

In the proof of Theorem 2.1, with C instead of C_1, we can obtain:

Theorem 2.2. Let A generates a local C-cosine function on X. If $B \in L(X)$ and $R(C^{-1}B) \subset D(A)$, then for each $x \in D(A)$ with $Ax \in R(C)$ and $y \in X$, the Cauchy problem (*) has a unique solution $u(\cdot) \in C^2([-\tau, \tau], X)$.

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