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Linear quivers, generic extensions and Kashiwara operators

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ABSTRACT

In the present paper, we introduce the generic extension graph G of a Dynkin or cyclic quiver Q and then compare this graph with the crystal graph C for the quantized enveloping algebra associated to Q via two maps $\wp_Q, \kappa_Q : \Omega \to \Lambda_Q$ induced by generic extensions and Kashiwara operators, respectively, where Λ_Q is the set of isoclasses of nilpotent representations of Q, and Ω is the set of all words on the alphabet I, the vertex set of Q. We prove that, if Q is a (finite or infinite) linear quiver, then the intersection of the fibres $\wp_Q^{-1}(\lambda)$ and $\kappa_Q^{-1}(\lambda)$ is non-empty for every $\lambda \in \Lambda_Q$. We will also show that this non-emptyness property fails for cyclic quivers.

1. INTRODUCTION

Let Q be a Dynkin or cyclic quiver with vertex set $I = \{1, 2, \dots, n\}$, and let Ω be the set of all words on the alphabet I. Let Λ_Q denote the set of isoclasses of (finite-dimensional) nilpotent representations of Q. In [2,3], the first two authors introduced a map, the *generic extension map*,

$$\wp = \wp_Q : \Omega \to \Lambda_Q.$$

The fibre of $\wp_Q$ is described as those words $w = i_1 \cdots i_m$ in Ω which define the same generic extension $S_{i_1} * \cdots * S_{i_m}$ of simple modules $S_{i_1}, \dots, S_{i_m}$. Also, the

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map $\wp$ gives a strong monomial basis property for the $\pm$-part $\mathbf{U}^{\pm}$ of the associated quantized enveloping algebra $\mathbf{U} = \mathbf{U}_v(Q)$.

In [11], M. Kashiwara introduced certain operators $\tilde{E}_i \in \mathrm{End}(\mathbf{U}^+)$ $(i \in I)$ in the construction of his crystal bases for $\mathbf{U}^+$. Kashiwara proved that, if

$$\mathcal{L} = \sum_{w \in \Omega} \mathcal{A}_\infty(\tilde{E}_w \cdot 1) \subseteq \mathbf{U}^+,$$

where $\tilde{E}_w = \tilde{E}_{i_1} \cdots \tilde{E}_{i_m}$ for $w = i_1 \cdots i_m$ and

$$\mathcal{A}_\infty = \{f(v) \in \mathbb{Q}(v) \mid f(v^{-1}) \text{ is regular at } v = 0\},$$

then the set $B = \{\tilde{E}_w \cdot 1 + v^{-1}\mathcal{L} \mid w \in \Omega\}$ forms a basis for $\mathcal{L}/v^{-1}\mathcal{L}$ which in turn determines the canonical basis $\mathbf{B} = \{\mathfrak{b}_\lambda \mid \lambda \in \Lambda\} \subset \mathcal{L}$ of $\mathbf{U}^+$. Here the index set Λ can be identified as Λ_Q. Thus, one obtains a map

$$\kappa = \kappa_Q : \Omega \to \Lambda_Q$$

defined by $\tilde{E}_w \cdot 1 \equiv \mathfrak{b}_{\kappa(w)} \pmod{v^{-1}\mathcal{L}}$. The description of κ is a hard problem. In particular, the fibres of κ are not known.

In this paper, we investigate the relations between the two maps κ_Q and $\wp_Q$. We shall see that in general, κ_Q as a map is not equal[1] to $\wp_Q$. However, it is always true that $\kappa_Q(w) \leqslant \wp_Q(w)$ for all $w \in \Omega$ and given Q. Here $\leqslant$ is the partial ordering on Λ_Q. Further relations will be discussed when Q is one of the following quivers:

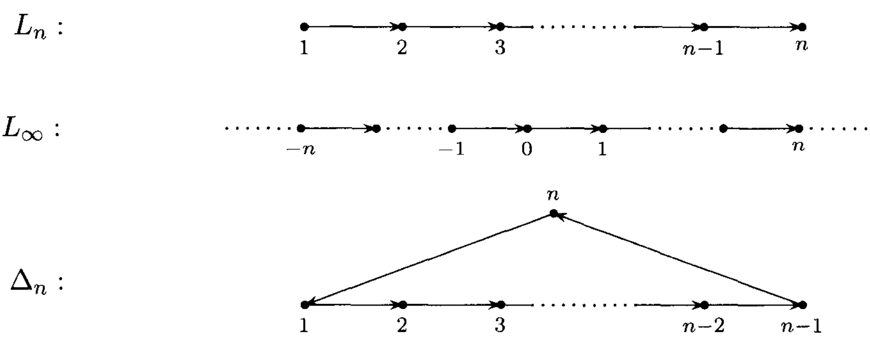

Figure 1. Linear and cyclic quivers.

In particular, we prove that, if Q is a finite linear or infinite linear quiver, then $\wp_Q^{-1}(\lambda) \cap \kappa_Q^{-1}(\lambda) \neq \emptyset$ for all $\lambda \in \Lambda_Q$, and show by counter-example that this fails if Q is a cyclic quiver.

We may interpret these relations in terms of graphs. There are two graphs with the same vertex set Λ_Q – the crystal graph C and the generic extension graph G – associated to κ and $\wp$, respectively. For $w \in \Omega$, $\kappa(w) = \lambda$ (resp. $\wp(w) = \lambda$) means that there is a path from vertex 0 to the vertex λ in C (resp. G) and every such a path is given by an element in Ω; so $\kappa \neq \wp$ means that the two graphs are not identical

[1] This is contrary to an earlier incorrect computation in the type A case announced in [6].

(i.e., they have different edge sets). However, for linear quivers, there is at least one path in common among all paths from 0 to any given vertex $\lambda \in \Lambda_Q$.

We organize the paper as follows. We present the basics for quiver representations and the generic extension map in Section 2, and Ringel–Hall algebras and quantum groups in Section 3. In Section 4, we introduce the map κ and discuss its properties, built on a work by Reineke in [16]. The last two sections are devoted to proving the main results mentioned above for linear quivers.

2. QUIVER REPRESENTATIONS AND THE GENERIC EXTENSION MAP

Let $Q = (I, Q_1)$ be a quiver, i.e., a finite directed graph, where $I = Q_0$ is the set of vertices $\{1, 2, \dots, n\}$ and Q_1 is the set of arrows. If $\rho \in Q_1$ is an arrow from tail i to head j, we write $h(\rho)$ for j and $t(\rho)$ for i.

A (finite-dimensional) *representation* $V = (V_i, V_\rho)$ of Q, consisting of a set of finite-dimensional vector spaces V_i for each $i \in I$ and a set of linear transformations $V_\rho : V_{t(\rho)} \to V_{h(\rho)}$ for each $\rho \in Q_1$, is identified with a (left) module over the path algebra kQ of Q. We call $\mathbf{dim}\, V := (\dim_k V_1, \dots, \dim_k V_n)$ the *dimension vector* of V and $d(V) := \sum_{i=1}^{n} \dim V_i$ the dimension of V. The representation $V = (V_i, V_\rho)$ is called *nilpotent* if for each oriented cycle $\rho_m \dots \rho_1$ at a vertex i, the k-linear map $V_{\rho_m} \cdots V_{\rho_1} : V_i \to V_i$ is nilpotent. Obviously, each vertex $i \in Q_0$ gives rise to a one-dimensional nilpotent representation S_i. Note that if Q contains no oriented cycles, then every representation is nilpotent.

Example 2.1. Let Q be a Dynkin quiver, that is, its underlying graph is a (simply laced) Dynkin graph of type ADE. By Gabriel's theorem [7], there is a bijection between the set of isoclasses of indecomposable representations of Q and the positive system $\Phi^+ = \Phi^+(Q)$ of the root system $\Phi(Q)$ associated with Q. For any $\alpha \in \Phi^+$, let $M(\alpha) = M_k(\alpha)$ denote the corresponding indecomposable representation of Q over k. By the Krull–Remak–Schmidt theorem, every representation M of Q is isomorphic to

$$M(\lambda) = M_k(\lambda) := \bigoplus_{\alpha \in \Phi^+} \lambda(\alpha) M_k(\alpha),$$

for some function $\lambda : \Phi^+ \to \mathbb{N}$. Thus, the isoclasses of representations of Q are indexed by the set

$$\Lambda = \Lambda_Q = \{\lambda : \Phi^+ \to \mathbb{N}\} \cong \mathbb{N}^{\Phi^+}.$$

We embed Φ^+ into Λ_Q naturally.

Example 2.2. Let Q be the cyclic quiver Δ_n $(n \geqslant 2)$. For each integer $l \geqslant 1$, there is a unique (up to isomorphism) indecomposable nilpotent representation $S_i[l]$ of length l with top S_i. It is well known that $S_i[l]$ $(i \in I, l \geqslant 1)$ yield all isoclasses of indecomposable nilpotent representations of Q, and this classification is independent of the field k. Thus, one way to describe the isoclasses of nilpotent representations of Q is to use n-tuples of partitions. Namely, for each n-tuple $\pi = (\pi^{(1)}, \pi^{(2)}, \dots, \pi^{(n)})$ of partitions $\pi^{(i)} = (\pi_1^{(i)} \geqslant \pi_2^{(i)} \geqslant \cdots)$, $1 \leqslant i \leqslant n$, we define a nilpotent representation[2] of Q

$$M(\pi) = M_k(\pi) = \bigoplus_{i\in I, j\geqslant 1} S_i\big[\pi_j^{(i)}\big].$$

If we write the partition $\pi^{(i)}$ as

$$\pi^{(i)} = \big(1^{\lambda_{i,1}}, 2^{\lambda_{i,2}}, \dots, l^{\lambda_{i,l}}, \dots\big).$$

Then π is uniquely determined by $\lambda = (\lambda_{i,l})$. Thus, we obtain a bijection between the set of n-tuples of partitions and the set

$$\Lambda_Q = \big\{\lambda : I \times \mathbb{Z}_+ \to \mathbb{N} \mid \operatorname{supp}(\lambda) \text{ is finite}\big\},$$

where $\mathbb{Z}_+$ denotes the set of all positive integers, and $\operatorname{supp}(\lambda) = \{(i,l) \mid \lambda(i,l) \neq 0\}$. We may also rewrite $M(\pi)$ as

$$M(\pi) = M(\lambda) := \bigoplus_{i\in I, l\geqslant 1} \lambda_{i,l} S_i[l].$$

We now assume for the rest of the section that k is *algebraically closed*. Fix $\mathbf{d} = (d_i)_i \in \mathbb{N}^n$ and define the affine space

$$R(\mathbf{d}) = R(Q, \mathbf{d}) := \prod_{\rho\in Q_1} \operatorname{Hom}_k\big(k^{d_{t(\rho)}}, k^{d_{h(\rho)}}\big) \cong \prod_{\rho\in Q_1} k^{d_{h(\rho)}\times d_{t(\rho)}}.$$

Thus, a point $x = (x_\rho)_\rho$ of $R(\mathbf{d})$ determines a representation $V(x)$ of Q. The algebraic group $GL(\mathbf{d}) = \prod_{i=1}^n GL_{d_i}(k)$ acts on $R(\mathbf{d})$ by conjugation

$$(g_i)_i \cdot (x_\rho)_\rho = \big(g_{h(\rho)} x_\rho g_{t(\rho)}^{-1}\big)_\rho,$$

and the $GL(\mathbf{d})$-orbits $\mathcal{O}_x$ in $R(\mathbf{d})$ correspond bijectively to the isoclasses $[V(x)]$ of representations of Q with dimension vector $\mathbf{d}$.

The stabilizer $GL(\mathbf{d})_x = \{g \in GL(\mathbf{d}) \mid gx = x\}$ of x is the group of automorphisms of $M := V(x)$ which is Zariski-open in $\operatorname{End}_{kQ}(M)$ and has dimension equal to $\dim \operatorname{End}_{kQ}(M)$. It follows that the orbit $\mathcal{O}_M := \mathcal{O}_x$ of M has dimension

$$\dim \mathcal{O}_M = \dim GL(\mathbf{d}) - \dim \operatorname{End}_{kQ}(M).$$

For two representations M, N of Q, define $[N] \leqslant [M]$ (or simply $N \leqslant M$) if $\mathcal{O}_N \subseteq \overline{\mathcal{O}_M}$, the closure of $\mathcal{O}_M$. This gives rise to a partial order on Λ_Q by

$$\lambda \leqslant \mu \iff M_k(\lambda) \leqslant M_k(\mu).$$

[2] The module $M(\pi)$ defined here is indeed the module $M(\tilde{\pi})$ defined in [20], where $\tilde{\pi} = (\tilde{\pi}^{(1)}, \dots, \tilde{\pi}^{(n)})$ and $\tilde{\pi}^{(i)}$ denotes the dual partition of $\pi^{(i)}$, $i \in I$.

From now on, we assume that Q is a Dynkin or cyclic quiver. Given nilpotent representations M, N of Q, consider the extensions

$$0 \longrightarrow N \longrightarrow E \longrightarrow M \longrightarrow 0$$

of M by N. Note that E is again nilpotent. By [1,17,2], there is a unique (up to isomorphism) such extension G with $\dim \mathcal{O}_G$ maximal (or equivalently, with $\dim \mathrm{End}_{kQ}(G)$ minimal). We call G the *generic extension* of M by N, denoted by $M * N$.

Let Ω be the set of all words on the alphabet $I = \{1, 2, \dots, n\}$. For $w = i_1 i_2 \cdots i_m \in \Omega$, let $\wp(w) \in \Lambda_Q$ be the element defined by

$$(2.2.1) \qquad S_{i_1} * S_{i_2} * \cdots * S_{i_m} \cong M\big(\wp(w)\big).$$

Thus, we obtain a map

$$\wp = \wp_Q : \Omega \longrightarrow \Lambda_Q, \quad w \longmapsto \wp(w).$$

Moreover, if Q is a Dynkin quiver, then $\wp$ is surjective; if Q is a cyclic quiver, then $\wp$ is not surjective with $\mathrm{Im}\,\wp = \Lambda_Q^a$, the set of all *aperiodic* n-tuples of partitions, that is, those $\pi = (\pi^{(1)}, \pi^{(2)}, \dots, \pi^{(n)})$ satisfying that for each $t \geqslant 1$, there is some $i \in I$ such that t is not a part in $\pi^{(i)}$.

For each $i \in I$, there is a map

$$(2.2.2) \qquad \sigma_i : \Lambda_Q \longrightarrow \Lambda_Q; \quad \lambda \longmapsto \sigma_i \lambda$$

defined by

$$M(\sigma_i \lambda) \cong S_i * M(\lambda).$$

It is clear that for each $w = i_1 i_2 \dots i_m$, it holds

$$(2.2.3) \qquad \wp(w) = \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_m}(0),$$

where 0 is the element in Λ_Q corresponding to the isoclass of zero module.

We end this section with the definition of the generic extension graph.

Definition 2.3. Let Q be a Dynkin or cyclic quiver. The *generic extension graph* associated to Q is the directed graph G with vertices $\lambda \in \Lambda_Q$ and arrows $\lambda \xrightarrow{i} \mu$, where $\lambda, \mu \in \Lambda_Q$ and $\sigma_i \lambda = \mu$ for some $i \in I$. Indeed, it is an I-colored graph.

Thus, by (2.2.3), every word $w \in \Omega$ defines a path from 0 to $\lambda = \wp(w)$. Clearly, $\wp^{-1}(\lambda)$ consists of all such paths, and the map $\wp$ sends every such a path to the other endpoint λ.

Let Q be a Dynkin or cyclic quiver. By [19,20,9], for λ, μ, ν in Λ_Q, there is a polynomial $\varphi^{\lambda}_{\mu,\nu}(T) \in \mathbb{Z}[T]$, called a *Hall polynomial*, such that for any finite field k of q_k elements, $\varphi^{\lambda}_{\mu,\nu}(q_k)$ is the number of submodules X of $M_k(\lambda)$ satisfying $X \cong M_k(\nu)$ and $M_k(\lambda)/X \cong M_k(\mu)$.

Let $\mathcal{Z} = \mathbb{Z}[v, v^{-1}]$ be the Laurent polynomial ring over $\mathbb{Z}$ in indeterminate v. The (*twisted generic*) *Ringel–Hall algebra* $\mathcal{H}_v(Q)$ of Q is by definition the free $\mathcal{Z}$-module having basis $\{u_\lambda = u_{[M(\lambda)]} \mid \lambda \in \Lambda_Q\}$ and satisfying the multiplication rules

$$u_\mu u_\nu = v^{\langle \mu,\nu\rangle} \sum_{\lambda\in\Lambda_Q} \varphi^{\lambda}_{\mu,\nu}(v^2) u_\lambda,$$

where $\langle \mu, \nu\rangle = \dim_k \operatorname{Hom}_{kQ}(M(\mu), N(\nu)) - \dim_k \operatorname{Ext}^1_{kQ}(M(\mu), N(\nu))$ is the Euler form associated to the quiver Q. For each $i \in I$, we simply write $u_i = u_{[S_i]}$.

On the other hand, every quiver Q determines uniquely a symmetric (generalized) Cartan matrix $C(Q)$. Let $\mathbf{U} = \mathbf{U}_v(Q)$ be the quantized enveloping algebra associated to $C(Q)$. Note that $\mathbf{U}$ is a deformation of the universal enveloping algebra of the semisimple Lie algebra $\mathfrak{g}(Q)$ (resp. $\widehat{\mathfrak{sl}}_n$) with Cartan matrix $C(Q)$ if Q is a Dynkin (resp. cyclic) quiver. Let $\mathbf{U}^+ = \mathbf{U}^+_v(Q)$ be the positive part of $\mathbf{U}$, that is, the $\mathbb{Q}(v)$-subalgebra of $\mathbf{U}$ generated by E_i, $i \in I$, with quantum Serre relations. Further, for each $m \geqslant 1$, let

$$[m] = \frac{v^m - v^{-m}}{v - v^{-1}} \quad \text{and} \quad [m]^! = [1][2]\cdots[m].$$

The Lusztig integral form $U^+ = U^+_v(Q)$ is the $\mathcal{Z}$-subalgebra of $\mathbf{U}^+$ generated by divided powers $E_i^{(m)} = \frac{E_i^m}{[m]^!}$, where $i \in I$ and $m \geqslant 1$.

Theorem 3.1 [19,20]. *Let Q be a Dynkin or cyclic quiver and $U^+_v(Q)$ be the Lusztig integral form of $\mathbf{U}^+_v(Q)$.*

(1) *If Q is a Dynkin quiver, then $\mathcal{H}_v(Q)$ is generated by $u_i^{(m)} := \frac{u_i^m}{[m]^!}$, $i \in I$ and $m \geqslant 1$, and there is a $\mathcal{Z}$-algebra isomorphism*

$$\Psi : U^+_v(Q) \xrightarrow{\sim} \mathcal{H}_v(Q); \quad E_i^{(m)} \longmapsto u_i^{(m)} \quad (i \in I, m \geqslant 1).$$

(2) *If Q is a cyclic quiver, then there is a $\mathcal{Z}$-algebra isomorphism*

$$\Psi : U^+_v(Q) \xrightarrow{\sim} \mathcal{C}_v(Q); \quad E_i^{(m)} \longmapsto u_i^{(m)} \quad (i \in I, m \geqslant 1),$$

where $\mathcal{C}_v(Q)$ is the $\mathcal{Z}$-subalgebra of $\mathcal{H}_v(Q)$ generated by all $u_i^{(m)}$, called the composition algebra of Q.

In the sequel, we shall identify $U^+_v(Q)$ with $\mathcal{H}_v(Q)$ (resp. $\mathcal{C}_v(Q)$) in case Q is a Dynkin (resp. cyclic) quiver. We finally note that the generic extension map $\wp$

can be applied to obtain a strong monomial basis property for the corresponding quantized enveloping algebra $\mathbf{U} = \mathbf{U}_v(Q)$ in [2,3].

4. CANONICAL BASES, GLOBAL CRYSTAL BASES AND THE MAP κ

In this section we review the canonical and global crystal bases for $\mathbf{U}^+ = \mathbf{U}_v^+(Q)$ defined by Lusztig and Kashiwara, respectively. The map $\kappa : \Omega \to \Lambda_Q$ will be defined in terms of Kashiwara operators.

Again, let Q be a Dynkin or cyclic quiver. For each $\lambda \in \Lambda_Q$, define

$$\tilde{u}_\lambda = v^{-\dim M(\lambda) + \dim \mathrm{End}(M(\lambda))} u_\lambda \in \mathcal{H}_v(Q).$$

Note that $\tilde{u}_i = u_i$.

By [13,15] (see also [22,18,3,5]), for each $\lambda \in \Lambda_Q$, there is a unique element

$$\mathfrak{b}_\lambda = \tilde{u}_\lambda + \sum_{\mu < \lambda} p_{\lambda,\mu} \tilde{u}_\mu \in \mathcal{H}_v(Q), \tag{4.0.1}$$

where $p_{\lambda,\mu} \in v^{-1}\mathbb{Z}[v^{-1}]$. All the elements $\mathfrak{b}_\lambda$ are invariant under the $\mathbb{Z}$-algebra involution

$$^{-} : U^+ \longrightarrow U^+; \quad E_i^{(m)} \longmapsto E_i^{(m)}, \quad v \longmapsto v^{-1}.$$

If Q is a Dynkin quiver, then $\mathbf{B}_Q = \{\mathfrak{b}_\lambda \mid \lambda \in \Lambda_Q\}$ is a $\mathcal{Z}$-basis of $\mathcal{H}_v(Q) = U^+$; if Q is a cyclic quiver, then $\mathbf{B}_Q = \{\mathfrak{b}_\pi \mid \pi \in \Lambda_Q^a\}$ is a $\mathcal{Z}$-basis of $\mathcal{C}_v(Q) = U^+$. In both cases, $\mathbf{B} = \mathbf{B}_Q$ is called the *canonical basis* of $\mathbf{U}_v^+(Q)$. Lusztig [13] shows that this basis has many remarkable properties.

Independently, Kashiwara [10,11] introduces the global crystal basis for a quantized enveloping algebra, which is shown in [14,8] to coincide with Lusztig's canonical basis. We follow [11, 2.2] to define the global crystal basis[3] of $\mathbf{U}^+$. For each $i \in I$, let $\tilde{E}_i : \mathbf{U}^+ \to \mathbf{U}^+$ be the Kashiwara operator. Let $\mathcal{A}_\infty$ be the subring of $\mathbb{Q}(v)$ consisting of all rational functions $f(v)$ such that $f(v^{-1})$ is regular at $v = 0$ and define $\mathcal{L}$ to be the $\mathcal{A}_\infty$-submodule of $\mathbf{U}^+$ generated by the elements

$$\tilde{E}_{i_1} \tilde{E}_{i_2} \cdots \tilde{E}_{i_m} \cdot 1 \tag{4.0.2}$$

for all words $w = i_1 i_2 \cdots i_m \in \Omega$, where 1 is the identity element of $\mathbf{U}^+$. By B we denote the subset of $\mathcal{L}/v^{-1}\mathcal{L}$ consisting of the images of all elements of the form (4.0.2) under the canonical projection $\mathcal{L} \to \mathcal{L}/v^{-1}\mathcal{L}$. Kashiwara shows the following facts:

(1) The set B is a $\mathbb{Q}$-basis of $\mathcal{L}/v^{-1}\mathcal{L}$, called the global crystal basis of $\mathbf{U}^+$.
(2) For each $b \in B$, there is a unique element $\tilde{b} \in U^+$ such that $\tilde{b} \in \mathcal{L} \cap \overline{\mathcal{L}}$ and that the image of $\tilde{b}$ under canonical map $\mathcal{L} \to \mathcal{L}/v^{-1}\mathcal{L}$ is b.
(3) The set $\mathbf{B}' := \{\tilde{b} \mid b \in B\}$ is a $\mathcal{Z}$-basis of U^+.

[3] Note that Kashiwara defines the crystal basis of $\mathbf{U}^-$ at $v = 0$. By using the isomorphism $\mathbf{U} \to \mathbf{U}$, $E_i \mapsto F_i$, $F_i \mapsto E_i$, $K_i \mapsto K_i$, $v \mapsto v^{-1}$, this can be reformulated as a basis of $\mathbf{U}^+$ at $v = \infty$.

Following [14, Theorem 2.3], we have $\mathbf{B} = \mathbf{B}'$. Thus, for each word $w = i_1 i_2 \dots i_m$, there is a unique $\kappa(w) = \lambda \in \Lambda_Q$ such that

$$\tilde{E}_{i_1} \tilde{E}_{i_2} \cdots \tilde{E}_{i_m} \cdot 1 \equiv \mathfrak{b}_\lambda \ (\mathrm{mod}\ v^{-1}\mathcal{L}).$$

This gives rise to a map

$$\kappa = \kappa_Q : \Omega \longrightarrow \Lambda_Q; \quad w \longmapsto \lambda = \kappa(w).$$

Clearly, if Q is a cyclic quiver, then $\mathrm{Im}\,\kappa = \Lambda_Q^a$.

Further, for each $i \in I$, there is a map

(4.0.3) $\quad \tau_i : \Lambda_Q \longrightarrow \Lambda_Q; \quad \lambda \longmapsto \tau_i \lambda$

define by

$$\tilde{E}_i(\mathfrak{b}_\lambda) \equiv \mathfrak{b}_{\tau_i \lambda} \ (\mathrm{mod}\ v^{-1}\mathcal{L}).$$

In case Q is a Dynkin quiver, by (4.0.1), this is equivalent to

$$\tilde{E}_i(\tilde{u}_\lambda) \equiv \tilde{u}_{\tau_i \lambda} \ (\mathrm{mod}\ v^{-1}\mathcal{L}).$$

Note that in this case all $\tilde{u}_\lambda$ lie in $\mathcal{L}$.

From the definition, for each $w = i_1 i_2 \dots i_m \in \Omega$, we have

(4.0.4) $\quad \kappa(w) = \tau_{i_1} \tau_{i_2} \cdots \tau_{i_m}(0).$

Now the *crystal graph* C (see [10,11]) is the directed graph with vertices $\lambda \in \Lambda_Q$ and arrows $\lambda \xrightarrow{i} \mu$, where $\lambda, \mu \in \Lambda_Q$ and $\tau_i \lambda = \mu$ for some $i \in I$. It is also an I-colored graph. Like the generic extension graph G, (4.0.4) can be used to identify the word set Ω with the set of all paths starting at 0.

It is interesting to compare the crystal graph C with the generic extension graph G. By definition, it suffices to compare the maps $\tau_i, \sigma_i : \Lambda_Q \to \Lambda_Q$. We will see shortly that τ_i and σ_i are not equal in general. However, by [14, Corollary 2.5] and the definition of σ_i, we have the following.

Proposition 4.1. *Let Q be a Dynkin quiver and i be a sink of Q, i.e., there is no arrow ρ with $t(\rho) = i$. Then $\tau_i = \sigma_i$ and, for each $\lambda \in \Lambda_Q$, $\tau_i \lambda = \sigma_i \lambda$ is defined by $M(\tau_i \lambda) \cong S_i \oplus M(\lambda)$.*

Based on this fact, Reineke [16] obtains the following characterization of each τ_i if Q is a Dynkin quiver. Given a Laurent polynomial $f(v) \in \mathbb{Z}[v, v^{-1}]$, the degree of $f(v)$ is defined to be the smallest integer d such that $v^{-d} f(v) \in \mathbb{Z}[v^{-1}]$. For $i \in I$ and $\lambda \in \Lambda_Q$, define

$$a_i(\lambda) := \max_\mu \deg f_{i,\lambda;\mu},$$

where $\tilde{u}_i \tilde{u}_\lambda = \sum_\mu f_{i,\lambda;\mu} \tilde{u}_\mu$.

The following result is a combination of [16, 3.2] and [16, 6.1, 7.1], from which the cyclic quiver case also follows (see the discussion in [12, 4.4] or the comments after Proposition 7.2).

Theorem 4.2. *Let Q be a Dynkin quiver of type other than E_8 or a cyclic quiver. Then, for $i \in I$ and $\lambda, \mu \in \Lambda_Q$, $\tilde{E}_i(\tilde{u}_\lambda) \equiv \tilde{u}_\mu \pmod{v^{-1}\mathcal{L}}$ (i.e., $\tau_i \lambda = \mu$) if and only if $f_{i,\lambda\mu} \neq 0$ and $\deg f_{i,\lambda;\mu} = a_i(\lambda) \geqslant a_i(\mu) - 1$.*

The theorem together with the fact that $f_{i,\lambda;\mu} \neq 0$ implies $\mu \leqslant \sigma_i(\lambda)$, i.e., $M(\mu) \leqslant S_i * M(\lambda) \cong M(\sigma_i \lambda)$ gives the following nice relation.

Corollary 4.3. *Let Q be a Dynkin quiver of type other than E_8 or a cyclic quiver. Then for each $w \in \Omega$, we have*

$$\kappa_Q(w) \leqslant \wp_Q(w).$$

Proof. Let $w = i_1 i_2 \dots i_m \in \Omega$. If $m = 0$ or 1, then $\wp(w) = \kappa(w)$. Let $m \geqslant 1$ and set $w_1 = i_2 \dots i_m$, $\lambda = \kappa(w_1)$ and $\mu = \kappa(w)$. By Theorem 4.2, we have

$$\kappa(w) = \tau_{i_1} \kappa(w_1) \leqslant \sigma_{i_1} \kappa(w_1).$$

Using an inductive argument, we may suppose that $\kappa(w_1) \leqslant \wp(w_1)$. This together with [17, Proposition 2] and [2, Proposition 3.4] implies

$$\kappa(w) \leqslant \sigma_{i_1} \kappa(w_1) \leqslant \sigma_{i_1} \wp(w_1) = \wp(w),$$

as required. $\square$

However, the two maps κ_Q and $\wp_Q$ are not equal in general as seen from the following example.

Example 4.4. Let Q be the quiver $Q = L_2 : \bullet \longrightarrow \bullet$. Let α_1 and α_2 denote the simple roots. Then $\Phi^+(Q) = \{\alpha_1, \alpha_1 + \alpha_2, \alpha_2\}$. Hence, each $\lambda \in \Lambda_Q$ can be identified with a triple (a, b, c) of non-negative integers, where $a = \lambda(\alpha_1)$, $b = \lambda(\alpha_1 + \alpha_2)$, and $c = \lambda(\alpha_2)$. For each $\lambda = (a, b, c) \in \Lambda_Q$, we have by [3, 7.1]

$$\sigma_1(a, b, c) = \begin{cases} (a+1, b, c) & \text{if } c = 0, \\ (a, b+1, c-1) & \text{if } c \geqslant 1 \end{cases}$$

and by [16, 3.2]

$$\tau_1(a, b, c) = \begin{cases} (a+1, b, c) & \text{if } a \geqslant c, \\ (a, b+1, c-1) & \text{if } a < c. \end{cases}$$

Thus, $\sigma_1 \neq \tau_1$. On the other hand, we have by Proposition 4.1 $\sigma_2 = \tau_2$. This shows that the edge sets of the generic extension graph and the crystal graph are different.

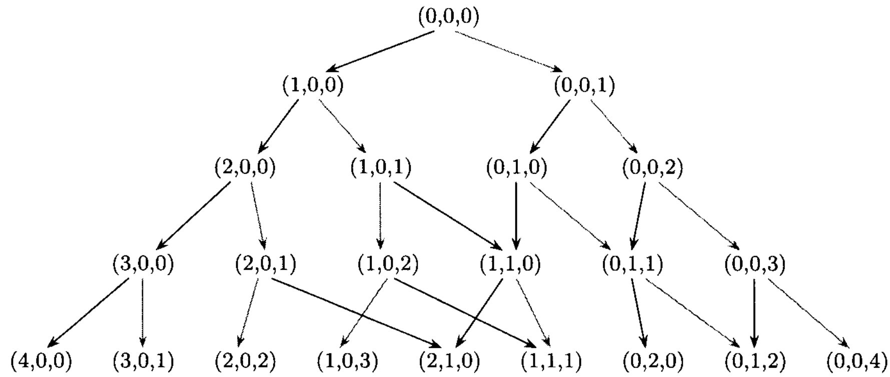

Figure 2. The graph $\mathsf{G}(A_2)$.

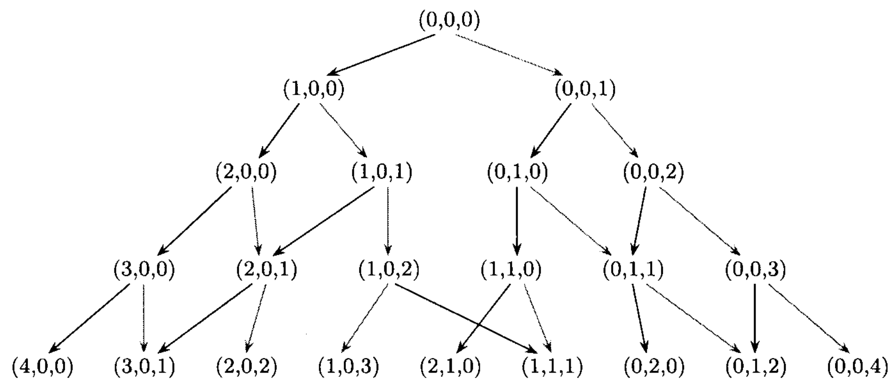

Figure 3. The graph $\mathsf{C}(A_2)$.

The generic extension graph G of Q consisting of all λ with $\dim M(\lambda) \leqslant 4$ is illustrated in Figure 2. Whereas, the crystal graph C of Q is given in Figure 3.

Here, in both cases, the color 1 is represented by a dark arrow, while the color 2 is represented by a gray arrow. Note that the two graphs have the same gray arrows since $\sigma_2 = \tau_2$.

For $\lambda, \mu \in \Lambda_Q$, we set

$$h_{\lambda,\mu} = \dim \mathrm{Hom}_{kQ}\big(M(\lambda), M(\mu)\big) \quad \text{and} \quad d_\lambda = \dim M(\lambda).$$

In particular, we set $h_\lambda = h_{\lambda,\lambda} = \dim \mathrm{End}_{kQ}(M(\lambda))$. We need the following result [16, Section 5] in the next section.

Proposition 4.5. *Let Q be a Dynkin quiver. Let $\lambda, \mu \in \Lambda_Q$ and $i \in I$ be such that $M(\lambda)$ is an extension of S_i by $M(\mu)$. Then*

$$\deg \varphi^{\lambda}_{i,\mu}(T) = h_{\mu,\lambda} - h_\mu.$$

In this section, we assume that Q is the finite linear quiver L_n given in Figure 1. Then the set $\Phi^+(Q)$ of positive roots can be identified with $\{(i,j) \mid 1 \leqslant i \leqslant j \leqslant n\}$. Let $M_{i,j}$ be the indecomposable representation of Q associated to the pair (i,j). Its top and socle are isomorphic to S_i and S_j, respectively. For each $\lambda \in \Lambda_Q$, we set $\lambda_{i,j} = \lambda(i,j)$ for all $1 \leqslant i \leqslant j \leqslant n$.

Lemma 5.1. *Let $i < j$ be in $I = \{1, 2, \dots, n\}$ and $\lambda \in \Lambda_Q$ be such that*

$$M(\lambda) \cong \bigoplus_{s=j}^{n} \lambda_{i,s} M_{i,s} \oplus \lambda_{i+1,j} M_{i+1,j} \oplus N,$$

where N satisfies that its top contains no summands isomorphic to S_i or S_{i+1}. Then

$$a_i(\lambda) = \sum_{s=j}^{n} \lambda_{i,s} \quad \text{and} \quad \sigma_i \lambda = \tau_i \lambda.$$

Moreover, if $\lambda_{i+1,j} = 0$, then $\sigma_i\lambda = \tau_i\lambda = \nu$ is defined by $M(\nu) \cong S_i \oplus M(\lambda)$; if $\lambda_{i+1,j} > 0$, then $\sigma_i\lambda = \tau_i\lambda = \mu$ is defined by

$$\mu_{s,t} = \begin{cases} \lambda_{s,t} + 1 & \text{if } (s,t) = (i,j), \\ \lambda_{s,t} - 1 & \text{if } (s,t) = (i+1,j), \\ \lambda_{s,t} & \text{otherwise}. \end{cases}$$

Proof. Let λ be given in the lemma. Since $\mathrm{Ext}^1_{kQ}(S_i, \bigoplus_{s=j}^{n} \lambda_{i,s} M_{i,s}) = 0$ and $\mathrm{Ext}^1_{kQ}(S_i, N) = 0$, we have

$$S_i * M(\lambda) = \left(\bigoplus_{s=j}^{n} \lambda_{i,s} M_{i,s} \oplus N \right) \oplus (S_i * \lambda_{i+1,j} M_{i+1,j}).$$

It is easy to see that

$$S_i * (\lambda_{i+1,j} M_{i+1,j}) \cong \begin{cases} S_i & \text{if } \lambda_{i+1,j} = 0, \\ M_{i,j} \oplus (\lambda_{i+1,j} - 1) M_{i+1,j} & \text{if } \lambda_{i+1,j} > 0. \end{cases}$$

Hence, $S_i * M(\lambda) \cong M(\nu)$ (resp. $M(\mu)$) if $\lambda_{i+1,j} = 0$ (resp. > 0), that is, $\sigma_i\lambda = \nu$ (resp. μ).

If $\lambda_{i+1,j} = 0$, then $\mathrm{Ext}^1(S_i, M(\lambda)) = 0$, that is, $M(\nu)$ is the unique extension of $M(\lambda)$ by S_i, and so

$$\tilde{u}_i \tilde{u}_\lambda = \sum_{\mu} f_{i,\lambda;\mu} \tilde{u}_\mu = f_{i,\lambda;\nu} \tilde{u}_\nu.$$

On the other hand, by definition, we have

$$\tilde{u}_i \tilde{u}_\lambda = v^{h_\lambda - d_\lambda} u_i u_\lambda = v^{h_\lambda - d_\lambda + \langle i,\lambda \rangle} \varphi^{\nu}_{i,\lambda}(v^2) u_\nu.$$

Thus, $f_{i,\lambda;\nu} = v^{h_\lambda - d_\lambda + \langle i,\lambda\rangle - h_\nu + d_\nu}\varphi^{\nu}_{i,\lambda}(v^2)$. By Proposition 4.5, we get

$$\begin{aligned} a_i(\lambda) &= \deg f_{i,\lambda;\nu} = h_\lambda - d_\lambda + \langle i,\lambda\rangle - h_\nu + d_\nu + 2(h_{\lambda,\nu} - h_\lambda) \\ &= h_{\lambda,i} = \dim \mathrm{Hom}_{kQ}\big(M(\lambda), S_i\big) = \sum_{s=j}^{n} \lambda_{i,s}. \end{aligned}$$

Replacing λ by ν in the above yields $a_i(\nu) = \sum_{s=j}^{n} \lambda_{i,s} + 1 = a_i(\lambda) + 1$. Hence, by Theorem 4.2, we get $\tilde{E}_i(\tilde{u}_\lambda) \equiv \tilde{u}_\nu \pmod{v^{-1}\mathcal{L}}$, i.e., $\tau_i\lambda = \nu$.

Suppose now $\lambda_{i+1,j} > 0$. Again, by definition, we have

$$\begin{aligned} \tilde{u}_i\tilde{u}_\lambda = v^{h_\lambda - d_\lambda} u_i u_\lambda &= v^{h_\lambda - d_\lambda + \langle i,\lambda\rangle}\big(\varphi^{\mu}_{i,\lambda}(v^2)u_\mu + \varphi^{\nu}_{i,\lambda}(v^2)u_\nu\big) \\ &= f_{i,\lambda;\mu}\tilde{u}_\mu + f_{i,\lambda;\nu}\tilde{u}_\nu, \end{aligned}$$

where

$$f_{i,\lambda;\mu} = v^{h_\lambda - d_\lambda + \langle i,\lambda\rangle - h_\mu + d_\mu}\varphi^{\mu}_{i,\lambda}\big(v^2\big) \quad \text{and} \quad f_{i,\lambda;\nu} = v^{h_\lambda - d_\lambda + \langle i,\lambda\rangle - h_\nu + d_\nu}\varphi^{\nu}_{i,\lambda}\big(v^2\big).$$

Then, by Proposition 4.5, we get

$$\begin{aligned} &\deg f_{i,\lambda;\mu} = 2h_{\lambda,\mu} - h_\lambda - h_\mu + 1 + \langle i,\lambda\rangle \quad \text{and} \\ &\deg f_{i,\lambda;\nu} = 2h_{\lambda,\nu} - h_\lambda - h_\nu + 1 + \langle i,\lambda\rangle. \end{aligned}$$

Set

$$M = \bigoplus_{s=j}^{n} \lambda_{i,s} M_{i,s} \oplus (\lambda_{i+1,j} - 1)M_{i+1,j} \oplus N.$$

Then

$$M(\lambda) = M \oplus M_{i+1,j} \quad \text{and} \quad M(\mu) = M \oplus M_{i,j}.$$

Hence, we get

$$\begin{aligned} \deg f_{i,\lambda;\mu} = {}& \dim \mathrm{Hom}_{kQ}(M, M_{ij}) - \dim \mathrm{Hom}_{kQ}(M, M_{i+1,j}) - \dim \mathrm{Hom}_{kQ}(M_{ij}, M) \\ &+ \dim \mathrm{Hom}_{kQ}(M_{i+1,j}, M) + \big\langle \mathbf{dim} S_i, \mathbf{dim} M(\lambda)\big\rangle + 1. \end{aligned}$$

An easy calculation shows that

$$\begin{aligned} &\dim \mathrm{Hom}_{kQ}(M, M_{ij}) = \dim \mathrm{Hom}_{kQ}(M, M_{i+1,j}) + \sum_{s=j}^{n} \lambda_{i,s}, \\ &\dim \mathrm{Hom}_{kQ}(M_{ij}, M) - \dim \mathrm{Hom}_{kQ}(S_i, M) \\ &\quad = \dim \mathrm{Hom}_{kQ}(M_{i+1,j}, M) - \lambda_{i+1,j} + 1, \\ &\big\langle \mathbf{dim} S_i, \mathbf{dim} M(\lambda)\big\rangle = \dim \mathrm{Hom}_{kQ}(S_i, M) - \lambda_{i+1,j}. \end{aligned}$$

This gives $\deg f_{i,\lambda;\mu} = \sum_{s=j}^{n} \lambda_{i,s}$. Similarly, we have $\deg f_{i,\lambda;\nu} = \sum_{s=j}^{n} \lambda_{i,s} - \lambda_{i+1,j}$. Thus, $a_i(\lambda) = \sum_{s=j}^{n} \lambda_{i,s}$.

Repeating all the arguments above for μ replaced by λ, we obtain that

$$a_i(\mu) = \sum_{s=j}^{n} \lambda_{i,s} + 1 = a_i(\lambda) + 1.$$

This together with Theorem 4.2 implies $\tau_i \lambda = \mu$.

We conclude that $\sigma_i \lambda = \tau_i \lambda$ in both cases. $\square$

Remark 5.2. Although one may use the explicit formula $\tau_i \lambda$ (for the linear quiver case only) given in [17, Section 1] to prove the lemma, we provided a proof which directly uses Theorem 4.2.

Let $\lambda \in \Lambda_Q$ and define for $1 \leqslant i \leqslant j \leqslant n$

$$w_{i,j} = w_{i,j}(\lambda) = \underbrace{i \dots i}_{\lambda_{ij}} \underbrace{i+1 \dots i+1}_{\lambda_{ij}} \dots \underbrace{j \dots j}_{\lambda_{ij}},$$

and

$$w = w(\lambda) = w_{n,n} w_{n-1,n-1} w_{n-1,n} \dots w_{1,1} w_{1,2} \dots w_{1,n}.$$

We now show that the paths corresponding to w in the generic extension graph and the crystal graph have the same endpoints.

Theorem 5.3. *Let $\lambda \in \Lambda_Q$ and $w = w(\lambda)$ be defined as above. Then*

$$\wp(w) = \lambda = \kappa(w).$$

In particular, $\wp^{-1}(\lambda) \cap \kappa^{-1}(\lambda) \neq \emptyset$.

Proof. Since $\mathrm{Ext}^1_{kQ}(M_{i,j}, M_{i,j}) = 0$, we get

$$M\big(\wp(w_{i,j})\big) \cong \lambda_{i,j} M_{i,j}.$$

Further, from the fact $\mathrm{Ext}^1_{kQ}(M_{i,j}, M_{s,t}) = 0$ for $i \geqslant s$ it follows

$$\begin{aligned} M\big(\wp(w)\big) &\cong \lambda_{n,n} M_{n,n} \oplus \lambda_{n-1,n-1} M_{n-1,n-1} \oplus \lambda_{n-1,n} M_{n-1,n} \oplus \cdots \\ &\quad \oplus \lambda_{1,1} M_{1,1} \oplus \cdots \oplus \lambda_{1,n} M_{1,n} \cong M(\lambda), \end{aligned}$$

i.e., $\wp(w) = \lambda$.

For $1 \leqslant i \leqslant j \leqslant n$, we define $\lambda^{(i,j)} \in \Lambda_Q$ by

$$\lambda^{(i,j)}_{s,t} = \begin{cases} \lambda_{s,t} & \text{if } s < i \text{ or } s = i, t \geqslant j, \\ 0 & \text{otherwise.} \end{cases}$$

We use induction on the length $l(\lambda) := \sum_{i,j} \lambda_{ij}$ to show $\kappa(w) = \lambda$. If $l(\lambda) = 0$ or 1, it is clear. Let now $l(\lambda) \geqslant 1$. Then there are $1 \leqslant i \leqslant j \leqslant n$ such that $\lambda_{i,j} \neq 0$, but $\lambda_{s,t} = 0$ for all $s > i$ or $s = i, t < j$, hence, $\lambda = \lambda^{(i,j)}$. For simplicity, we set $m = \lambda_{i,j}$ and $\pi = \lambda^{(i,j+1)}$. Then $l(\pi) < l(\lambda)$ and

$$w = w(\lambda) = \underbrace{i \dots i}_{m} \dots \underbrace{j \dots j}_{m}\, w(\pi).$$

By induction hypothesis, we may suppose $\kappa(w(\pi)) = \pi$. Then

$$\kappa(w) = \tau_i^m \cdots \tau_{j-1}^m \tau_j^m \pi.$$

Since

$$M(\pi) = \bigoplus_{s=j+1}^{n} \lambda_{i,s} M_{i,s} \oplus \bigoplus_{\substack{s<i \\ s \leqslant t \leqslant n}} \lambda_{s,t} M_{s,t},$$

we have by repeatedly applying Lemma 5.1 that $M(\tau_j^m \pi) = mS_j \oplus M(\pi)$. By further applying Lemma 5.1 to $\tau_j^m \pi$, we get $M(\tau_{j-1}^m \tau_j^m \pi) = mM_{j-1,j} \oplus M(\pi)$. Inductively, we finally deduce

$$M\big(\tau_i^m \cdots \tau_{j-1}^m \tau_j^m \pi\big) \cong mM_{i,j} \oplus M(\pi) \cong M(\lambda),$$

that is, $\kappa(w) = \lambda$.

Example 5.4. We use Figure 2 and Figure 3 to compute the intersection $\wp^{-1}(\lambda) \cap \kappa^{-1}(\lambda)$ for all λ with $\dim M(\lambda) = 4$ in the following table:

λ	$\wp^{-1}(\lambda) \cap \kappa^{-1}(\lambda)$	λ	$\wp^{-1}(\lambda) \cap \kappa^{-1}(\lambda)$	λ	$\wp^{-1}(\lambda) \cap \kappa^{-1}(\lambda)$
(4,0,0)	1^4	(1,0,3)	$2^3 1$	(0,2,0)	$1212, 1^2 2^2$
(3,0,1)	21^3	(2,1,0)	$1^3 2$	(0,1,2)	$2^2 12, 212^2, 12^3$
(2,0,2)	$2^2 1^2$	(1,1,1)	$12^2 1, 21^2 2$	(0,0,4)	2^4

6. THE INFINITE LINEAR QUIVER CASE

The consideration for linear quivers L_n can be easily transferred to the infinite linear quiver L_∞. As in the finite case, for all $i, j \in \mathbb{Z}$ with $i \leqslant j$, there is an indecomposable representation $M_{i,j}$ of L_∞ with top S_i and socle S_j, where S_i and S_j are simple representations corresponding to vertices i and j, respectively. Thus, the set of isoclasses of (finite-dimensional) indecomposable representations of L_∞ is identified with

$$\Phi_\infty^+ = \big\{(i, j) \mid i, j \in \mathbb{Z},\ i \leqslant j\big\},$$

and the isoclasses of representations of L_∞ are indexed by the set

$$\Lambda_\infty = \big\{\lambda : \Phi_\infty^+ \longrightarrow \mathbb{N} \mid \text{supp}(\lambda) \text{ is finite}\big\},$$

where $\text{supp}(\lambda) = \{(i, j) \in \Phi_\infty^+ \mid \lambda(i, j) \neq 0\}$.

The existence of Hall polynomials allows us to define the Ringel–Hall algebra $\mathcal{H}_v(L_\infty)$ of L_∞, which can be identified with the Lusztig integral form $U_v^+(\mathfrak{sl}_\infty)$ of the positive part $\mathbf{U}_v^+(\mathfrak{sl}_\infty)$ of the quantized enveloping algebra of $\mathfrak{sl}_\infty$ (see [12]).

As in the L_n case, for each $i \in \mathbb{Z}$, the generic extension of S_i by a representation and the Kashiwara operator $\tilde{E}_i^\infty$, respectively, induce maps

$$\sigma_i^\infty, \tau_i^\infty : \Lambda_\infty \longrightarrow \Lambda_\infty.$$

Let Ω_∞ denote the set of all words on the alphabet $I = \mathbb{Z}$. We further define maps

$$\wp^\infty, \kappa^\infty : \Omega_\infty \longrightarrow \Lambda_\infty$$

by respectively

$$\wp^\infty(w) = \sigma_{i_1}^\infty \sigma_{i_2}^\infty \cdots \sigma_{i_m}^\infty(0) \quad \text{and} \quad \kappa^\infty(w) = \tau_{i_1}^\infty \tau_{i_2}^\infty \cdots \tau_{i_m}^\infty(0),$$

where $w = i_1 i_2 \ldots i_m \in \Omega_\infty$. Theorem 5.3 implies the following

Proposition 6.1. *For each $\lambda \in \Lambda_\infty$, we have $\wp^\infty(w) = \kappa^\infty(w) = \lambda$ for some $w \in \Omega_\infty$.*

7. THE CYCLIC QUIVER CASE

In this section, we assume that Q is the cyclic quiver $\Delta = \Delta_n$ $(n \geqslant 2)$ with vertex set $I = \{1, 2, \ldots, n\}$. We first compare maps σ_i and τ_i $(1 \leqslant i \leqslant n)$ for Δ_n with those for L_∞.

For all $1 \leqslant i \leqslant n-1$, we define

$$\psi = \psi_i : \Lambda_\Delta \longrightarrow \Lambda_\infty; \quad \lambda = (\lambda_{i',l})_{i' \in I, l \in \mathbb{Z}_+} \longmapsto \psi(\lambda)$$

by

$$M\big(\psi(\lambda)\big) = \bigoplus_{s=1}^{n} \bigoplus_{j \geqslant 1} \lambda_{s,j} M_{s,s+j-1}.$$

In other words, for $(s, t) \in \Phi_\infty^+$, we have

$$\psi(\lambda)(s,t) = \begin{cases} \lambda_{s,j} & \text{if } j = t - s + 1, \\ 0 & \text{otherwise.} \end{cases}$$

Further, we define

$$\psi_n : \Lambda_\Delta \longrightarrow \Lambda_\infty; \quad \lambda \longmapsto \psi_n(\lambda)$$

by

$$M\big(\psi_n(\lambda)\big) = \bigoplus_{s=1}^{n-1} \bigoplus_{j \geqslant 1} \lambda_{s,j} M_{s+1,s+j} \oplus \bigoplus_{j \geqslant 1} \lambda_{n,j} M_{1,j}.$$

Proposition 7.1.

(1) *We have the following commutative squares*

$$\begin{array}{ccc} \Lambda_\Delta & \xrightarrow{\psi_i} & \Lambda_\infty \\ \sigma_i \downarrow & & \downarrow \sigma_i^\infty \\ \Lambda_\Delta & \xrightarrow[\psi_i]{} & \Lambda_\infty \end{array} \quad \textit{and} \quad \begin{array}{ccc} \Lambda_\Delta & \xrightarrow{\psi_n} & \Lambda_\infty \\ \sigma_n \downarrow & & \downarrow \sigma_1^\infty \\ \Lambda_\Delta & \xrightarrow[\psi_n]{} & \Lambda_\infty \end{array}$$

where $1 \leqslant i \leqslant n-1$.

(2) *For* $\mu, \lambda \in \Lambda_\Delta$ *and* $1 \leqslant i \leqslant n$, *we have*

$$\varphi^{\mu}_{i,\lambda}(T) = \varphi^{\psi_i(\mu)}_{\psi_i(i),\psi_i(\lambda)}(T).$$

Proof. (1) This follows from [2, Proposition 3.7] and [3, Proposition 7.3].

(2) If there exists an exact sequence

$$(7.1.1) \quad 0 \longrightarrow S_i \longrightarrow M(\mu) \longrightarrow M(\lambda) \longrightarrow 0,$$

then there is an $l \geqslant 1$ such that

$$\lambda_{s,t} = \begin{cases} \mu_{i,l} - 1 & \text{if } (s,t) = (i,l), \\ \mu_{i+1,l-1} + 1 & \text{if } (s,t) = (i+1, l-1), \\ \mu_{s,t} & \text{otherwise.} \end{cases}$$

Let

$$a = \sum_{j>l} \mu_{i,j} \quad (\text{resp. } b = \sum_{j=l} \mu_{i,j}).$$

An easy calculation shows (see the proof of [2, Proposition 9.1])

$$\varphi^{\mu}_{i,\lambda}(T) = \varphi^{\psi_i(\mu)}_{\psi_i(i),\psi_i(\lambda)}(T) = T^a\left(1 + T + \cdots + T^{b-1}\right).$$

If there is no exact sequence of the form (7.1.1), then $\varphi^{\mu}_{i,\lambda}(T) = 0 = \varphi^{\psi_i(\mu)}_{\psi_i(i),\psi_i(\lambda)}(T)$. □

The following result has been given in [12, 4.4].[4]

Proposition 7.2. *The following squares commute*

$$\begin{array}{ccc} \Lambda_\Delta & \xrightarrow{\psi_i} & \Lambda_\infty \\ \tau_i \downarrow & & \downarrow \tau_i^\infty \\ \Lambda_\Delta & \xrightarrow[\psi_i]{} & \Lambda_\infty \end{array} \quad (1 \leqslant i \leqslant n-1), \quad \begin{array}{ccc} \Lambda_\Delta & \xrightarrow{\psi_n} & \Lambda_\infty \\ \tau_n \downarrow & & \downarrow \tau_1^\infty \\ \Lambda_\Delta & \xrightarrow[\psi_n]{} & \Lambda_\infty \end{array}$$

[4] It was not pointed out in [12, 4.4] that the case for $i = n$ is slightly different from the cases for $i = 1, 2, \ldots, n-1$. We separate this through the second commutative diagram.

We remark that Theorem 4.2 for the cyclic quiver case can be easily obtained form 7.1(2), 7.2 and the Dynkin case. Moreover, based on Proposition 7.2, Leclerc et al. [12, Theorem 4.1] have explicitly worked out $\tau_i\lambda$ which we now describe.

For each $\lambda \in \Lambda_\Delta$, let

$$s_{i,m} = \sum_{l \geqslant m} (\lambda_{i,l} - \lambda_{i+1,l}),$$

where $\lambda_{i+1,l} = \lambda_{1,l}$ in case $i = n$. Let m_0 be the minimal positive integer such that

$$s_{i,m_0} = \max\{s_{i,m} \mid m \geqslant 1\}.$$

By definition, if $m_0 > 1$, then $\lambda_{i+1,m_0-1} \geqslant 1$. In this case, we define $\mu \in \Lambda_\Delta$ by

$$\mu_{s,t} = \begin{cases} \lambda_{i,m_0} + 1 & \text{if } (s,t) = (i, m_0), \\ \lambda_{i+1,m_0-1} - 1 & \text{if } (s,t) = (i+1, m_0 - 1), \\ \lambda_{s,t} & \text{otherwise.} \end{cases}$$

In other words, $M(\mu)$ is obtained from $M(\lambda)$ by replacing a summand $S_{i+1}[m_0 - 1]$ with a summand $S_i[m_0]$. If $m_0 = 1$, μ is defined by $M(\mu) \cong S_i \oplus M(\lambda)$.

In contrast, for the given i and λ, if $\lambda_{i+1,j} = 0$ for all $j \geqslant 1$, we define $\nu \in \Lambda_\Delta$ by $M(\nu) \cong S_i \oplus M(\lambda)$. Otherwise, let m_1 be the maximal number j such that $\lambda_{i+1,j} \neq 0$ and define ν by

$$\nu_{s,t} = \begin{cases} \lambda_{i,m_1+1} + 1 & \text{if } (s,t) = (i, m_1 + 1), \\ \lambda_{i+1,m_1} - 1 & \text{if } (s,t) = (i+1, m_1), \\ \lambda_{s,t} & \text{otherwise.} \end{cases}$$

Theorem 7.3. *Let $\lambda \in \Lambda_\Delta$, $i \in I$ and let μ, ν be defined as above. Then*

(1) ([12, Thm 4.1]) $\tau_i\lambda = \mu$;
(2) ([2, 3.7]) $\sigma_i\lambda = \nu$.

We now apply the theorem to show that there exists $\lambda \in \Lambda_\Delta$ satisfying $\wp^{-1}(\lambda) \cap \kappa^{-1}(\lambda) = \emptyset$.

Example 7.4. Let $n = 3$ and Q be the cyclic quiver Δ_3. Let $\lambda \in \Lambda_\Delta$ be such that

$$M(\lambda) = S_1 \oplus S_1[2] \oplus S_1[3] \oplus S_2[2] \oplus S_3[3].$$

Let N be a maximal submodule with $M/N \cong S_i$ for $1 \leqslant i \leqslant 3$. If $i = 1$, then

$$N \cong \begin{cases} S_1 \oplus S_1[2] \oplus S_2[2] \oplus S_2[2] \oplus S_3[3] =: N_1 & \text{or} \\ S_1 \oplus S_1[3] \oplus S_2 \oplus S_2[2] \oplus S_3[3] =: N_2 & \text{or} \\ S_1[2] \oplus S_1[3] \oplus S_2[2] \oplus S_3[3] =: N_3. \end{cases}$$

By Theorem 7.3(2), we have

$$
\begin{aligned}
S_1 * N_1 &\cong S_1 \oplus S_1[2] \oplus S_1[3] \oplus S_2[2] \oplus S_3[3] = M(\lambda),\\
S_1 * N_2 &\cong S_1 \oplus S_1[3] \oplus S_1[3] \oplus S_2 \oplus S_3[3],\\
S_1 * N_3 &\cong S_1[2] \oplus S_1[3] \oplus S_1[3] \oplus S_3[3].
\end{aligned}
$$

In a similar way, for $i = 2$ or 3, we can show $S_i * N \ncong M(\lambda)$. Let $\mu \in \Lambda$ satisfy $M(\mu) \cong N_1$. Then $\sigma_1(\mu) = \lambda$ and

$$\wp^{-1}(\lambda) = \{1w \mid w \in \wp^{-1}(\mu)\}.$$

Repeating the arguments above, we finally obtain

$$\wp^{-1}(\lambda) = \{y = 132^2 1^3 3^2 2^2\}.$$

But, by Theorem 7.3(1), we have

$$M(\kappa(y)) \cong S_1 \oplus S_1 \oplus S_1[2] \oplus S_2[2] \oplus S_2[2] \oplus S_3[3] \ncong M(\lambda).$$

This means that $\wp^{-1}(\lambda) \cap \kappa^{-1}(\lambda) = \emptyset$. In fact, by applying Theorem 7.3(1) repeatedly, we get

$$
\begin{aligned}
\kappa^{-1}(\lambda) = \{&1^2 231213232,\ 1^2 231213^2 2^2,\ 1^2 231231232,\ 1^2 23123132^2,\\
&1^2 2321^2 3232,\ 1^2 2321^2 3^2 2^2,\ 1^2 232131232,\ 1^2 23213132^2,\\
&1^2 321213232,\ 1^2 321213^2 2^2,\ 1^2 321231232,\ 1^2 32123132^2\}.
\end{aligned}
$$

Remark 7.5. Although Theorem 5.3 fails for cyclic quivers, it is natural to expect that the same result holds for all Dynkin quivers. Furthermore, with the techniques of Frobenius morphisms on quiver representations developed in [4], the truth for the simply-laced case would imply that for the non-simply laced case.

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REFERENCES

[1] Bongartz K. – On degenerations and extensions of finite dimensional modules, Adv. Math. **121** (1996) 245–287.

[2] Deng B., Du J. – Monomial bases for quantum affine $\mathfrak{sl}_n$, Adv. Math. **191** (2005) 276–304.

[5] A few typos in the statement of Problem 2 should be corrected. For example, $\tilde{f}_y(1) = \tilde{f}_w(1)$ really meant that their images are equal.

[3] Deng B., Du J. – Bases of quantized enveloping algebras, Pacific J. Math. **202** (2005) 33–48.
[4] Deng B., Du J. – Frobenius morphisms and representations of algebras, Trans. Amer. Math. Soc. **358** (2006) 3591–3622.
[5] Deng B., Du J., Xiao J. – Generic extensions and canonical bases for cyclic quivers, Preprint, 2004.
[6] Du J. – Finite dimensional algebras, quantum groups and finite groups of Lie type, in: Fields Inst. Commun., vol. 40, Amer. Math. Soc., Providence, RI, 2004, pp. 225–244.
[7] Gabriel P. – Unzerlegbare Darstellungen I, Manuscripta Math. **6** (1972) 71–103.
[8] Grojnowski I., Lusztig G. – A comparison of bases of quantized enveloping algebras, Contemp. Math. **153** (1993) 11–19.
[9] Guo J.Y. – The Hall polynomials of a cyclic serial algebra, Comm. Algebra **23** (1995) 743–751.
[10] Kashiwara M. – Crystalizing the q-analogue of universal enveloping algebras, Comm. Math. Phys. **133** (1990) 249–260.
[11] Kashiwara M. – On crystal bases of the q-analogue of universal enveloping algebras, Duke Math. J. **63** (1991) 465–516.
[12] Leclerc B., Thibon J.-Y., Vasserot E. – Zelevinsky's involution at roots of unity, J. Reine Angew. Math. **513** (1999) 33–51.
[13] Lusztig G. – Canonical bases arising from quantized enveloping algebras, J. Amer. Math. Soc. **3** (1990) 447–498.
[14] Lusztig G. – Quivers, perverse sheaves, and the quantized enveloping algebras, J. Amer. Math. Soc. **4** (1991) 366–421.
[15] Lusztig G. – Affine quivers and canonical bases, Inst. Hautes Études Sci. Publ. Math. **76** (1992) 111–163.
[16] Reineke M. – On the coloured graph structure of Lusztig's canonical basis, Math. Ann. **307** (1997) 705–723.
[17] Reineke M. – Generic extensions and multiplicative bases of quantum groups at $q = 0$, Represent. Theory **5** (2001) 147–163.
[18] Reineke M. – Feigin's map and monomial bases for quantized enveloping algebras, Math. Z. **237** (2001) 639–667.
[19] Ringel C.M. – Hall algebras, in: Topics in Algebra, Banach Center Publications, vol. 66 (1990), pp. 433–447.
[20] Ringel C.M. – The composition algebra of a cyclic quiver, Proc. London Math. Soc. **66** (1993) 507–537.
[21] Ringel C.M. – Hall algebras revisited, Israel Math. Conference Proceedings **7** (1993) 171–176.
[22] Ringel C.M. – The Hall algebra approach to quantum groups, Aportaciones Mat. Comun. **15** (1995) 85–114.

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