J. Phys. A: Math. Theor. **40** (2007) 13121–13130 doi:10.1088/1751-8113/40/43/019

Multiparty-controlled teleportation of an arbitrary *m*-qudit state with a pure entangled quantum channel

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Received 22 May 2007, in final form 7 September 2007
Published 9 October 2007
Online at stacks.iop.org/JPhysA/40/13121

Abstract

We present a general scheme for multiparty-controlled teleportation of an arbitrary m-qudit (d-dimensional quantum system) state by using non-maximally entangled states as the quantum channel. The sender performs m generalized Bell-state measurements on her $2m$ particles, the controllers take some single-particle measurements with the measuring basis X_d and the receiver only needs to introduce one auxiliary two-level particle to extract quantum information probabilistically with the fidelity unit if he cooperates with all the controllers. All the parties can use some decoy photons to set up their quantum channel securely, which will forbid a dishonest party to eavesdrop freely. This scheme is optimal as the probability that the receiver obtains the originally unknown m-qudit state equals the entanglement of the quantum channel.

PACS numbers: 03.67.Hk, 03.65.Ud, 42.50.Dv

1. Introduction

The principle of quantum mechanics provides some novel ways for quantum information transmission, such as quantum key distribution [1–4], quantum secret sharing [5–9], quantum secure direct communication [10, 11], deterministic secure quantum communication [12, 13], quantum secret report [14], quantum secret conference [15], quantum dialogue [16], quantum teleportation [17], and so on. Quantum teleportation, a unique thing in quantum mechanics, provides a way for two parties to teleport an unknown quantum state $|\chi\rangle = a|0\rangle + b|1\rangle$, exploiting the nonlocal correlation of an Einstein–Podolsky–Rosen (EPR) state shared in advance. For this task, the sender performs a Bell-state measurement on the unknown quantum system χ and one of the EPR particles, and the receiver takes a unitary operation on the remaining EPR particle, according to the information of the Bell-state measurement. Since Bennet *et al* [17] first discovered that the information of an unknown qubit $|\chi\rangle$ can be disassembled into some pieces and then reconstructed with classical information and quantum correlations, researchers have devoted much interest to quantum teleportation. On one hand, several experiments have demonstrated the teleportation of a single qubit with entangled photons and ions [18–23]. On the other hand, a great number of theoretical schemes for teleporting an unknown state, especially an N-particle entangled state, have been proposed with different quantum channels [24–37].

Recently, the controlled teleportation for the single-qubit or the m-qubit state has been studied by some groups. The basic idea of a controlled teleportation scheme [38–41] is to let an unknown quantum state be recovered by a remote receiver only when he cooperates with the controllers. It is similar to another branch of quantum communication, quantum state sharing (QSTS) [42–48], whose task is to let several receivers share an unknown quantum state with cooperations. Essentially one receiver can reconstruct the originally unknown state with the help of others. In principle, almost all the QSTS schemes [42–48] can be used for controlled teleportation with or without a little modification, and vice versa [40, 42, 44]. In 1999, Karlsson and Bourennane proposed the first controlled teleportation protocol with a three-qubit Greenberger–Horne–Zeilinger (GHZ) state for teleporting a single-qubit state [38]. In 2004, Yang *et al* [39] presented a multiparty controlled teleportation protocol to teleport multi-qubit quantum information. In 2005, Deng *et al* [40] introduced a symmetric multiparty controlled teleportation scheme for an arbitrary two-particle entanglement state. Moreover, they presented another scheme for sharing an arbitrary two-particle state with EPR pairs and GHZ-state measurements [44] or Bell-state measurements [45]. Both those two QSTS schemes [40, 44] can be used for controlled teleportation directly without any modification. Also, Zhang, Jin and Zhang [46] presented a scheme for sharing an arbitrary two-particle state based on entanglement swapping. Zhang *et al* [47] proposed a multiparty QSTS scheme for sharing an unknown single-qubit state with photon pairs and a controlled teleportation scheme by using quantum secret sharing of classical message for teleporting arbitrary m-qubit quantum information. Recently, Li *et al* [42] have proposed an efficient symmetric multiparty QSTS protocol for sharing an arbitrary m-qubit state. All those three QSTS schemes, in principle, are equivalent to a secure scheme for teleportation with some controllers.

Although there are some schemes for controlled teleportation [38–41] or QSTS [42–47], all of them are based on a maximally entangled quantum channel, not a pure entangled one. A practical quantum signal source often produces a pure entangled state because of its unsymmetry to some extent. In this work, we will give a general form for controlled teleportation of an arbitrary m-qudit (d-level quantum system) state via the control of n controllers by using d-dimensional pure entangled states as the quantum channel, following some ideas in [42]. Except for the sender Alice, each of the controllers needs only to take m single-particle measurements on his particles, and the receiver can probabilistically reconstruct the unknown m-qudit state with an auxiliary qubit (two-level particle) and m unitary operations if she cooperates with all the controllers. This scheme for controlled teleportation of the m-qudit state is optimal as the probability that the receiver obtains the originally unknown m-qudit state with the fidelity unit equals the entanglement of the quantum channel.

2. Controlled teleportation of an arbitrary single-particle qudit with a pure entangled quantum channel

The generalized Bell states (GBS) of d-dimensional quantum systems (the analogue of the Bell state for spin-$\frac{1}{2}$ particles) are [17]

$$|\psi_{rs}\rangle = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} \mathrm{e}^{\frac{2\pi\mathrm{i}}{d} jr} |j\rangle |j \oplus s\rangle \tag{1}$$

where $r, s = 0, 1, \ldots, d-1$, are used to label the d^2 orthogonal GBS. $|0\rangle, |1\rangle, \ldots$, and $|d-1\rangle$ are the d eigenvectors of the measuring basis (MB) Z_d, and $j \oplus s$ means $j + s$ mod d. The d^2 unitary operations $U_{uv}(u, v = 0, 1, \ldots, d-1)$ can transfer one of the Bell states into each other:

$$U_{uv} = \sum_{j=0}^{d-1} \mathrm{e}^{\frac{2\pi\mathrm{i}}{d} uj} |j \oplus v\rangle\langle j|. \tag{2}$$

Another unbiased basis X_d which has d eigenvectors can be written as $\{|0\rangle_x, \ldots, |r\rangle_x, \ldots, |d-1\rangle_x\}$ [13, 49]:

$$|r\rangle_x = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} \mathrm{e}^{\frac{2\pi\mathrm{i}}{d} jr} |j\rangle \tag{3}$$

where $r \in \{0, 1, \ldots, d-1\}$. The two unbiased bases have the relation $|\langle k|r\rangle_x|^2 = \frac{1}{d}$. Here $|k\rangle$ is an eigenvector of the MB Z_d and $|r\rangle_x$ is an eigenvector of the MB X_d.

Now, let us describe the principle of our controlled teleportation of an arbitrary m-qudit state with m pure entangled states. For presenting the principle of our scheme clearly, we first consider the case to teleport an unknown single-particle qudit state and then generalize it to the case with an arbitrary m-particle qudit state.

Suppose the originally unknown single-particle qudit state teleported is

$$|\chi\rangle_{\chi_0} = \beta_0|0\rangle + \beta_1|1\rangle + \cdots + \beta_{d-1}|d-1\rangle, \tag{4}$$

where

$$|\beta_0|^2 + |\beta_1|^2 + \cdots + |\beta_{d-1}|^2 = 1. \tag{5}$$

The pure entangled $(n+2)$-particle state used for setting up the quantum channel is

$$|\Phi\rangle_{a_0a_1\cdots a_{n+1}} = c_0 \prod_{k=0}^{n+1} |0\rangle_{a_k} + \cdots + \prod_{k'=0}^{n+1} c_{d-1}|d-1\rangle_{a_{k'}}, \tag{6}$$

where a_k $(k = 0, 1, \ldots, n+1)$ are the $n+2$ particles in the pure entangled state $|\Phi\rangle$, and

$$\frac{1}{d} \sum_{j=0}^{d-1} |c_j|^2 = 1. \tag{7}$$

Similar to the controlled teleportation of qubits [40, 42], Alice should first set up a pure entangled quantum channel with the controllers, say Bob$_q$ $(q = 1, 2, \ldots, n)$ and the receiver, say Charlie. The way for sharing a sequence of pure entangled $(n+2)$-particle qubit states has been discussed in [13]. In detail, Alice prepares a sequence of pure entangled states $|\Phi\rangle_{a_0a_1\cdots a_{n+1}}$, and divides them into $n+2$ particle sequences, say S_k $(k = 0, 1, \ldots, n+1)$. That is, Alice picks up the particle a_k in each pure entangled state $|\Phi\rangle_{a_0a_1\cdots a_{n+1}}$ to make up the particle sequence S_k, in the same way as [10, 11, 40, 42]. To prevent a potentially dishonest controller from stealing the information freely or the receiver from recovering the unknown state without the control of the controllers [50], Alice has to replace some particles in the sequence S_k with her decoy photons [51, 52] before she sends the sequence S_k to a controller, say Bob$_k$ (or the receiver Charlie if $k = n+1$). The decoy photons can be prepared by measuring and manipulating some particles in pure entangled states [13]. For instance, Alice measures the particle a_0 in the state $|\Phi\rangle_{a_0a_1\cdots a_{n+1}}$ with the MB Z_d, and then obtains the state of all the other particles $|r\rangle$ if that of the particle a_0 is $|r\rangle$. Alice can manipulate the particle a_k with unitary operations $\{U'_{uv} = |u\rangle\langle v|\}$ and high-dimensional Hadamard operation H_d [13, 49, 52]:

$$H_d = \frac{1}{\sqrt{d}} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & \mathrm{e}^{2\pi \mathrm{i}/d} & \cdots & \mathrm{e}^{(d-1)2\pi \mathrm{i}/d} \\ 1 & \mathrm{e}^{4\pi \mathrm{i}/d} & \cdots & \mathrm{e}^{(d-1)4\pi \mathrm{i}/d} \\ \vdots & \vdots & \cdots & \vdots \\ 1 & \mathrm{e}^{2(d-1)\pi \mathrm{i}/d} & \cdots & \mathrm{e}^{(d-1)2(d-1)\pi \mathrm{i}/d} \end{pmatrix}. \tag{8}$$

That is, Alice can prepare her decoy photons randomly in one of the $2d$ states $\{|0\rangle, |1\rangle, \ldots, |d-1\rangle; |0\rangle_x, |1\rangle_x, \ldots, |d-1\rangle_x\}$ without an ideal high-dimension single-photon source [13, 52].

After setting up the pure entangled quantum channel securely, Alice performs a generalized Bell-state measurement on her particles χ_0 and a_0, the quantum correlation will be transferred into the quantum system composed of the other $n+1$ particles $a_1, a_2, \ldots, a_{n+1}$. For reconstructing the original qudit state $|\chi\rangle_{\chi_0}$, the n controllers Bob$_k$ perform X_d measurements on their particles and the receiver can probabilistically extract the information of the original state $|\chi\rangle_{\chi_0}$ by introducing one auxiliary two-level particle. In detail, one can rewrite the state of the composite quantum system composed of all the particles $\chi_0, a_0, a_1, \ldots, a_{n+1}$ as follows:

$$\begin{aligned} |\chi\rangle_{\chi_0} \otimes |\Phi\rangle_{a_0 a_1 \cdots a_{n+1}} &= \left(\sum_{j=0}^{d-1} \beta_j |j\rangle\right)_{\chi_0} \otimes \left(\sum_{j'=0}^{d-1} c_{j'} \prod_{k=0}^{n+1} |j'\rangle_{a_k}\right) \\ &= \frac{1}{\sqrt{d}} \sum_{r,s} \left[|\psi_{rs}\rangle_{\chi_0 a_0} \otimes \sum_{j=0}^{d-1} \mathrm{e}^{-\frac{2\pi \mathrm{i}}{d} jr} \beta_j c_{j\oplus s} \prod_{k=1}^{n+1} |j \oplus s\rangle_{a_k} \right]. \end{aligned} \tag{9}$$

After Alice performs the generalized Bell-state (GBS) measurement on the particles χ_0 and a_0, the remaining particles $(a_1, a_2, \ldots, a_{n+1})$ collapse to the state $|\varphi\rangle_{a_1 \cdots a_{n+1}}$ (without being normalized) if Alice gets the outcome $|\psi_{rs}\rangle_{\chi_0 a_0}$:

$$|\varphi\rangle_{a_1, \ldots, a_{n+1}} = \sum_{j=0}^{d-1} \mathrm{e}^{-\frac{2\pi \mathrm{i}}{d} jr} \beta_j c_{j\oplus s} \prod_{k=1}^{n+1} |j \oplus s\rangle_{a_k}. \tag{10}$$

To probabilistically reconstruct the original state, the controllers Bob$_k$ perform measurements with the MB X_d on their particles independently. The measurements done by all the controllers can be expressed as M, similar to [42, 44]:

$$M = (\langle 0|_x)^{n-t_1-\cdots t_{d-1}} \otimes (\langle 1|_x)^{t_1} \otimes \cdots \otimes (\langle d-1|_x)^{t_{d-1}}. \tag{11}$$

Here t_j $(j = 1, 2, \ldots, d-1)$ represents the number of the controllers that obtain the result $|j\rangle_x$. After controllers perform M measurements on their particles, the state of the particle in the hand of the receiver Charlie becomes (neglect a whole factor $\frac{1}{d^{n/2}}$)

$$\begin{aligned} |\varphi\rangle_{a_{n+1}} &= M \left(\sum_{j=0}^{d-1} \mathrm{e}^{-\frac{2\pi \mathrm{i}}{d} jr} \beta_j c_{j\oplus s} \prod_{k=1}^{n+1} |j \oplus s\rangle_{a_k} \right) \\ &= \sum_{j=0}^{d-1} \mathrm{e}^{-\frac{2\pi \mathrm{i}}{d} [jr + (j\oplus s) r']} \beta_j c_{j\oplus s} |j \oplus s\rangle_{a_{n+1}}, \end{aligned} \tag{12}$$

where

$$r' = 1 \cdot t_1 + 2 \cdot t_2 + \cdots + (d-1) \cdot t_{d-1}. \tag{13}$$

That is, the state of the receiver's particle a_{n+1} is determined by the measurement results of the sender and all the controllers. Suppose $|c_k|^2 = \min\{|c_i|^2, i = 0, \ldots, d-1\}$. For extracting information of the original state $|\chi\rangle_{\chi_0}$ from $|\varphi\rangle_{a_{n+2}}$ probabilistically, Charlie can perform a general evolution $U_{\max}$ on particle a_{n+1} and an auxiliary qubit a_{aux} whose original state is $|0\rangle_{\text{aux}}$. In detail, under the basis $\{|0\rangle|0\rangle_{\text{aux}}, |1\rangle|0\rangle_{\text{aux}}, \ldots, |d-1\rangle|0\rangle_{\text{aux}}, |0\rangle|1\rangle_{\text{aux}}, \ldots, |d-1\rangle|1\rangle_{\text{aux}}\}$, the collective unitary transformation $U_{\max}$ can be chosen as

$$U_{\max} = \begin{pmatrix} \frac{c_k}{c_0} & 0 & \cdots & 0 & \sqrt{1-(\frac{c_k}{c_0})^2} & 0 & \cdots & 0 \\ 0 & \frac{c_k}{c_1} & \cdots & 0 & 0 & \sqrt{1-(\frac{c_k}{c_1})^2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{c_k}{c_{d-1}} & 0 & 0 & \cdots & \sqrt{1-(\frac{c_k}{c_{d-1}})^2} \\ \sqrt{1-(\frac{c_k}{c_0})^2} & 0 & \cdots & 0 & -\frac{c_k}{c_0} & 0 & \cdots & 0 \\ 0 & \sqrt{1-(\frac{c_k}{c_1})^2} & \cdots & 0 & 0 & -\frac{c_k}{c_1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sqrt{1-(\frac{c_k}{c_{d-1}})^2} & 0 & 0 & \cdots & -\frac{c_k}{c_{d-1}} \end{pmatrix}, \tag{14}$$

i.e.,

$$U_{\max}|\varphi\rangle_{a_{n+1}}|0\rangle_{\text{aux}} = \sum_{j=0}^{d-1} \mathrm{e}^{-\frac{2\pi \mathrm{i}}{d}[jr+(j\oplus s)r']} \beta_j c_{j\oplus s} |j \oplus s\rangle_{a_{n+1}} \times \left(\frac{c_k}{c_{j\oplus s}} |0\rangle_{\text{aux}} + \sqrt{1 - \left(\frac{c_k}{c_{j\oplus s}}\right)^2} |1\rangle_{\text{aux}} \right). \tag{15}$$

Charlie measures his auxiliary particle after the unitary transformation $U_{\max}$. The controlled teleportation succeeds if the measurement result is $|0\rangle_{\text{aux}}$; otherwise the teleportation fails, and the information of the original state is disappeared. If the controlled teleportation succeeds, Charlie gets the state of the particle a_{n+1}:

$$\begin{aligned} |\varphi'\rangle_{a_{n+1}} &= \sum_{j'=0}^{d-1} \mathrm{e}^{-\frac{2\pi \mathrm{i}}{d}[(j'r'+(j'-s)r]} \beta_{d+j'-s\oplus d} c_k |j'\rangle_{a_{n+1}} \\ &= c_k \sum_{j=0}^{d-1} \mathrm{e}^{-\frac{2\pi \mathrm{i}}{d}[(j\oplus s)r'+jr]} \beta_j |j \oplus s\rangle_{a_{n+1}}. \end{aligned} \tag{16}$$

Charlie can reconstruct the originally unknown state $|\chi\rangle$ by performing a unitary operation

$$U_{r'+r,d-s} = \sum_{j'=0}^{d-1} \mathrm{e}^{\frac{2\pi \mathrm{i}}{d} j'(r+r')} |j' \oplus d - s\rangle\langle j'| \tag{17}$$

on his particle a_{n+1}, i.e.,

$$U_{r'+r,d-s}|\varphi'\rangle_{a_{n+1}} = A\sum_{j=0}^{d-1}\beta_j|j\rangle_{a_{n+1}}, \tag{18}$$

where $A = c_k\,\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}rs}$ is a whole factor which does not change the feature of the state.

As discussed in [53, 54], the maximal probability P_s for extracting the unknown state $|\chi\rangle$ with the fidelity unit from the state $|\varphi\rangle_{a_{n+1}}$ is the square of the minimal coefficient in c_j ($j = 0, 1, \ldots, d-1$). That is, the receiver Charlie can recover the unknown state $|\chi\rangle$ with the probability $P_s = |c_k|^2$.

3. Controlled teleportation of *m* qudits

Now, let us generalize this scheme to the case with an unknown m-qudit state. In this time, the agents should first share m pure entangled states $|\Phi\rangle^{\otimes m}$ in the same way discussed above. Similar to the case with an unknown single-particle qudit state, the sender (Alice) performs m generalized Bell-state measurements, and then the controllers (Bob$_s$) make X_d measurements on their particles. The receiver Charlie first probabilistically extracts the information via a unitary transformation on his particles and an auxiliary two-level particle, and then reconstructs the original state by performing some unitary operations on his particles kept.

In detail, the quantum channel is a sequence of pure entangled $(n+2)$-particle states (the same m quantum systems), i.e.,

$$|\Phi'\rangle \equiv \prod_{l=1}^{m}\left(c_0\prod_{k=0}^{n+1}|0\rangle_{a_k} + \cdots + \prod_{k'=0}^{n+1}c_{d-1}|d-1\rangle_{a_{k'}}\right)_l. \tag{19}$$

Alice sends the kth ($k = 1, 2, \ldots, n$) particle a_{kl} in the lth ($l = 1, 2, \ldots, m$) pure entangled state to Bob$_k$ and the $(n+1)$ th particle $a_{n+1,l}$ to the receiver Charlie, and she keeps the first particle a_{0l} in each pure entangled state. Also all the parties can set up this quantum channel with decoy photons [13, 14, 51, 52], the same as that discussed above.

Suppose an arbitrary m-qudit state can be described as

$$|\chi'\rangle_{\chi_1\chi_2\cdots\chi_m} = \sum_{n'_1n'_2\cdots n'_m=0}^{d-1}\beta_{n'_1n'_2\cdots n'_m}|n'_1n'_2\cdots n'_m\rangle, \tag{20}$$

and

$$\sum_{n'_1n'_2\cdots n'_m=0}^{d-1}\left|\beta_{n'_1n'_2\cdots n'_m}\right|^2 = 1, \tag{21}$$

where $\chi_1, \chi_2, \ldots, \chi_m$ are the m particles in the originally unknown state $|\chi'\rangle$. For the controlled teleportation, Alice first takes the generalized Bell-state measurement on the particles χ_l and a_{0l} ($l = 1, 2, \ldots, m$), and then the controllers Bob$_k$ ($k = 1, 2, \ldots, n$) perform X_d measurements on their particles. The measurements done by all the controllers Bob$_s$ can be written as

$$M' = \prod_{l=1}^{m}M_l, \tag{22}$$

where

$$M_l = (\langle 0|_x)^{n-t_1^l-\cdots-t_{d-1}^l}\otimes\left(\langle 1|_x^{t_1^l}\right)\otimes\cdots\otimes(\langle d-1|_x)^{t_{d-1}^l} \tag{23}$$

represent the single-particle measurements done by all the controllers on the particles in the lth pure entangled state $|\Phi\rangle_l \equiv \left(c_0 \prod_{k=0}^{n+1} |0\rangle_{a_k} + \cdots + \prod_{k'=0}^{n+1} c_{d-1}|d-1\rangle_{a_{k'}}\right)_l$, and t_j^l represents the number of the controllers who obtain the outcomes $|j\rangle_x$.

The state of the composite system composed of particles $\chi_1, \chi_2, \ldots, \chi_m$ and a_{kl} ($k = 0, 1, \ldots, n+1$ and $l = 1, 2, \ldots, m$) can be described as

$$\begin{aligned}
|\chi'\rangle \otimes |\Phi'\rangle = &\sum_{n_1' n_2' \cdots n_m'=0}^{d-1} \beta_{n_1' n_2' \cdots n_m'} |n_1' n_2' \cdots n_m'\rangle_{\chi_1 \chi_2 \cdots \chi_m} \\
&\otimes \prod_{l=1}^{m} \left(c_0 \prod_{k=0}^{n+1} |0\rangle_{a_k} + \cdots + \prod_{k'=0}^{n+1} c_{d-1}|d-1\rangle_{a_{k'}} \right)_l \\
= &\frac{1}{d^{m/2}} \sum_{\substack{r_1 \cdots r_m, \\ s_1 \cdots s_m, j_1 \cdots j_m}}^{d-1} \left|\psi_{r_1 s_1}\right\rangle_{\chi_1 a_{01}} \otimes \left|\psi_{r_2 s_2}\right\rangle_{\chi_2 a_{02}} \otimes \cdots \otimes \left|\psi_{r_m s_m}\right\rangle_{\chi_m a_{0m}} \\
&\otimes \mathrm{e}^{-\frac{2\pi \mathrm{i}}{d}(j_1 r_1 + j_2 r_2 + \cdots + j_m r_m)} \otimes \beta_{j_1 j_2 \cdots j_m} \otimes c_{j_1 \oplus s_1} c_{j_2 \oplus s_2} \cdots c_{j_m \oplus s_m} \\
&\times \left(\prod_{k_1=1}^{n+1} |j_1 \oplus s_1\rangle_{k_1} \right) \left(\prod_{k_2=1}^{n+1} |j_2 \oplus s_2\rangle_{k_2} \right) \cdots \left(\prod_{k_m=1}^{n+1} |j_m \oplus s_m\rangle_{k_m} \right). \qquad (24)
\end{aligned}$$

That is, after Alice performs m GBS measurements on her $2m$ particles $\chi_l a_{0l}$ ($l = 1, 2, \ldots, m$), the subsystem composed of the particles remained collapses to the corresponding state $|\xi\rangle_{a_{11} a_{12} \cdots a_{n+1,m}}$. If the outcomes of the GBS measurements obtained by Alice are $|\psi_{r_l s_l}\rangle_{\chi_l a_{0l}}$ ($l = 1, 2, \ldots, m$), the state of the subsystem can be written as (without normalization)

$$\begin{aligned}
|\xi\rangle_{a_{11} a_{12} \cdots a_{n+1,m}} = &\sum_{j_1 \cdots j_m=0}^{d-1} \mathrm{e}^{-\frac{2\pi \mathrm{i}}{d}(j_1 r_1 + j_2 r_2 + \cdots + j_m r_m)} \beta_{j_1 j_2 \cdots j_m} c_{j_1 \oplus s_1} c_{j_2 \oplus s_2} \cdots c_{j_m \oplus s_m} \\
&\otimes \left(\prod_{k_1=1}^{n+1} |j_1 \oplus s_1\rangle_{k_1} \right) \left(\prod_{k_2=1}^{n+1} |j_2 \oplus s_2\rangle_{k_2} \right) \cdots \left(\prod_{k_m=1}^{n+1} |j_m \oplus s_m\rangle_{k_m} \right). \qquad (25)
\end{aligned}$$

After all the controllers Bob$_s$ take single-particle measurements on their particles with the MB X_d, the state of the particles $a_{n+1,l}$ ($l = 1, 2, \ldots, m$) kept by the receiver Charlie becomes

$$\begin{aligned}
|\theta\rangle_{a_{n+1,1} a_{n+1,2} \cdots a_{n+1,m}} &\equiv M' |\xi\rangle_{a_{11} a_{12} \cdots a_{n+1,m}} \\
&= \sum_{j_1 \cdots j_m=0}^{d-1} \mathrm{e}^{-\frac{2\pi \mathrm{i}}{d}\{[j_1 r_1 + (j_1 \oplus s_1) r_1''] + [j_2 r_2 + (j_2 \oplus s_2) r_2''] + \cdots + [j_m r_m + (j_m \oplus s_m) r_m'']\}} \otimes \beta_{j_1 j_2 \cdots j_m} \\
&\quad \otimes c_{j_1 \oplus s_1} c_{j_2 \oplus s_2} \cdots c_{j_m \oplus s_m} \otimes |j_1 \oplus s_1\rangle_{a_{n+1,1}} |j_2 \oplus s_2\rangle_{a_{n+1,2}} \cdots |j_m \oplus s_m\rangle_{a_{n+1,m}}. \qquad (26)
\end{aligned}$$

Here $r_l'' = t_1^l + 2t_2^l + \cdots + (d-1)t_{d-1}^l$. To reconstruct the original state probabilistically, Charlie first performs a unitary transformation on his particles and an auxiliary particle whose original state is $|0\rangle_{\mathrm{aux}}$. In essence, the auxiliary particle is used to select the useful information from the unknown state, no matter what the useless information is. That is, Charlie can use a two-dimension qubit (a two-level quantum system) for extracting the useful information. One level is used to map the useful information after a unitary evolution, and the other is used to map some useless information. Similar to the case of controlled teleportation of an unknown single qudit, under the basis $\left\{|fg \cdots h\rangle_{a_{n+1,1} a_{n+1,2} \cdots a_{n+1,m}} |0\rangle_{\mathrm{aux}};\ |fg \cdots h\rangle_{a_{n+1,1} a_{n+1,2} \cdots a_{n+1,m}} |1\rangle_{\mathrm{aux}}\right\}$

$(f, g, h \in \{0, 1, \ldots, d-1\})$ the unitary evolution ($2d^m \times 2d^m$ matrix) can be chosen as

$$U'_{\max} = \begin{pmatrix} \frac{(c_k)^m}{(c_0)^m} & \cdots & 0 & \cdots & 0 & \sqrt{1-(\frac{c_k}{c_0})^{2m}} & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & \Gamma_{fg\cdots h} & \cdots & 0 & 0 & \cdots & \Gamma^{+}_{fg\cdots h} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & \frac{(c_k)^m}{(c_{d-1})^m} & 0 & \cdots & 0 & \cdots & \sqrt{1-(\frac{c_k}{c_{d-1}})^{2m}} \\ \sqrt{1-(\frac{c_k}{c_0})^{2m}} & \cdots & 0 & \cdots & 0 & -\frac{(c_k)^m}{(c_0)^m} & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & \Gamma^{+}_{fg\cdots h} & \cdots & 0 & 0 & \cdots & -\Gamma_{fg\cdots h} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & \sqrt{1-(\frac{c_k}{c_{d-1}})^{2m}} & 0 & \cdots & 0 & \cdots & -\frac{(c_k)^m}{(c_{d-1})^m} \end{pmatrix}, \tag{27}$$

where

$$\Gamma_{fg\cdots h} \equiv \frac{(c_k)^m}{c_f c_g \cdots c_h}, \qquad \Gamma^{+}_{fg\cdots h} \equiv \sqrt{1-(\Gamma_{fg\cdots h})^2}. \tag{28}$$

That is, the unitary evolution $U'_{\max}$ can transfer the state $|\theta\rangle_{a_{n+1,1}a_{n+1,2}\cdots a_{n+1,m}}$ into the unknown state $|\chi'\rangle_{\chi_1\chi_2\cdots\chi_m}$ probabilistically, i.e.,

$$\begin{aligned} U'_{\max}|\theta\rangle_{a_{n+1,1}a_{n+1,2}\cdots a_{n+1,m}}|0\rangle_{\text{aux}} = \sum_{j_1\cdots j_m} & \mathrm{e}^{-\frac{2\pi \mathrm{i}}{d}\{[j_1r_1+(j_1\oplus s_1)r''_1]+[j_2r_2+(j_2\oplus s_2)r''_2]+\cdots+[j_mr_m+(j_m\oplus s_m)r''_m]\}} \\ & \otimes \beta_{j_1j_2\cdots j_m} \otimes c_{j_1\oplus s_1}c_{j_2\oplus s_2}\cdots c_{j_m\oplus s_m} \otimes |j_1\oplus s_1\rangle_{a_{n+1,1}}|j_2\oplus s_2\rangle_{a_{n+1,2}}\cdots|j_m\oplus s_m\rangle_{a_{n+1,m}} \\ & \times \left(\frac{c_k^m}{c_{j_1\oplus s_1}c_{j_2\oplus s_2}\cdots c_{j_m\oplus s_m}}|0\rangle_{\text{aux}} + \sqrt{1-\left(\frac{c_k^m}{c_{j_1\oplus s_1}c_{j_2\oplus s_2}\cdots c_{j_m\oplus s_m}}\right)^2}|1\rangle_{\text{aux}}\right). \end{aligned} \tag{29}$$

Same as the case for controlled teleportation of a single qudit, Charlie performs a measurement on the auxiliary qubit with the MB $\{|0\rangle, |1\rangle\}$. The controlled teleportation fails if the measurement result is $|1\rangle_{\text{aux}}$; otherwise, the teleportation succeeds and the particles kept by Charlie will collapse to the state

$$\begin{aligned} |\theta'\rangle_{a_{n+1,1}a_{n+1,2}\cdots a_{n+1,m}} &= c_k^m \sum_{j_1\cdots j_m} \mathrm{e}^{-\frac{2\pi \mathrm{i}}{d}\{[j_1r_1+(j_1\oplus s_1)r''_1]+[j_2r_2+(j_2\oplus s_2)r''_2]+\cdots+[j_mr_m+(j_m\oplus s_m)r''_m]\}} \otimes \beta_{j_1j_2\cdots j_m} \\ &\quad \otimes |j_1\oplus s_1\rangle_{a_{n+1,1}}|j_2\oplus s_2\rangle_{a_{n+1,2}}\cdots|j_m\oplus s_m\rangle_{a_{n+1,m}} \\ &= \alpha \sum_{j_1\cdots j_m} \mathrm{e}^{-\frac{2\pi \mathrm{i}}{d}(j_1r'''_1+j_2r'''_2+\cdots+j_mr'''_m)}\beta_{j_1j_2\cdots j_m}|j_1\oplus s_1\rangle_{a_{n+1,1}} \\ &\quad \times |j_2\oplus s_2\rangle_{a_{n+1,2}}\cdots|j_m\oplus s_m\rangle_{a_{n+1,m}} \end{aligned} \tag{30}$$

where

$$\alpha = c_k^m\,\mathrm{e}^{-\frac{2\pi \mathrm{i}}{d}(s_1r''_1\oplus s_2r''_2\oplus\cdots\oplus s_mr''_m)}, \tag{31}$$

$$r'''_l = r_l + r''_l. \tag{32}$$

Charlie can reconstruct the originally unknown state $|\chi'\rangle_{\chi_1\chi_2\cdots\chi_m}$ with a unitary transformation determined by the measurement results published by Alice and the controllers Bob_s if the controlled teleportation succeeds. Under the basis $\left\{|fg\cdots h\rangle_{a_{n+1,1}a_{n+1,2}\cdots a_{n+1,m}}\right\}$ $(f, g, h \in \{0, 1, \ldots, d-1\})$, the unitary transformation is

$$U_{r_1'''r_2'''\cdots r_m''',s_1s_2\cdots s_m} = \sum_{j_1'j_2'\cdots j_m'} \mathrm{e}^{\frac{2\pi \mathrm{i}}{d}(j_1'r_1'''+j_2'r_2'''+\cdots+j_m'r_m''')} |j_1'\rangle|j_2'\rangle\cdots|j_m'\rangle\langle j_1'\oplus s_1|\langle j_2'\oplus s_2|\cdots\langle j_m'\oplus s_m|, \tag{33}$$

i.e.,

$$U_{r_1'''r_2'''\cdots r_m''',s_1s_2\cdots s_m}|\theta'\rangle = \alpha|\chi'\rangle_{\chi_1\chi_2\cdots\chi_m}. \tag{34}$$

From equation (29), one can see the maximal probability that Charlie can reconstruct the originally unknown state $|\chi'\rangle_{\chi_1\chi_2\cdots\chi_m}$ with the fidelity unit is $P_{sm} = |c_k|^{2m}$. Here $|c_k|^2 = \min\{|c_j|^2, j = 0, 1, \ldots, d-1\}$.

4. Discussion and summary

If $c_j = 1$ for all the j from 0 to $d-1$, the quantum channel is composed of m maximally entangled $(n+2)$-particle states. The receiver can reconstruct the unknown state with probability 100% in principle if he cooperates with all the controllers, similar to the case with two-level quantum systems in [42]. Moreover, the unitary evolution $U'_{\max}$ is the identity matrix $I_{2d^m\times 2d^m}$ which means doing nothing on the particles controlled by the receiver and his auxiliary two-level particle. The receiver can obtain the unknown state with m single-particle unitary operations on his m particles.

In summary, we have presented a general scheme for multiparty-controlled teleportation of an arbitrary m-qudit state by using m pure entangled $(n+2)$-particle quantum systems as the quantum channel. The sender Alice can share a sequence of pure entangled states with all the other parties by inserting some decoy photons randomly in the particle sequences transmitted to the controllers and the receiver. The receiver can probabilistically extract the information of the originally unknown state by performing a general evolution on his particle and an auxiliary two-level particle. Charlie can reconstruct the originally unknown state with m unitary transformations on his particles according to the measurement results obtained by all the parties. The optimal probability of successful teleportation is $p = |c_k|^{2m}$ which is just the entanglement of the quantum channel.

Acknowledgments

This work was supported by the National Natural Science Foundation of China under Grant Nos. 10604008 and 10435020, and Beijing Education Committee under Grant No. XK100270454.

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