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Multiple positive solutions of systems of Hammerstein integral equations with applications to fractional differential equations

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Abstract

Positive solutions of systems of Hammerstein integral equations are studied by using the theory of the fixed-point index for compact maps defined on cones in Banach spaces. Criteria for the fixed-point index of the Hammerstein integral operators being 1 or 0 are given. These criteria are generalizations of previous results on a single Hammerstein integral operator. Some of criteria are new and involve the first eigenvalues of the corresponding systems of linear Hammerstein operators. The existence and estimates of the first eigenvalues are given. Applications are given to systems of fractional differential equations with two-point boundary conditions. The Green's functions of the boundary value problems are derived and their useful properties are provided. As illustrations, the existence of nonzero positive solutions of two specific such boundary value problems is studied.

1. *Introduction*

We study the existence of positive solutions of systems of Hammerstein integral equations of the form

$$\mathbf{z}(t) = (A_1\mathbf{z}(t), \ldots, A_n\mathbf{z}(t)) := A\mathbf{z}(t) \quad \text{for } t \in [0,1], \tag{1.1}$$

where $A_i\mathbf{z}(t) = \int_0^1 k(t,s)g_i(s)f_i(s,\mathbf{z}(s))\,ds$ and $i \in \{1,\ldots,n\}$. In applications, the kernels k are the corresponding Green's functions arising from the boundary value problems.

Equation (1.1) was studied in [**2**, **10**] and the references therein. Agarwal, O'Regan and Wong [**2**] studied the existence of one or multiple positive solutions of (1.1) when $k = k_i$ and f_i or $-f_i$ are positive and applied their results to a variety of integer-order boundary value problems (BVPs). Franco, Infante and O'Regan [**10**] studied systems of perturbed Hammerstein integral equations, where $k = k_i$ and f_i are allowed to take negative values, and applied their results to treat some second-order BVPs. The main tool used in [**2**, **10**] is the standard theory of the fixed-point index for compact maps defined on cones in the Banach space $C([0,1];\mathbb{R}^n)$; see [**3**, **11**] for the index theory. Some suitable conditions imposed on f_i are given to ensure that the fixed-point index of the nonlinear operators involved is 1 or 0. None of these earlier results use the first eigenvalues of the corresponding system of the linear Hammerstein integral operators, denoted by $\mathcal{L}_n$, and deal with the fractional differential equations.

It is known that, when $n = 1$, there are very good conditions imposed on f_1 that ensure that the fixed-point index of the Hammerstein integral operators is 1 or 0. In particular, some of those involving the first eigenvalues of the linear operator $\mathcal{L}_1$ obtained recently by Webb and Lan [**40**] are sharp conditions. Webb and Lan's results are generalizations of those obtained by Erbe [**9**] and Liu and Li [**30**], where k is required to be symmetric. Some of Webb and Lan's results on zero index require the uniqueness of the positive eigenvalues and are proved by the

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permanence property of the fixed-point index. The uniqueness of the positive eigenvalues can be dropped using Nussbaum's result on the continuity of radii of the spectra for compact linear operators (see [**37**, Remark 1.4]). Lan [**24**] obtained results on the eigenvalue problems for semi-positone Hammerstein integral equations, where the uniqueness of the positive eigenvalues and the permanence property are not used, but the results on the index being 1 are obtained only for some open subsets K_ρ with ρ larger than some $\rho_0 > 0$. Hence, some of results in [**40**] cannot be generalized to the semi-positone cases. The first eigenvalue principles were also used by Li [**28**], who worked in the space L^2, and by Zhang and Sun [**43**], who treated m-point BVPs.

In order to show that our results are generalizations of previous ones, we mention some of the conditions used in [**24**, **40**] below. For example, if f_1 depends only on u, then some of these conditions are

$$\lim_{u\to 0+} f_1(u)/u > M_1, \quad \lim_{u\to\infty} f_1(u)/u < m_1, \tag{1.2}$$

or

$$\lim_{u\to 0+} f_1(u)/u > \mu_1, \quad \lim_{u\to\infty} f_1(u)/u < \mu_1, \tag{1.3}$$

where $\mu_1 = 1/r(\mathcal{L}_1)$ with $r(\mathcal{L}_1)$ being the first eigenvalue of $\mathcal{L}_1$, and m_1 and M_1 are computable constants related to $k(t,s)g_1(s)$ (precise definitions of the symbols in this section will be given later). It is known [**40**] that

$$m_1 \leqslant \mu_1 \leqslant M_1. \tag{1.4}$$

In this paper, we investigate the existence of positive solutions of system (1.1), where k and f_i are required to be positive. We first work on the existence of the first eigenvalues of the linear operator $\mathcal{L}_n$. We shall provide conditions on k that ensure that the first eigenvalues exist and generalize (1.4). We shall show that μ_1 is greater than some of the m_i and smaller than some of the M_i, but, in general, the inequalities $m_i \leqslant \mu_1 \leqslant M_i$ for all $i \in \{1, \dots, n\}$ may not hold.

Next, we generalize the results on the fixed-point indices obtained in [**40**] to the case when $n > 1$. Like in [**24**], we do not use the uniqueness of the positive eigenvalues and the permanence property of the fixed-point index. It is worth pointing out that we shall see that, when $n > 1$, the limits in (1.2) and the first inequality of (1.3) can be replaced by the more general limits $\lim_{|\mathbf{z}|\to 0+} f_i(\mathbf{z})/|\mathbf{z}|$ or $\lim_{|\mathbf{z}|\to\infty} f_i(\mathbf{z})/|\mathbf{z}|$, while, in general, there is difficulty in replacing the second inequality of (1.3) by the weaker inequality $\lim_{|\mathbf{z}|\to\infty} f_i(\mathbf{z})/|\mathbf{z}| < \mu_1$, where $\mathbf{z} \in \mathbb{R}^n_+$. However, in some superlinear cases, some suitable conditions related to the weaker inequality apply; we refer to [**36**, **42**, **46**] for the study when $n = 2$. We shall provide stronger conditions involving μ_1 to replace such inequalities as the second inequality of (1.3) and show in our applications that these stronger conditions are easily verified. Some similar conditions that are stronger than ours in some cases were used in [**5**, **6**], where only results on the existence of one solution were obtained.

Finally, by combining our results on the fixed-point index of A with the theory of the fixed-point index, we give results on the existence of one or multiple positive solutions of (1.1). These results are generalizations of some earlier results obtained in [**9**, **20**, **40**] from $n = 1$ to $n > 1$.

As applications of our results on (1.1), we consider the existence of positive solutions of systems of fractional differential equations

$$\begin{aligned} &-D^\alpha z_i(t) = g_i(t) f_i(t, \mathbf{z}(t)), \\ &z_i(0) = 0, \quad \gamma z_i(1) + \delta z_i'(1) = 0, \end{aligned} \tag{1.5}$$

where $i \in \{1, \dots, n\}$, $1 < \alpha < 2$, $\delta > 0$ and $\gamma > (2-\alpha)\delta$.

When $n = 1$, equation (1.5) with $\delta = 0$ or $\gamma = 0$ was studied in [**4**, **14**] by using both Leggett–William fixed-point theorems [**27**] and the fixed-point index. We refer to [**7**, **8**, **13**, **17**, **18**, **20**,

29, **35**, **40**, **41**, **44**, **45**] and the references therein for other boundary conditions and other order α.

We shall derive the Green's functions k and prove that they satisfy the required upper and lower bounds that will be found. These facts show that results on (1.1) can be applied to treat (1.5). As illustrations, we shall consider the existence of positive solutions of (1.5) when $g_i \equiv 1$ and

$$f_i(s, \mathbf{z}) = \sum_{j=1}^{n} a_{ij}(s) z_j^{\mu_{ij}} \quad \text{or} \quad f_i(s, \mathbf{z}) = \lambda(z_i^{\alpha_i} + z_i^{\beta_i}) h_i(\hat{z}_i).$$

When $n = 1$ and $\alpha = 2$, such types of equations were studied in [**12**, **22**, **31**, **32**].

2. *Characteristic values of linear operators*

In this section, we shall study the characteristic values of the linear Hammerstein integral operator

$$L\mathbf{u}(t) = \left(\int_0^1 k(t,s) g_1(s) u_1(s)\, ds, \ldots, \int_0^1 k(t,s) g_n(s) u_n(s)\, ds \right) \quad \text{on } [0,1], \tag{2.1}$$

where $\mathbf{u}(t) = (u_1(t), \ldots, u_n(t))$. When $n = 1$, the characteristic values of L were studied in [**24**, **40**].

Let $I_n = \{1, \ldots, n\}$. We list the following conditions.

(C_1) The function $k : [0,1] \times (0,1) \to \mathbb{R}_+$ satisfies the following conditions:

(i) for each $t \in [0,1]$, we have that $k(t, \cdot) : (0,1) \to \mathbb{R}_+$ is measurable;

(ii) there exist a measurable function $\Phi : (0,1) \to \mathbb{R}_+$ and a continuous function $C : [0,1] \to [0,1]$ such that $\|C\| \in (0,1]$ and

$$C(t)\Phi(s) \leqslant k(t,s) \leqslant \Phi(s) \quad \text{for } t \in [0,1] \text{ and } s \in (0,1).$$

(C_2) For each $i \in I_n$, we have that $g_i : [0,1] \to \mathbb{R}_+$ is measurable and $\int_0^1 k(t,s) g_i(s)\, ds < \infty$ for $t \in [0,1]$.

(C_3) For each $i \in I_n$ and $\tau \in [0,1]$, we have that $\lim_{t \to \tau} \int_0^1 |k(t,s) - k(\tau,s)| g_i(s)\, ds = 0$.

(P) There exist $a, b \in [0,1]$ with $a < b$ such that

$$c := c(a,b) = \min\{C(t) : t \in [a,b]\} > 0.$$

(P^*) For any $\{a_m\}, \{b_m\} \subset (0,1)$ with $\lim_{m\to\infty} a_m = 0$ and $\lim_{m\to\infty} b_m = 1$, there exists $m_0 \in \mathbb{N}$ such that

$$c_m := c(a_m, b_m) = \min\{C(t) : t \in [a_m, b_m]\} > 0 \quad \text{for } m \geqslant m_0.$$

When $n = 1$, the above conditions were used, for example, in [**19**, **23**, **24**, **40**].

We always use the norm $|\mathbf{x}| = \max\{|x_i| : i \in I_n\}$ in $\mathbb{R}^n$. We denote by $C([0,1];\mathbb{R}^n)$ the Banach space of continuous functions from $[0,1]$ into $\mathbb{R}^n$ with the norm $\|x\| = \max\{\|x_i\| : i \in I_n\}$, where

$$\|x_i\| = \max\{|x_i(t)| : t \in [0,1]\}.$$

To study the characteristic values of L defined in (2.1), we need to consider a more general operator $L_{\alpha,\beta} : C([0,1];\mathbb{R}^n) \to C([0,1];\mathbb{R}^n)$ defined by

$$L_{\alpha,\beta}\mathbf{u}(t) := \left(\int_\alpha^\beta k(t,s) g_1(s) u_1(s)\, ds, \ldots, \int_\alpha^\beta k(t,s) g_n(s) u_n(s)\, ds \right), \tag{2.2}$$

where $\alpha, \beta \in [0,1]$ with $\alpha < \beta$.

Recall that a real number λ is called an eigenvalue of the linear operator $L : C([0,1];\mathbb{R}^n) \to C([0,1];\mathbb{R}^n)$ if there exists a nonzero function $\varphi \in C([0,1];\mathbb{R}^n)$ such that $\lambda\varphi = L\varphi$. The reciprocals of eigenvalues are called *characteristic values* of L. The radius of the spectrum of L, denoted by $r(L)$, is given by the well-known spectral radius formula

$$r(L) = \lim_{m\to\infty} \sqrt[m]{\|L^m\|},$$

where $\|L\|$ is the norm of L. The well-known Krein–Rutman theorem (see [**3**, Theorem 3.1] or [**16**, **34**]) shows that, if K is a total cone in a real Banach space X, that is, $X = \overline{K - K}$, and $L : X \to X$ is a compact linear operator such that $L(K) \subset K$ and $r(L) > 0$, then there exists an eigenvector $\varphi \in K \setminus \{0\}$ such that $r(L)\varphi = L\varphi$.

Let $P = C([0, 1]; \mathbb{R}^n_+)$. Then P is a reproducing cone in $C([0, 1]; \mathbb{R}^n)$. We introduce a smaller cone K than P defined by

$$K = \{\mathbf{x} \in P : x_i(t) \geqslant C(t)\|x_i\| \text{ for } t \in [0, 1] \text{ and } i \in I_n\}. \tag{2.3}$$

This type of cone with $n = 1$ were used in [**1**, **21**, **23**, **24**] to study semi-positone problems. We note that, when $n = 1$, under the assumption (P), the cone K defined in (2.3) is smaller than those used, for example, in [**11**, **15**, **19**, **25**, **40**]. Solutions in smaller cones have better properties.

When $n = 1$, it is shown in [**24**] that, if $\|C\| < 1$, then K is reproducing. The same technique can be used to show that the conclusion holds for $n \geqslant 1$. In Section 4, we shall provide a cone K with $\|C\| < 1$, and so it is reproducing. There is an example given in [**24**] that shows that, if $n = 1$ and $\|C\| = 1$, then K need not be total.

Using Lemma 2.1 in [**19**] and the Krein–Rutman theorem mentioned above, we can show the following result. Its proof is similar to that of Theorem 2.1 in [**24**] and is omitted.

THEOREM 2.1. *Under the hypotheses* (C_1)(i), (C_2) *and* (C_3), *the operator* $L_{\alpha,\beta}$ *defined in* (2.2) *maps* $C([0, 1]; \mathbb{R}^n)$ *into* $C([0, 1]; \mathbb{R}^n)$ *and is compact. In addition, if* (C_1)(ii) *holds, then* $L_{\alpha,\beta}$ *maps* P *into* K *and is compact. If we assume further that*

$$\gamma := \gamma(\alpha, \beta) = \min\left\{\int_\alpha^\beta \Phi(s)g_i(s)C(s)\,ds : i \in I_n\right\} > 0, \tag{2.4}$$

then $r(L_{\alpha,\beta}) \geqslant \gamma\|C\|$ *and there exists* $\varphi \in K \setminus \{0\}$ *such that* $L_{\alpha,\beta}\varphi = r(L_{\alpha,\beta})\varphi$.

Theorem 2.1 generalizes Theorem 2.1 in [**24**] from $n = 1$ to $n > 1$ and improves Lemma 1.2 in [**5**] with $\Omega = [\alpha, \beta]$, where $n = 2$ and each k_i is continuous. When $n = 1$, we refer to [**40**, Lemma 2.5 and Theorem 2.6] for similar results, where the linear operator and the cone involved are different.

Let $m \in \mathbb{N}$ with $m \geqslant 2$ and $a_m, b_m \in (0, 1)$ with $a_m < b_m$ satisfy $a_m \to 0$ and $b_m \to 1$. We write

$$\mu_1 = 1/r(L) \quad \text{and} \quad \mu_m = 1/r(L_m) \quad \text{for } m \geqslant 2, \tag{2.5}$$

where L is defined in (2.1) and $L_m = L_{a_m,b_m}$.

It was proved by Nussbaum (see [**33**, Lemma 2]) that the radius of the spectrum is continuous, that is, if $L, L_m : X \to X$ are compact linear operators and $\lim_{m\to\infty} \|L_m - L\| = 0$, then $\lim_{m\to\infty} r(L_m) = r(L)$. We use this result to prove the following result that will be used in Section 3.

THEOREM 2.2. *Assume that* (C_1)–(C_3) *hold and* $\gamma(0, 1) > 0$. *Then there exists* $m_0 > 1$ *such that, for each* $m \geqslant m_0$, *the value* μ_m *defined in* (2.5) *is a characteristic value of* L_m. *Moreover,* $\mu_m \to \mu_1$ *as* $m \to \infty$.

Proof. Since $\gamma(a_m,b_m)\to\gamma(0,1)$ as $m\to\infty$ and $\gamma(0,1)>0$, there exists $m_0\in\mathbb{N}$ such that $\gamma(a_m,b_m)>0$ for $m\geqslant m_0$. It follows from Theorem 2.1 that $\mu_1,\mu_m\in(0,\infty)$ and there exists $\varphi_m\in K\setminus\{0\}$ with $\|\varphi_m\|=1$ such that $\varphi_m=\mu_m L_m\varphi_m$ for each $m\geqslant m_0$. It is easy to see that $\|(L_m-L)\mathbf{u}\|\leqslant\|\mathbf{u}\|\xi_m$ for $\mathbf{u}\in C([0,1];\mathbb{R}^n)$, where $\xi_m=\max\{(\xi_m)_i : i\in I_n\}$ and

$$(\xi_m)_i=\max_{0\leqslant t\leqslant 1}\int_0^{a_m}k(t,s)g_i(s)\,ds+\max_{0\leqslant t\leqslant 1}\int_{b_m}^1 k(t,s)g_i(s)\,ds.$$

Since $\xi_m\to 0$, we have $\lim_{m\to\infty}\|L_m-L\|=0$. It follows from the continuity of the radius of the spectrum mentioned above that $\mu_m\to\mu_1$ as $m\to\infty$. □

Theorem 2.2 generalizes Theorem 2.2 in [**24**] from $n=1$ to $n>1$. When $n=1$, we refer to [**37**, Remark 1.4; **40**, Theorem 3.7] for similar results.

Let $a,b\in[0,1]$ with $a<b$. For $i\in I_n$, let

$$m_i=\left(\max_{t\in[0,1]}\int_0^1 k(t,s)g_i(s)\,ds\right)^{-1}\quad\text{and}\quad M_i(a,b)=\left(\min_{t\in[a,b]}\int_a^b k(t,s)g_i(s)\,ds\right)^{-1}.$$

The following result gives upper and lower bounds for μ_1.

THEOREM 2.3. *Assume that (C_1)–(C_3) and (P) hold and $\int_a^b\Phi(s)g_i(s)\,ds>0$ for $i\in I_n$. Then the following assertions hold.*

(i) *We have that $\gamma(0,1)>0$, $\mu_1\in(0,\infty)$ and there exists $\varphi=(\varphi_1,\dots,\varphi_n)\in K\setminus\{0\}$ such that $\varphi=\mu_1 L\varphi$.*

(ii) *Let $I^*=\{i\in I_n : \varphi_i\neq 0\}$. Then*

$$m\leqslant\mu_1\leqslant M(a,b), \tag{2.6}$$

where $m=\max\{m_i : i\in I^*\}$ and $M(a,b)=\min\{M_i(a,b) : i\in I^*\}$.

Proof. (i) Let $i\in I_n$. By (C_1)(ii) and (P), we have, for $t\in[a,b]$, that

$$\int_a^b k(t,s)g_i(s)\,ds\geqslant C(t)\int_a^b\Phi(s)g_i(s)\,ds\geqslant c(a,b)\int_a^b\Phi(s)g_i(s)\,ds>0.$$

It follows that $M_i(a,b)$ and m_i are well defined. Moreover, it is easy to show that $\gamma(0,1)>0$. The result (i) follows from Theorem 2.1.

(ii) Let $i\in I^*$. Then $\|\varphi_i\|>0$ and

$$\sigma_i:=\min\{\varphi_i(s) : s\in[a,b]\}\geqslant c(a,b)\|\varphi_i\|>0.$$

Since $\varphi=\mu_1 L\varphi$, we have, for $t\in[0,1]$, that

$$\varphi_i(t)=\mu_1\int_0^1 k(t,s)g_i(s)\varphi_i(s)\,ds\leqslant\mu_1\|\varphi_i\|/m_i.$$

It follows that $m_i\leqslant\mu_1$ for $i\in I^*$ and $m\leqslant\mu_1$. Let $t\in[a,b]$. Then

$$\varphi_i(t)\geqslant\mu_1\sigma_i\int_a^b k(t,s)g_i(s)\,ds\geqslant\mu_1\sigma_i/M_i(a,b)$$

and $\sigma_i\geqslant\mu_1\sigma_i/M_i(a,b)$. Hence, $\mu_1\leqslant M_i(a,b)$ for $i\in I^*$ and $\mu_1\leqslant M(a,b)$. □

Theorem 2.3(ii) generalizes Theorem 2.8 in [**40**] from $n=1$ to $n>1$. It is possible that some of the φ_i are zero although $\varphi\neq 0$ and, in general, $I^*\neq I_n$. Hence, in Theorem 2.3(ii), one cannot replace I^* by I_n.

3. *Hammerstein integral equations*

In this section, we study the existence of positive solutions of systems of Hammerstein integral equations of the form

$$\mathbf{z}(t) = (A_1\mathbf{z}(t), \dots, A_n\mathbf{z}(t)) := A\mathbf{z}(t) \quad \text{for } t \in [0,1], \tag{3.1}$$

where $\mathbf{z}(t) = (z_1(t), \dots, z_n(t))$ and

$$A_i\mathbf{z}(t) = \int_0^1 k(t,s)g_i(s)f_i(s,\mathbf{z}(s))\,ds \quad \text{for } t \in [0,1] \text{ and } i \in I_n. \tag{3.2}$$

Equation (3.1) was studied in [**2**], where $k = k_i$ and the f_i or $-f_i$ are positive, and in [**10**], where systems of perturbed Hammerstein integral equations are involved and $k = k_i$ and f_i are allowed to take negative values. None of these papers use the first eigenvalues of the corresponding linear Hammerstein integral operators obtained in Section 2. Here, we shall assume that k and f_i are positive and employ the first eigenvalues.

We always assume that (C_1)–(C_3) and the following condition holds.

(C_4) For each $i \in I_n$, we have that $f_i : [0,1] \times \mathbb{R}^n_+ \to \mathbb{R}_+$ satisfies Carathéodory conditions on $[0,1] \times \mathbb{R}^n_+$, that is, $f_i(\cdot, \mathbf{z})$ is measurable for each fixed $\mathbf{z} \in \mathbb{R}^n_+$ and $f_i(t, \cdot)$ is continuous on $\mathbb{R}^n_+$ for almost every (a.e.) $t \in [0,1]$, and for each $r > 0$ there exists $(g_r)_i \in L^\infty(0,1)$ such that

$$f_i(s,\mathbf{z}) \leqslant (g_r)_i(s) \quad \text{for a.e. } s \in [0,1] \text{ and all } \mathbf{z} \in \mathbb{R}^n_+ \text{ with } |\mathbf{z}| \leqslant r.$$

The following result shows that A is compact from K to K, that is, A is continuous and $\overline{A(D)}$ is compact for each bounded subset $D \subset K$. Its proof follows from Theorem 2.1 and is omitted.

LEMMA 3.1. *Under the hypotheses (C_1)–(C_4), the map A defined in (3.1) maps K into K and is compact.*

We need some results from the theory of the fixed-point index for compact maps [**3**, **11**]. Let D be a bounded open set in a Banach space X and let K be a cone in X. We denote by $\bar{D}_K$ and ∂D_K the closure and the boundary, respectively, of $D_K = D \cap K$ relative to K. We shall use the following known results (see, for example, [**23**, Lemma 1] or [**20**, Lemma 2.4]).

LEMMA 3.2. *Assume that $D_K \neq \emptyset$ and $A : \bar{D}_K \to K$ is a compact map. Then the following results hold.*

(i) *If $x \neq \varrho Ax$ for $x \in \partial D_K$ and $\varrho \in (0,1]$, then $i_K(A, D_K) = 1$.*

(ii) *If there exists $e \in K \setminus \{0\}$ such that $x \neq Ax + \nu e$ for $x \in \partial D_K$ and $\nu \geqslant 0$, then $i_K(A, D_K) = 0$.*

(iii) *Let D^1 be an open subset in X such that $\overline{D^1_K} \subset D_K$. If $i_K(A, D_K) = 1$ and $i_K(A, D^1_K) = 0$, then A has a fixed point in $D_K \setminus \overline{D^1_K}$. The same result holds if $i_K(A, D_K) = 0$ and $i_K(A, D^1_K) = 1$.*

NOTATION 3.3. *For each $i \in I_n$, we make the following definitions:*

$$m_\phi = \left(\max_{t\in[0,1]} \int_0^1 k(t,s)g_i(s)\phi(s)\,ds \right)^{-1}, \quad M_\psi = \left(\min_{t\in[a,b]} \int_a^b k(t,s)g_i(s)\psi(s)\,ds \right)^{-1}.$$

Let E be a fixed subset of $[0,1]$ of measure zero. Let

$$\overline{f_i}(\mathbf{z}) = \sup_{s\in[0,1]\setminus E} f_i(s,\mathbf{z}), \quad \underline{f_i}(\mathbf{z}) = \inf_{s\in[a,b]\setminus E} f_i(s,\mathbf{z}),$$
$$f_i^0 = \limsup_{|\mathbf{z}|\to 0+} \overline{f_i}(\mathbf{z})/|\mathbf{z}|, \quad f_i^\infty = \limsup_{|\mathbf{z}|\to\infty} \overline{f_i}(\mathbf{z})/|\mathbf{z}|,$$
$$(f_i)_0 = \liminf_{|\mathbf{z}|\to 0+} \underline{f_i}(\mathbf{z})/|\mathbf{z}|, \quad (f_i)_\infty = \liminf_{|\mathbf{z}|\to\infty} \underline{f_i}(\mathbf{z})/|\mathbf{z}|.$$

Let $\rho > 0$ and let $K_\rho = \{x \in K : \|x\| < \rho\}$, $\partial K_\rho = \{x \in K : \|x\| = \rho\}$ and $\overline{K}_\rho = \{x \in K : \|x\| \leqslant \rho\}$.

The following result provides conditions that ensure that $i_K(A, K_\rho) = 1$ and generalizes Lemma 2.6 in [**19**] and Lemma 2.8 in [**20**] from $n = 1$ to $n > 1$.

THEOREM 3.4. *Assume that there exists $\rho > 0$ such that $\mathbf{z} \neq A\mathbf{z}$ for $\mathbf{z} \in \partial K_\rho$ and the following condition holds.*

$(H^1_{\leqslant})_{\phi_\rho}$ For each $i \in I_n$, there exists a measurable function $\phi^i_\rho : [0,1] \to \mathbb{R}_+$ such that $\int_0^1 \Phi(s) g_i(s) \phi^i_\rho(s)\, ds > 0$ and

$$f_i(s,\mathbf{z}) \leqslant \phi^i_\rho(s) m_{\phi^i_\rho} \rho \quad \text{for a.e. } s \in [0,1] \text{ and all } \mathbf{z} \in \mathbb{R}^n_+ \text{ with } |\mathbf{z}| \in [0,\rho].$$

Then $i_K(A, K_\rho) = 1$.

Proof. By $(H^1_{\leqslant})_{\phi_\rho}$, we have, for each $i \in I_n$ and $\mathbf{z} \in \partial K_\rho$, that

$$A_i\mathbf{z}(t) \leqslant m_{\phi^i_\rho} \rho \int_0^1 k(t,s) g_i(s) \phi^i_\rho(s)\, ds \leqslant \rho = \|\mathbf{z}\|.$$

This implies that $\|A_i\mathbf{z}\| \leqslant \|\mathbf{z}\|$ for $i \in I_n$ and $\|A\mathbf{z}\| \leqslant \|\mathbf{z}\|$ for $\mathbf{z} \in \partial K_\rho$. By Lemma 3.2($i$), we have $i_K(A, K_\rho) = 1$. □

The following condition implies that $(H^1_{\leqslant})_{\phi_\rho}$ holds and that $\mathbf{z} \neq A\mathbf{z}$ for $\mathbf{z} \in \partial K_\rho$.

$(H^1_{<})_{\phi_\rho}$ For each $i \in I_n$, there exist a measurable function $\phi^i_\rho : [0,1] \to \mathbb{R}_+$ and $\tau_i \in (0, m_{\phi^i_\rho})$ such that $\int_0^1 \Phi(s) g_i(s) \phi^i_\rho(s)\, ds > 0$ and

$$f_i(s,\mathbf{z}) \leqslant \phi^i_\rho(s) \tau_i \rho \quad \text{for a.e. } s \in [0,1] \text{ and all } \mathbf{z} \in \mathbb{R}^n_+ \text{ with } |\mathbf{z}| \in [0,\rho].$$

COROLLARY 3.5. *Assume that $\int_0^1 \Phi(s) g_i(s)\, ds > 0$ for $i \in I_n$ and the following condition holds*:

$$0 \leqslant f_i^0 < m_i \quad \text{for } i \in I_n. \tag{3.3}$$

Then there exists $\rho_0 > 0$ such that $i_K(A, K_\rho) = 1$ for $\rho \in (0, \rho_0)$.

Proof. By (3.3), there exist $\varepsilon > 0$ and $\rho_0 > 0$ such that $f_i^0 \leqslant m_i - \varepsilon$ for $i \in I_n$ and

$$f_i(s,\mathbf{z}) \leqslant (m_i - \varepsilon)|\mathbf{z}| \quad \text{for a.e. } s \in [0,1] \text{ and } \mathbf{z} \in \mathbb{R}^n_+ \text{ with } |\mathbf{z}| \leqslant \rho_0.$$

The result follows from Theorem 3.4 with $\phi^i_\rho \equiv 1$. □

Corollary 3.6. *Assume that the following condition holds.*

$(H^1_<)^\infty_{\phi_r}$ *There exists $r > 0$ such that, for each $i \in I_n$, there exist a measurable function $\phi^i_r : [0,1] \to \mathbb{R}_+$ with $\int_0^1 \Phi(s) g_i(s) \phi^i_r(s)\, ds > 0$ and $\tau_i \in (0, m_{\phi^i_r})$ such that*

$$f_i(s, \mathbf{z}) \leqslant \phi^i_r(s) \tau_i |\mathbf{z}| \quad \text{for a.e. } s \in [0,1] \text{ and all } \mathbf{z} \in \mathbb{R}^n_+ \text{ with } |\mathbf{z}| \geqslant r.$$

Then there exists $\rho_0 \geqslant r$ such that $i_K(A, K_\rho) = 1$ for $\rho > \rho_0$.

Proof. Let $i \in I_n$. By (C_4), there exists $(g_r)_i \in L^\infty(0,1)$ such that

$$f_i(s, \mathbf{z}) \leqslant (g_r)_i(s) \quad \text{for a.e. } s \in [0,1] \text{ and all } \mathbf{z} \in \mathbb{R}^n_+ \text{ with } |\mathbf{z}| \in [0, r].$$

This, together with $(H^1_<)^\infty_{\phi_r}$, implies that

$$f_i(s, \mathbf{z}) \leqslant \phi^i_r(s) \tau_i |\mathbf{z}| + (g_r)_i(s) \quad \text{for a.e. } s \in [0,1] \text{ and all } \mathbf{z} \in \mathbb{R}^n_+. \tag{3.4}$$

Let $\rho_0 = \max\{r, \max\{m_{\phi^i_r}/(m_{g^i_r}(m_{\phi^i_r} - \tau_i)) : i \in I_n\}\}$, $\rho > \rho_0$ and

$$\phi^i_\rho(s) = \phi^i_r(s)\tau_i + \frac{(g_r)_i(s)}{\rho} \quad \text{for } s \in [0,1].$$

Then

$$\max_{t \in [0,1]} \int_0^1 k(t,s) g_i(s) \phi^i_\rho(s)\, ds \leqslant \frac{\tau_i}{m_{\phi^i_r}} + \frac{1}{m_{\phi^i_r}\rho} < 1$$

and $m^i_{\phi_\rho} > 1$. Let $\xi^i \in (1, m^i_{\phi_\rho})$. By (3.4), we have

$$f_i(s, \mathbf{z}) \leqslant \phi^i_\rho(s)\rho \leqslant \phi^i_\rho(s)\xi^i\rho \quad \text{for a.e. } s \in [0,1] \text{ and all } \mathbf{z} \in \mathbb{R}^n_+ \text{ with } |\mathbf{z}| \in [0, \rho]$$

and $(H^1_<)_{\phi_\rho}$ holds. The result follows from Theorem 3.4. □

By using Corollary 3.6 with $\phi^i_r \equiv 1$, we obtain the following result.

Corollary 3.7. *Assume that $\int_0^1 \Phi(s) g_i(s)\, ds > 0$ for $i \in I_n$ and*

$$0 \leqslant f^\infty_i < m_i \quad \text{for } i \in I_n.$$

Then there exists $\rho_0 > 0$ such that $i_K(A, K_\rho) = 1$ for $\rho > \rho_0$.

By Theorem 2.3(ii), we see that μ_1 is greater than or equal to some of the m_i. In particular, when $n = 1$, we have that μ_1 is greater than or equal to m_1. Therefore, replacing m_1 by μ_1 produces a weaker condition; see [**40**, Theorems 3.2 and 3.3]. However, when $n > 1$, it seems difficult to prove that the fixed-point index of A is 1 under one of the following hypotheses:

$$0 \leqslant f^0_i < \mu_1 \quad \text{or} \quad 0 \leqslant f^\infty_i < \mu_1 \quad \text{for } i \in I_n.$$

Hence, we give stronger conditions in the following two theorems that generalize Theorems 3.2 and 3.3 in [**40**] from $n = 1$ to $n > 1$.

Theorem 3.8. *Assume that $\gamma(0,1) > 0$ and the following condition holds.*

$(f^0_i)_{\mu_1}$ *There exist $\varepsilon > 0$ and $\rho_0 > 0$ such that, for $i \in I_n$, we have*

$$f_i(s, \mathbf{z}) \leqslant (\mu_1 - \varepsilon) z_i \quad \text{for a.e. } s \in [0,1] \text{ and all } \mathbf{z} \in \mathbb{R}^n_+ \text{ with } |\mathbf{z}| \in [0, \rho_0].$$

Then $i_K(A, K_\rho) = 1$ for each $\rho \in (0, \rho_0]$.

Proof. Let $\rho \in (0, \rho_0]$. We prove that

$$\mathbf{z} \neq \varrho A\mathbf{z} \quad \text{for } \mathbf{z} \in \partial K_\rho \text{ and } \varrho \in [0,1]. \tag{3.5}$$

In fact, if (3.5) does not hold, then there exist $\mathbf{z} \in \partial K_\rho$ and $\varrho \in [0,1]$ such that $\mathbf{z} = \varrho A\mathbf{z}$. Hence, we have, for $i \in I_n$ and $t \in [0,1]$, that

$$z_i(t) \leqslant \int_0^1 k(t,s) g_i(s) f_i(s, \mathbf{z}(s))\, ds \leqslant (\mu_1 - \varepsilon) \int_0^1 k(t,s) g_i(s) z_i(s)\, ds.$$

This implies that $\mathbf{z}(t) \leqslant (\mu_1 - \varepsilon) L\mathbf{z}(t)$, $L\mathbf{z}(t) \leqslant (\mu_1 - \varepsilon) L^2 \mathbf{z}(t)$ and

$$\mathbf{z}(t) \leqslant (\mu_1 - \varepsilon) L\mathbf{z}(t) \leqslant (\mu_1 - \varepsilon)^2 L^2 \mathbf{z}(t) \quad \text{for } t \in [0,1].$$

Repeating the process gives

$$\mathbf{z}(t) \leqslant (\mu_1 - \varepsilon)^m L^m \mathbf{z}(t) \quad \text{for } t \in [0,1] \text{ and } m \in \mathbb{N}$$

and $1 \leqslant (\mu_1 - \varepsilon)^m \|L^m\|$ for $m \in \mathbb{N}$. Hence, we have

$$1 \leqslant (\mu_1 - \varepsilon) \lim_{m \to \infty} \|L^m\|^{1/m} = (\mu_1 - \varepsilon) \frac{1}{\mu_1} < 1,$$

which is a contradiction. It follows from (3.5) and Lemma 3.2(i) that $i_K(A, K_\rho) = 1$. □

THEOREM 3.9. *Assume that* $\gamma(0,1) > 0$ *and the following condition holds.*

$(f_i^\infty)_{\mu_1}$ *There exist* $\varepsilon > 0$ *and* $\rho_0 > 0$ *such that, for each* $i \in I_n$, *we have*

$$f_i(s, \mathbf{z}) \leqslant (\mu_1 - \varepsilon) z_i \quad \textit{for a.e. } s \in [0,1] \textit{ and all } \mathbf{z} \in \mathbb{R}_+^n \textit{ with } |\mathbf{z}| \geqslant \rho_0.$$

Then $i_K(A, K_\rho) = 1$ *for* $\rho > \rho_0$.

Proof. Since $\gamma(0,1) > 0$, it follows from Theorem 2.1 that $r(L) > 0$ and $\mu_1 \in (0, \infty)$. By (C_4), for each $i \in I_n$, there exists $(g_{\rho_0})_i \in L^\infty(0,1)$ such that

$$f_i(s, \mathbf{z}) \leqslant (g_{\rho_0})_i(s) \quad \text{for a.e. } s \in [0,1] \text{ and all } \mathbf{z} \in \mathbb{R}_+^n \text{ with } |\mathbf{z}| \leqslant \rho_0.$$

This, together with the hypothesis $(f_i^\infty)_{\mu_1}$, implies that

$$f_i(s, \mathbf{z}) \leqslant (g_{\rho_0})_i(s) + (\mu_1 - \varepsilon) z_i \quad \text{for a.e. } s \in [0,1] \text{ and all } \mathbf{z} \in \mathbb{R}_+^n. \tag{3.6}$$

Since $r((\mu_1 - \varepsilon)L) = (\mu_1 - \varepsilon) r(L) = (\mu_1 - \varepsilon)/\mu_1 < 1$, we have that $(I - (\mu_1 - \varepsilon)L)^{-1}$ exists, is bounded and satisfies $(I - (\mu_1 - \varepsilon)L)^{-1} K \subset K$. We define

$$\rho_1^*(t) = (\rho_1, \dots, \rho_1) \in \mathbb{R}^n \quad \text{for each } t \in [0,1],$$

where $\rho_1 = \max\{\int_0^1 \Phi(s) g_i(s) (g_{\rho_0})_i(s)\, ds : i \in I_n\}$. Then $\rho_1^* \in K \setminus \{0\}$. Let $\rho^* = \|(I - (\mu_1 - \varepsilon)L)^{-1} \rho_1^*\|$. Then $\rho^* > 0$. Let $\rho > \rho^*$. We prove that

$$\mathbf{z} \neq \varrho A\mathbf{z} \quad \text{for } \mathbf{z} \in \partial K_\rho \text{ and } \varrho \in [0,1]. \tag{3.7}$$

If not, then there exist $\mathbf{z} \in \partial K_\rho$ and $\varrho \in [0,1]$ such that $\mathbf{z} = \varrho A\mathbf{z}$. By (3.6) and (C_1)(ii), we have, for $i \in I_n$ and $t \in [0,1]$, that

$$\begin{aligned} z_i(t) \leqslant A_i\mathbf{z}(t) &\leqslant \int_0^1 k(t,s) g_i(s) (g_{\rho_0})_i(s)\, ds + \int_0^1 k(t,s) g_i(s) (\mu_1 - \varepsilon) z_i(s)\, ds \\ &\leqslant \rho_1 + (\mu_1 - \varepsilon) \int_0^1 k(t,s) g_i(s) z_i(s)\, ds \end{aligned}$$

and $\mathbf{z}(t) \leqslant \rho_1^* + (\mu_1 - \varepsilon) L\mathbf{z}(t)$. This implies that $(I - (\mu_1 - \varepsilon)L)\mathbf{z}(t) \leqslant \rho_1^*$ for $t \in [0,1]$ and

$$\mathbf{z} \leqslant (I - (\mu_1 - \varepsilon)L)^{-1} \rho_1^*.$$

Hence, we have $\rho = \|\mathbf{z}\| \leqslant \rho^* < \rho$, which is a contradiction. The result follows from (3.7) and Lemma 3.2(i). □

In order to prove that the fixed-point index of A is zero, we need to generalize a relatively open subset Ω_ρ, introduced in [20], from $n = 1$ to $n > 1$. Assume that (P) holds. We define a continuous function $q : C([0, 1]; \mathbb{R}_+) \to \mathbb{R}_+$ by

$$q(x) = \min\{x(t) : t \in [a, b]\}$$

and a continuous function $q_n : P \to \mathbb{R}_+$ by

$$q_n(\mathbf{z}) = \max\{q(z_i) : i \in I_n\}.$$

Let $\rho > 0$. With c given in (P), we define a relatively open set by

$$\Omega_\rho = \{\mathbf{z} \in K : q_n(\mathbf{z}) < c\rho\}.$$

A similar relatively open subset was introduced in [10], where a larger cone is used.

The following result gives properties of Ω_ρ and generalizes Lemma 2.3 in [23] or Lemma 3.3 in [24] from $n = 1$ to $n > 1$.

LEMMA 3.10. *The set Ω_ρ defined above has the following properties*:

(i) *Ω_ρ is open relative to K;*

(ii) *$K_{c\rho} \subset \Omega_\rho \subset K_\rho$;*

(iii) *$\mathbf{z} \in \partial\Omega_\rho$ if and only if $\mathbf{z} \in K$ and $q_n(\mathbf{z}) = c\rho$, where $\partial\Omega_\rho$ denotes the boundary of Ω_ρ relative to K;*

(iv) *if $\mathbf{z} \in \partial\Omega_\rho$, then there exists $i \in I_n$ such that $q(z_i) = q_n(\mathbf{z}) = c\rho$ and*

$$c\rho \leqslant z_i(t) \leqslant \rho \quad \textit{for } t \in [a, b].$$

Proof. It is obvious that (a), (c) and the first inclusion of (b) hold. Let $\mathbf{z} \in \Omega_\rho$. Then $q_n(\mathbf{z}) < c\rho$ and $\mathbf{z} \in K$. By (2.3), we have $c\|z_i\| \leqslant q(z_i) < c\rho$ for all $i \in I_n$ and $\|\mathbf{z}\| < \rho$. This implies that the second inclusion of (b) holds. Let $\mathbf{z} \in \partial\Omega_\rho$. Then, by (c), $q_n(\mathbf{z}) = c\rho$ and there exists $i \in I_n$ such that $c\rho = q(z_i) \leqslant z_i(t) \leqslant \rho$ for $t \in [a, b]$. Hence, (d) holds. □

For convenience, we write

$$\mathbf{z} = (z_1, \dots, z_n) = (z_i, \hat{z}_i), \tag{3.8}$$

where $\hat{z}_i = (z_1, \dots, z_{i-1}, z_{i+1}, \dots, z_n)$.

The following result gives conditions that ensure that $i_K(A, \Omega_\rho) = 0$ and generalizes Lemma 2.5 in [19] and Lemma 2.6 in [20] from $n = 1$ to $n > 1$.

THEOREM 3.11. *Assume that (P) holds and there exists $\rho > 0$ such that $\mathbf{z} \neq A\mathbf{z}$ for $\mathbf{z} \in \partial\Omega_\rho$ and the following condition holds.*

$(H^0_{\geqslant})_{\psi_\rho}$ *For each $i \in I$, there exists a measurable function $\psi^i_\rho : [a, b] \to \mathbb{R}_+$ such that $\int_a^b \Phi(s) g_i(s) \psi^i_\rho(s)\, ds > 0$ and*

$$f_i(s, \mathbf{z}) \geqslant \psi^i_\rho(s) M_{\psi^i_\rho} c\rho \quad \textit{for a.e. } s \in [a, b] \textit{ and all } \mathbf{z} = (z_i, \hat{z}_i) \in [c\rho, \rho] \times [0, \rho]^{n-1}.$$

Then $i_K(A, \Omega_\rho) = 0$.

Proof. Let $\mathbf{e}(t) \equiv (1, \dots, 1) \in \mathbb{R}^n$ for $t \in [0, 1]$. We prove that

$$\mathbf{z} \neq A\mathbf{z} + \mu\mathbf{e} \quad \text{for } x \in \partial\Omega_\rho \text{ and } \mu \geqslant 0. \tag{3.9}$$

In fact, if not, then there exist $\mathbf{z} = (z_1, \dots, z_n) \in \partial\Omega_\rho$ and $\mu > 0$ such that $\mathbf{z} = A\mathbf{z} + \nu\mathbf{e}$. By Lemma 3.10(iv), there exists $i \in I_n$ such that $q(z_i) = q_n(\mathbf{z}) = c\rho$ and $c\rho \leqslant z_i(t) \leqslant \rho$ for $t \in [a,b]$. By $(H^0_{\geqslant})_{\psi_\rho}$, we have, for $t \in [a,b]$, that

$$\begin{aligned} z_i(t) &= \int_0^1 k(t,s)g_i(s)f_i(s,\mathbf{z}(s))\,ds + \nu \geqslant \int_a^b k(t,s)g_i(s)f_i(s,\mathbf{z}(s))\,ds + \nu \\ &\geqslant c\rho M_{\psi_\rho^i} \int_a^b k(t,s)g_i(s)\psi_\rho^i(s)\,ds + \nu \geqslant c\rho + \nu. \end{aligned}$$

This implies that $q(z_i) \geqslant c\rho + \nu > c\rho$, which contradicts $q(z_i) = c\rho$. It follows from (3.9) and Lemma 3.2(ii) that $i_K(A, \Omega_\rho) = 0$. □

Similar to Theorems 3.8 and 3.9, the characteristic value μ_1 can be employed to show that the fixed-point index of A is zero.

THEOREM 3.12. *Assume that $\gamma(0,1) > 0$ and the following condition holds.*

$((f_i)_0)_{\mu_1}$ *There exist $\varepsilon > 0$ and $\rho_0 > 0$ such that, for each $i \in I_n$, we have*

$$f_i(s,\mathbf{z}) \geqslant (\mu_1 + \varepsilon)z_i \quad \text{for a.e. } s \in [0,1] \text{ and all } \mathbf{z} \in \mathbb{R}^n_+ \text{ with } |\mathbf{z}| \in [0,\rho_0]. \tag{3.10}$$

Then, for each $\rho \in (0,\rho_0]$, if $\mathbf{z} \neq A\mathbf{z}$ for $\mathbf{z} \in \partial K_\rho$, then $i_K(A,K_\rho) = 0$.

Proof. Let $\rho \in (0,\rho_0]$. We prove that

$$\mathbf{z} \neq A\mathbf{z} + \nu\varphi_1 \quad \text{for all } \mathbf{z} \in \partial K_\rho \text{ and } \nu > 0, \tag{3.11}$$

where $\varphi_1 \in K \setminus \{0\}$ with $\|\varphi_1\| = 1$ and $\varphi_1 = \mu_1 L\varphi_1$. In fact, if not, then there exist $\mathbf{z} \in \partial K_\rho$ and $\nu > 0$ such that $\mathbf{z} = A\mathbf{z} + \nu\varphi_1$. This implies that $\mathbf{z} \geqslant \nu\varphi_1$. Let $\tau_1 = \sup\{\omega > 0 : \mathbf{z} \geqslant \omega\varphi_1\}$. Then $0 < \nu \leqslant \tau_1 < \infty$ and

$$\mathbf{z} \geqslant \tau_1\varphi_1. \tag{3.12}$$

By (3.10) and (3.12), we have, for $i \in I_n$ and $t \in [0,1]$, that

$$\begin{aligned} z_i(t) &= \int_0^1 k(t,s)g_i(s)f_i(s,\mathbf{z}(s))\,ds + \nu(\varphi_1)_i(t) \geqslant \int_0^1 k(t,s)g_i(s)f_i(s,\mathbf{z}(s))\,ds \\ &\geqslant \int_0^1 k(t,s)g_i(s)(\mu_1+\varepsilon)z_i(s)\,ds \geqslant (\mu_1+\varepsilon)\tau_1 \int_0^1 k(t,s)g_i(s)(\varphi_1)_i(s)\,ds \\ &= ((\mu_1+\varepsilon)\tau_1/\mu_1)(\varphi_1)_i(t) \end{aligned}$$

and $\mathbf{z} \geqslant ((\mu_1+\varepsilon)\tau_1/\mu_1)\varphi_1$. By (3.12), we have $\tau_1 \geqslant (\mu_1+\varepsilon)\tau_1/\mu_1 > \tau_1$, which is a contradiction. The result follows from (3.11) and Lemma 3.2(ii). □

As a special case of Theorem 3.12, we obtain the following result that generalizes Theorem 3.4 in [**40**] from $n = 1$ to $n > 1$.

COROLLARY 3.13. *Assume that $\gamma(0,1) > 0$ and the following condition holds*:

$$\mu_1 < (f_i)_0 \leqslant \infty \quad \text{for each } i \in I_n. \tag{3.13}$$

Then there exists $\rho_0 > 0$ such that, for each $\rho \in (0,\rho_0]$, if $\mathbf{z} \neq A\mathbf{z}$ for $\mathbf{z} \in \partial K_\rho$, then i_K $(A,K_\rho) = 0$.

Proof. Since $\mu_1 < (f_i)_0 \leqslant \infty$ for each $i \in I_n$, there exist $\varepsilon > 0$ and $\rho_0 > 0$ such that, for each $i \in I_n$ and a.e. $s \in [0,1]$, we have

$$f_i(s,\mathbf{z}) \geqslant (\mu_1+\varepsilon)|\mathbf{z}| \geqslant (\mu_1+\varepsilon)|z_i| = (\mu_1+\varepsilon)z_i \quad \text{for all } \mathbf{z} \in \mathbb{R}^n_+ \text{ with } |\mathbf{z}| \in [0,\rho_0].$$

Hence, $((f_i)_0)_{\mu_1}$ holds. The result follows from Theorem 3.12. □

We shall see that Theorem 4.6 of Section 4 shows that $((f_i)_0)_{\mu_1}$ holds, but (3.13) may not hold.

To prove the following result, we need to use (P^*) and Theorem 2.2.

THEOREM 3.14. *Assume that $\gamma(0,1) > 0$, (P^*) and the following condition holds.*

$((f_i)_\infty)_{\mu_1}$ *There exist $\varepsilon > 0$ and $\rho_0 > 0$ such that, for each $i \in I_n$, we have*

$$f_i(s,\mathbf{z}) \geqslant (\mu_1+\varepsilon)z_i \quad \text{for a.e. } s \in [0,1] \text{ and all } \mathbf{z} \in \mathbb{R}^n_+ \text{ with } |\mathbf{z}| \geqslant \rho_0. \tag{3.14}$$

Then there exists $\rho_1 \geqslant \rho_0$ such that, for each $\rho \geqslant \rho_1$, if $\mathbf{z} \neq A\mathbf{z}$ for $\mathbf{z} \in \partial K_\rho$, then $i_K(A,K_\rho) = 0$.

Proof. By Theorem 2.2, $\mu_1 \in (0,\infty)$ and there exist $m^* \geqslant 2$ and $\varphi_m \in K$ with $\|\varphi_m\| = 1$ such that $\mu_m \in (0,\infty)$ for $m \geqslant m^*$, $\mu_m L_m \varphi_m = \varphi_m$ and $\mu_m \to \mu_1$. Moreover, there exist $\varepsilon_0 > 0$ and $m_0 \geqslant m^*$ such that, for each $i \in I_n$, we have

$$f_i(s,\mathbf{z}) \geqslant (\mu_{m_0}+\varepsilon_0)z_i \quad \text{for a.e. } s \in [0,1] \text{ and all } \mathbf{z} \in \mathbb{R}^n_+ \text{ with } |\mathbf{z}| \geqslant \rho_0. \tag{3.15}$$

By (P^*), we have $c_{m_0} = c(a_{m_0}, b_{m_0}) > 0$. Let $\rho \geqslant \rho_0/c_{m_0}$. We prove that

$$\mathbf{z} \neq A\mathbf{z} + \nu\varphi_{m_0} \quad \text{for } \mathbf{z} \in \partial K_\rho \text{ and } \nu > 0. \tag{3.16}$$

In fact, if not, then there exist $\mathbf{z} \in \partial K_\rho$ and $\nu > 0$ such that

$$\mathbf{z}(t) = A\mathbf{z}(t) + \nu\varphi_{m_0}(t) \quad \text{for } t \in [0,1]. \tag{3.17}$$

Then $\mathbf{z} \geqslant \nu\varphi_{m_0}$. Let $\tau = \sup\{\omega > 0 : \mathbf{z} \geqslant \omega\varphi_{m_0}\}$. Then $\tau \geqslant \nu > 0$ and

$$\mathbf{z} \geqslant \tau\varphi_{m_0}. \tag{3.18}$$

Since $\mathbf{z} \in \partial K_\rho$, we have, for each $i \in I_n$ and $s \in [a_{m_0}, b_{m_0}]$, that

$$z_i(s) \geqslant C(s)\|z_i\| \geqslant c_{m_0}\|z_i\|.$$

Hence, we obtain

$$|\mathbf{z}(s)| \geqslant c_{m_0}\|\mathbf{z}\| = c_{m_0}\rho \geqslant \rho_0 \quad \text{for } s \in [a_{m_0}, b_{m_0}].$$

This, together with (3.15), implies that

$$f_i(s,\mathbf{z}(s)) \geqslant (\mu_{m_0}+\varepsilon_0)z_i(s) \quad \text{for a.e. } s \in [a_{m_0}, b_{m_0}]. \tag{3.19}$$

By (3.17)–(3.19), we have, for $i \in I_n$ and $t \in [0,1]$, that

$$\begin{aligned} z_i(t) &\geqslant \int_{a_{m_0}}^{b_{m_0}} k(t,s)g_i(s)f_i(s,\mathbf{z}(s))\,ds \geqslant \int_{a_{m_0}}^{b_{m_0}} k(t,s)g_i(s)(\mu_{m_0}+\varepsilon_0)z_i(s)\,ds \\ &\geqslant ((\mu_{m_0}+\varepsilon_0)\tau/\mu_{m_0})(\varphi_{m_0})_i(t) \end{aligned}$$

and $\mathbf{z} \geqslant ((\mu_{m_0}+\varepsilon_0)\tau/\mu_{m_0})\varphi_m$. By (3.18), we have $\tau \geqslant (\mu_{m_0}+\varepsilon_0)\tau/\mu_{m_0} > \tau$, which is a contradiction. The result follows from (3.16) and Lemma 3.2(ii). □

As a special case of Theorem 3.14, the following result generalizes Theorem 3.8 in [**40**], which uses the uniqueness of positive eigenvalues and the permanence property.

COROLLARY 3.15. *Assume that* $\gamma(0,1) > 0$ *and* (P^*) *hold and*

$$\mu_1 < (f_i)_\infty \leqslant \infty \quad \text{for } i \in I_n.$$

Then there exists $\rho_1 > 0$ *such that, for each* $\rho \geqslant \rho_1$, *if* $\mathbf{z} \neq A\mathbf{z}$ *for* $\mathbf{z} \in \partial K_\rho$, *then* $i_K(A, K_\rho) = 0$.

Now, we are in a position to consider the existence of positive solutions of (3.1). Using Lemma 3.2(iii), combining the results on the fixed-point index obtained above implies results on the existence of one or several positive solutions of (3.1). Here we only state a few of these results and omit the proofs. We refer to [**19**, **23**, **24**, **38**–**40**] for some related results.

THEOREM 3.16. (i) *Assume that* (P) *and one of the following conditions holds*:

(H_1) *there exist* $\rho_1, \rho_2 > 0$ *with* $\rho_1 < c\rho_2$ *such that* $(H^1_{\leqslant})_{\phi_{\rho_1}}$ *and* $(H^0_{\geqslant})_{\psi_{\rho_2}}$ *hold*;

(H_2) *there exist* $\rho_1, \rho_2 > 0$ *with* $\rho_1 < \rho_2$ *such that* $(H^0_{\geqslant})_{\psi_{\rho_1}}$ *and* $(H^1_{\leqslant})_{\phi_{\rho_2}}$ *hold.*

Then (3.1) *has a solution* $x \in K$ *with* $\rho_1 \leqslant \|x\| \leqslant \rho_2$.

(ii) *Assume that* $\gamma(0,1) > 0$ *and one of the following conditions holds*:

(H_3) *for* $i \in I_n$, *we have that* $((f_i)_0)_{\mu_1}$ *and* $((f_i)^\infty)_{\mu_1}$ *hold*;

(H_4) *for* $i \in I_n$, *we have that* $((f_i)^0)_{\mu_1}$, $((f_i)_\infty)_{\mu_1}$ *and* (P^*) *hold.*

Then (3.1) *has a nonzero positive solution in* K.

When $n = 2$, Theorem 3.16(H_3) or (H_4) improves Remarks 1.6 or 1.7 in [**5**], where k_i is symmetric and the superlinear or sublinear conditions are stronger than those of (H_3) or (H_4), respectively.

THEOREM 3.17. (i) *Assume that* (P) *and one of the following conditions holds*:

(S_1) *there exist* $\rho_1, \rho_2, \rho_3 \in (0, \infty)$ *with* $\rho_1 < c\rho_2$ *and* $\rho_2 < \rho_3$ *such that* $(H^1_{\leqslant})_{\phi_{\rho_1}}$, $(H^0_{\geqslant})_{\psi_{\rho_2}}$, $x \neq Ax$ *for* $x \in \partial\Omega_{\rho_2}$ *and* $(H^1_{\leqslant})_{\phi_{\rho_3}}$ *hold*;

(S_2) *there exist* $\rho_1, \rho_2, \rho_3 \in (0, \infty)$ *with* $\rho_1 < \rho_2 < c\rho_3$ *such that* $(H^0_{\geqslant})_{\psi_{\rho_1}}$, $(H^1_{\leqslant})_{\phi_{\rho_2}}$, $x \neq Ax$ *for* $x \in \partial K_{\rho_2}$ *and* $(H^0_{\geqslant})_{\psi_{\rho_3}}$ *hold.*

Then (3.1) *has two nonzero solutions in* K. *Moreover, in* (S_1), *if* $(H^1_{\leqslant})_{\phi_{\rho_1}}$ *is replaced by* $(H^1_{<})_{\phi_{\rho_1}}$, *then* (3.1) *has the third solution* $x_0 \in K_{\rho_1}$.

(ii) *Assume that* $\gamma(0,1) > 0$ *and one of the following conditions holds*:

(S_3) *assume that* $((f_i)^0)_{\mu_1}$, $((f_i)^\infty)_{\mu_1}$ *and* (P) *hold and there exists* $\rho \in (0, \infty)$ *such that* $(H^0_{\geqslant})_{\psi_\rho}$ *holds and* $x \neq Ax$ *for* $x \in \partial\Omega_\rho$;

(S_4) *assume that* $((f_i)_0)_{\mu_1}$, $((f_i)_\infty)_{\mu_1}$ *and* (P^*) *hold and there exists* $\rho \in (0, \infty)$ *such that* $(H^1_{\leqslant})_{\phi_\rho}$ *holds and* $x \neq Ax$ *for* $x \in \partial K_\rho$;

(S_5) *assume that* $((f_i)_0)_{\mu_1}$ *and* (P) *hold and there exist* $\rho_2, \rho_3 \in (0, \infty)$ *with* $\rho_2 < c\rho_3$ *such that* $(H^1_{\leqslant})_{\phi_{\rho_2}}$, $x \neq Ax$ *for* $x \in \partial K_{\rho_2}$ *and* $(H^0_{\geqslant})_{\psi_{\rho_3}}$ *hold.*

Then (3.1) *has two nonzero solutions in* K.

4. *Fractional differential equations*

In this section, we apply the results obtained in Section 3 to study the existence of positive solutions of systems of fractional differential equations of the form

$$-D^\alpha z_i(t) = g_i(t) f_i(t, \mathbf{z}(t)) \quad \text{for a.e. } t \in [0,1] \tag{4.1}$$

subject to the following two-point boundary condition:

$$z_i(0) = 0, \quad \gamma z_i(1) + \delta z_i'(1) = 0, \tag{4.2}$$

where $i \in I_n$, $\mathbf{z}(t) = (z_1(t), \ldots, z_n(t))$, $\gamma, \delta \geqslant 0$ with $\gamma + \delta > 0$, $1 < \alpha < 2$ and D^α is the Riemann–Liouville differential operator of order α, namely,

$$D^\alpha w(t) = \frac{1}{\Gamma(2-\alpha)} \frac{d^2}{dt^2} \int_0^t \frac{w(s)}{(t-s)^{\alpha-1}}\, ds. \tag{4.3}$$

When $n = 1$, the existence of one or three positive solutions of (4.1) and (4.2) with $\delta = 0$ or $\gamma = 0$ was studied by Bai and Lü [**4**] and Kaufmann and Mboumi [**14**], respectively. We refer to [**7**, **8**, **17**, **18**, **29**, **35**, **41**, **44**, **45**] and the references therein for other boundary conditions and other order α.

The boundary condition (4.2) is a special case of the well-known general separated boundary conditions that have been widely studied, for example, in [**19**, **20**, **40**]. Because there is difficulty in deriving the Green's function subject to the general separated boundary conditions, we work only on (4.2).

The following new result provides the Green's function subject to (4.2) that generalizes Lemma 2.3 in [**4**], where $\delta = 0$, and Lemma 2.3 in [**14**], where $\gamma = 0$.

Lemma 4.1. *Let $1 < \alpha < 2$, $\gamma, \delta \geqslant 0$ with $\gamma + \delta > 0$ and $\beta = (\alpha-1)\delta/[\gamma + (\alpha-1)\delta]$. Let $y : (0,1) \to \mathbb{R}$ be measurable such that $\int_0^1 s^{\alpha-1}(1-s)^{\alpha-2}(1+\beta s - s)y(s)\, ds < \infty$. Then the boundary value problem*

$$-D^\alpha w(t) = y(t),$$
$$w(0) = 0, \quad \gamma w(1) + \delta w'(1) = 0$$

has a unique solution

$$w(t) = \int_0^1 k(t,s) y(s)\, ds,$$

where $k : [0,1] \times [0,1) \to \mathbb{R}$ is defined by

$$k(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1}(1-s)^{\alpha-2}(1+\beta s - s) - (t-s)^{\alpha-1} & \text{if } s \leqslant t, \\ t^{\alpha-1}(1-s)^{\alpha-2}(1+\beta s - s) & \text{if } t < s. \end{cases} \tag{4.4}$$

Proof. It is well known that, if $-D^\alpha w(t) = y(t)$, then we have, for $t \in (0,1]$, that

$$w(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\, ds - C_1 t^{\alpha-1} - C_2 t^{\alpha-2} \quad \text{for } C_1, C_2 \in \mathbb{R}; \tag{4.5}$$

see, for example, [**4**, Lemma 2.2]. Since $w(0) = 0$, $\alpha - 1 > 0$ and $\alpha - 2 < 0$, it follows from (4.5) that $C_2 = 0$ and

$$w(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\, ds - C_1 t^{\alpha-1} \quad \text{for } t \in [0,1] \text{ and } C_1 \in \mathbb{R}. \tag{4.6}$$

Hence, we have

$$w(1) = -\frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} y(s)\, ds - C_1,$$
$$w'(t) = -\frac{(\alpha-1)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-2} y(s)\, ds - (\alpha-1) C_1 t^{\alpha-2}$$

and

$$w'(1) = -\frac{(\alpha-1)}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-2} y(s)\, ds - (\alpha-1) C_1.$$

Let $\varsigma = \gamma + (\alpha - 1)\delta$. Since $\gamma w(1) + \delta w'(1) = 0$, we obtain

$$\begin{aligned} C_1 &= -\frac{1}{\varsigma\Gamma(\alpha)}\left[\gamma\int_0^1 (1-s)^{\alpha-1} y(s)\,ds + (\alpha-1)\delta\int_0^1 (1-s)^{\alpha-2} y(s)\,ds\right] \\ &= -\frac{1}{\varsigma\Gamma(\alpha)}\int_0^1 [\gamma(1-s)^{\alpha-1} + (\alpha-1)\delta(1-s)^{\alpha-2}] y(s)\,ds \\ &= -\frac{1}{\varsigma\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-2}[\gamma - \gamma s + (\alpha-1)\delta] y(s)\,ds \\ &= -\frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-2}(1+\beta s - s) y(s)\,ds. \end{aligned}$$

This, together with (4.6), implies that, for $t \in [0,1]$, we have

$$\begin{aligned} w(t) = \frac{1}{\Gamma(\alpha)}\bigg\{&\int_0^t [t^{\alpha-1}(1-s)^{\alpha-2}(1+\beta s - s) - (t-s)^{\alpha-1}] y(s)\,ds \\ &+ \int_t^1 t^{\alpha-1}(1-s)^{\alpha-2}(1+\beta s - s) y(s)\,ds\bigg\}. \end{aligned}$$

The result follows. □

It is obvious that $k : [0,1] \times [0,1) \to \mathbb{R}_+$ is continuous. To prove that k satisfies (C_1)(ii) under suitable conditions, we first give the following result.

LEMMA 4.2. *Let $\delta > 0$, $\gamma > (2-\alpha)\delta$ and $s_0 = 1 - [(2-\alpha)\delta/\gamma]$. Then*

$$g(s) := \frac{(1-s)^{2-\alpha}}{1+\beta s - s} \leqslant g(s_0) = \frac{\gamma + (\alpha-1)\delta}{\gamma + (2\alpha-3)\delta}\left[\frac{(2-\alpha)\delta}{\gamma}\right]^{2-\alpha} < 1 \quad \textit{for } s \in [0,1].$$

Proof. It is easy to verify that, for $s \in [0,1)$, we have

$$g'(s) = -\frac{(1-\beta)(\alpha-1)(1-s)^{1-\alpha}}{(1+\beta s - s)^2}(s - s^*) = -\frac{(1-\beta)(\alpha-1)(1-s)^{1-\alpha}}{(1+\beta s - s)^2}(s - s_0),$$

where $s^* = (\alpha - \beta - 1)/(1-\beta)(\alpha - 1) = s_0$. Since $\alpha > 1$ and $\gamma > (2-\alpha)\delta$, it follows that $s^* > 0$ and

$$1 - s^* = \frac{(2-\alpha)\beta}{(1-\beta)(\alpha-1)} > 0, \quad \frac{\gamma + (\alpha-1)\delta}{\gamma + (2\alpha-3)\delta} < 1 \quad \text{and} \quad \left[\frac{(2-\alpha)\delta}{\gamma}\right]^{2-\alpha} < 1.$$

Hence, $g(s) \leqslant g(s^*) = g(s_0) < 1$ for $s \in (0,1)$. □

Let

$$\Phi(s) = \frac{1}{\Gamma(\alpha)} s^{\alpha-1}(1-s)^{\alpha-2}(1+\beta s - s) \quad \text{for } s \in [0,1) \tag{4.7}$$

and

$$C(t) = t^{\alpha-1}[1 - g(s_0)] \quad \text{for } t \in [0,1], \tag{4.8}$$

where $g(s_0)$ is the same as in Lemma 4.2.

The following new result shows that k, Φ and C defined in (4.4), (4.7) and (4.8) satisfy (C_1)(ii).

LEMMA 4.3. *The kernel k defined in (4.4) has the following properties:*

(i) *$k(t,s) \leqslant \Phi(s)$ for $t \in [0,1]$ and $s \in [0,1)$;*

(ii) *If $\delta > 0$ and $\gamma > (2-\alpha)\delta$, then*

$$k(t,s) \geqslant C(t)\Phi(s) \quad \text{for } t \in [0,1] \text{ and } s \in [0,1). \tag{4.9}$$

Proof. (i) It is obvious that $k(t,s) \leqslant k(s,s) = \Phi(s)$ for $t \leqslant s$. Let $s \in [0,1)$ and

$$h(t) = t^{\alpha-1}(1-s)^{\alpha-2}(1+\beta s - s) - (t-s)^{\alpha-1} \quad \text{for } t \in (s,1).$$

We rewrite h as follows:

$$h(t) = t^{\alpha-1}[(1-\beta)(1-s)^{\alpha-1} + \beta 1 - s)^{\alpha-2}] - (t-s)^{\alpha-1} \quad \text{for } t \in (s,1).$$

Then we have, for $t \in (s,1)$, that

$$\begin{aligned} h'(t) &= (\alpha-1)t^{\alpha-2}[(1-\beta)(1-s)^{\alpha-1} + \beta(1-s)^{\alpha-2}] - (\alpha-1)(t-s)^{\alpha-2} \\ &= \frac{(\alpha-1)}{(t-s)^{2-\alpha}}\left\{(1-\beta)(1-s)^{\alpha-1}(1-s/t)^{2-\alpha} + \beta\left[\frac{t-s}{t(1-s)}\right]^{2-\alpha} - 1\right\} \\ &\leqslant \frac{(\alpha-1)}{(t-s)^{2-\alpha}}[(1-\beta)+\beta-1] = 0. \end{aligned}$$

Hence, h is decreasing on $(s,1)$ and $h(t) \leqslant h(s) = \Gamma(\alpha)\Phi(s)$ for $t \in (s,1)$. It follows that $k(t,s) = (1/\Gamma(\alpha))h(t) \leqslant \Phi(s)$ for $s \leqslant t$.

(ii) If $t < s$, then, since $s^{\alpha-1} \leqslant 1$ for $s \in [0,1]$, we have by (4.4) that

$$k(t,s) = \frac{1}{\Gamma(\alpha)}t^{\alpha-1}(1-s)^{\alpha-2}(1+\beta s - s) \geqslant t^{\alpha-1}\Phi(s) \geqslant C(t)\Phi(s).$$

If $s \leqslant t$, then, by Lemma 4.2, we obtain

$$\begin{aligned} k(t,s) &= \frac{1}{\Gamma(\alpha)}[t^{\alpha-1}(1-s)^{\alpha-2}(1+\beta s - s) - (t-s)^{\alpha-1}] \\ &\geqslant \frac{1}{\Gamma(\alpha)}[t^{\alpha-1}(1-s)^{\alpha-2}(1+\beta s - s) - t^{\alpha-1}] \\ &= \frac{1}{\Gamma(\alpha)}t^{\alpha-1}(1-s)^{\alpha-2}(1+\beta s - s)[1-g(s)] \\ &\geqslant t^{\alpha-1}[1-g(s_0)]\Phi(s) = C(t)\Phi(s). \end{aligned}$$

It follows that $k(t,s) \geqslant C(t)\Phi(s)$ for $t \in [0,1]$ and $s \in [0,1)$. □

Even when $\delta = 0$ or $\gamma = 0$, it seems difficult to find a suitable function $C(t)$ such that (4.9) holds.

In order to apply results in the above section, one needs to compute some of the following three values:

$$m_{\phi_\rho^i},\ M_{\psi_\rho^i} \text{ and } \mu_1.$$

When $\alpha = 2$ and all of these functions ϕ_ρ^i, ψ_ρ^i and g_i are 1, these constants have been widely studied, for example, in [**26**, **40**] and the references therein. If $1 < \alpha < 2$, then, even when these functions are 1, it may not be easy to determine the second or third value or find formulas for these values. However, when $\phi_\rho^i = g_i \equiv 1$, we can provide a formula for the first value and give an upper bound for the second value under suitable assumptions. If $\gamma(0,1) > 0$, then it follows from Theorem 2.1 that μ_1 exists. We do not know the exact value of μ_1, even when $\delta = 0$ or $\gamma = 0$. When $\alpha = 2$, we refer to [**40**] for the exact value of μ_1 and its estimates.

Let

$$m^* = \left(\max_{0\leqslant t\leqslant 1}\int_0^1 k(t,s)\,ds\right)^{-1} \quad \text{and} \quad M^*(a,b) = \left(\min_{a\leqslant t\leqslant b}\int_a^b k(t,s)\,ds\right)^{-1}. \tag{4.10}$$

LEMMA 4.4. *(i) We have*

$$m^* = \frac{\alpha^{\alpha+1}\Gamma(\alpha)}{(\alpha-1)^{\alpha-1}}\left[\frac{\gamma+(\alpha-1)\delta}{\gamma+\alpha\delta}\right]^{\alpha}.$$

(ii) *Let $a \in (0,1)$ and $\omega(a) = \int_a^1 (1-s)^{\alpha-2}(1+\beta s - s)\,ds$. Then*

$$M(a,1) \leqslant \frac{\Gamma(a)}{\min\left\{a^{\alpha-1}\omega(a), \omega(a) - \dfrac{(1-a)^{\alpha}}{\alpha}\right\}}.$$

Proof. (i) Let $h(t) = \Gamma(\alpha)\int_0^1 k(t,s)\,ds$ for $t \in [0,1]$. By (4.4), we have, for $t \in [0,1]$, that

$$\begin{aligned} h(t) &= \int_0^1 t^{\alpha-1}(1-s)^{\alpha-2}(1+\beta s - s)\,ds - \int_0^t (t-s)^{\alpha-1}\,ds \\ &= t^{\alpha-1}\int_0^1 [(1-\beta)(1-s)^{\alpha-1} + \beta(1-s)^{\alpha-2}]\,ds - \int_0^t (t-s)^{\alpha-1}\,ds \\ &= t^{\alpha-1}\left(\frac{1-\beta}{\alpha} + \frac{\beta}{\alpha-1}\right) - \frac{t^{\alpha}}{\alpha} = \frac{\alpha+\beta-1}{\alpha(\alpha-1)}t^{\alpha-1} - \frac{t^{\alpha}}{\alpha}. \end{aligned}$$

Let $t_0 = (\alpha+\beta-1)/\alpha$. Then $t_0 \in [0,1]$ and $h'(t_0) = 0$. Hence,

$$h(t) \leqslant h(t_0) = \frac{t_0^{\alpha}}{\alpha(\alpha-1)} = \frac{(\alpha-1)^{\alpha-1}}{\alpha^{\alpha+1}}\left[\frac{\gamma+\alpha\delta}{\gamma+(\alpha-1)\delta}\right]^{\alpha} \quad \text{for } t \in [0,1].$$

It follows that

$$\max_{0\leqslant t\leqslant 1}\int_0^1 k(t,s)\,ds = \frac{1}{\Gamma(\alpha)}\frac{(\alpha-1)^{\alpha-1}}{\alpha^{\alpha+1}}\left[\frac{\gamma+\alpha\delta}{\gamma+(\alpha-1)\delta}\right]^{\alpha},$$

and the result holds.

ii Let $g(t) = \Gamma(\alpha)\int_a^1 k(t,s)\,ds$ for $t \in [a,1]$. Then we have, for $t \in [a,1]$, that

$$g(t) = t^{\alpha-1}\omega(a) - \int_a^t (t-s)^{\alpha-1}\,ds = t^{\alpha-1}\omega(a) - \frac{(t-a)^{\alpha}}{\alpha}$$

and

$$g''(t) = -(\alpha-1)(2-\alpha)t^{\alpha-3}\omega(a) - (\alpha-1)(t-a)^{\alpha-2} \leqslant 0.$$

Hence, g is concave down on $[a,1]$ and

$$g(t) \geqslant \min\{g(a), g(1)\} = \min\left\{a^{\alpha-1}\omega(a), \omega(a) - \frac{(1-a)^{\alpha}}{\alpha}\right\} \quad \text{for } t \in [a,1].$$

The result follows. □

In the following, we always assume that $\delta > 0$ and $\gamma > (2-\alpha)\delta$. By Lemma 4.3, (C_1) in Section 2 holds. By Lemma 4.2, $g(s_0) \in (0,1)$. By (4.8), we have $C(0) = 0$, $C(t) > 0$ for $t \in (0,1]$ and $\|C\| \in (0,1)$. Hence, for $a, b \in (0,1]$ with $a < b$, (P) holds and thus (P^*) holds.

We assume that $\{g_i\}$ and $\{f_i\}$ in (4.1) satisfy (C_2) and (C_3) with k defined in (4.4) and (C_4), respectively.

With C given by (4.8), the cone K defined in (2.3) is reproducing since $\|C\| < 1$. In this section, we always use the cone K defined in (2.3) with C given in (4.8).

By Lemma 4.1, equations (4.1) and (4.2) can be written as in (3.1) with k defined in (4.4). Hence, Theorems 3.16 and 3.17 hold for (4.1) and (4.2).

As applications of our results, we consider a system of fractional differential equations of the form

$$D^\alpha z_i(t) + \sum_{j=1}^{n} a_{ij}(t)(\operatorname{sgn} z_j)|z_j|^{\mu_{ij}} = 0 \quad \text{for a.e. } t \in [0,1] \text{ and } i \in I_n \tag{4.11}$$

subject to (4.2), where $1 < \alpha < 2$, $\delta > 0$ and $\gamma > (2-\alpha)\delta$.

When $\alpha = 2$, the above system with Dirichlet boundary conditions (that is, $\delta = 0$) was studied in [**12**], where $a_{ij} \in C([0,1], \mathbb{R}_+)$. Moreover, some results on the conjugacy of a second-order ordinary differential equation were employed to obtain a differential inequality that implies that a suitable fixed-point index is 0. In the following, we use Theorem 3.16(H_1), which is different from that used in [**12**] and allows $a_{ij} \in L^1(0,1)$.

THEOREM 4.5. *Let $i, j \in I_n$. Assume that the following conditions hold:*
 (i) $\mu_{ij} > 1$;
 (ii) $a_{ij} : (0,1) \to \mathbb{R}_+$ *is measurable and* $a_{ij}\Phi \in L^1(0,1)$;
 (iii) *there exist* $a, b \in (0,1]$ *with* $a < b$ *such that* $\int_a^b \Phi(s)a_{ii}(s)\,ds > 0$.
Then (4.11) *and* (4.2) *have a solution* $z \in K$ *with* $\|z\| > 0$.

Proof. For each $i \in I_n$, let $g_i \equiv 1$ and define a function $f_i : [0,1] \times \mathbb{R}^n_+ \to \mathbb{R}_+$ by

$$f_i(s, \mathbf{z}) = \sum_{j=1}^{n} a_{ij}(s) z_j^{\mu_{ij}}.$$

Let $\mu = \min\{\mu_{ij} : i, j \in I_n\}$, $\mathcal{M} = \max\{\sum_{j=1}^n \int_0^1 \Phi(s)a_{ij}(s)\,ds : i \in I_n\}$ and

$$0 < \rho_1 < \min\left\{1, \left(\frac{1}{\mathcal{M}}\right)^{1/(\mu-1)}\right\}.$$

Then $\rho_1^{\mu_{ij}-1} \leqslant \rho_1^{\mu-1}$ for $i, j \in I_n$. For each $i \in I_n$, we define $\phi^i_{\rho_1} : [0,1] \to \mathbb{R}_+$ by

$$\phi^i_{\rho_1}(s) = \sum_{j=1}^{n} a_{ij}(s)\rho_1^{\mu_{ij}-1}.$$

Then, by Lemma 4.3, we have, for $t \in [0,1]$, that

$$\begin{aligned}\int_0^1 k(t,s)\phi^i_{\rho_1}(s)\,ds &\leqslant \int_0^1 \Phi(s)\phi^i_{\rho_1}(s)\,ds \leqslant \rho_1^{\mu-1}\sum_{j=1}^{n}\int_0^1 \Phi(s)a_{ij}(s)\,ds \\ &\leqslant \rho_1^{\mu-1}\mathcal{M} < 1\end{aligned}$$

and $m_{\phi^i_{\rho_1}} > 1$. Hence, for a.e. $s \in [0,1]$ and $\mathbf{z} \in \mathbb{R}^n_+$ with $|\mathbf{z}| \leqslant \rho_1$, we have

$$f_i(s, \mathbf{z}) \leqslant \sum_{j=1}^{n} a_{ij}(s)\rho_1^{\mu_{ij}-1}\rho_1 = \phi^i_{\rho_1}(s)\rho_1 < \phi^i_{\rho_1}(s) m_{\phi^i_{\rho_1}}\rho_1$$

and $(H^1_<)_{\phi_{\rho_1}}$ holds.

Let $\mu_* = \min\{\mu_{ii} : i \in I_n\}$, $\mathcal{M}_* = \min\{\int_a^b \Phi(s)a_{ii}(s)\,ds : i \in I_n\}$ and $c := c(a,b) > 0$. Let

$$\rho_2 > \max\{\frac{1}{c}, (\frac{1}{c^{\mu_*}M_*})^{\frac{1}{\mu_*-1}}\}.$$

For each $i \in I_n$, we define $\psi^i_{\rho_2} : [0,1] \to \mathbb{R}_+$ by

$$\psi^i_{\rho_2}(s) = a_{ii}(s)(c\rho_2)^{\mu_{ii}-1}.$$

Then, for $t \in [a,b]$, we have

$$\int_a^b k(t,s)\psi^i_{\rho_2}(s)\,ds \geqslant c(c\rho_2)^{\mu_{ii}-1}\int_a^b \Phi(s)a_{ii}(s)\,ds \geqslant c(c\rho_2)^{\mu_*-1}\mathcal{M}_* > 1$$

and $M_{\psi^i_{\rho_2}} > 1$. Hence, for a.e. $s \in [0,1]$ and $\mathbf{z} = (z_i, \hat{z}_i) \in [c\rho_2, \rho_2] \times [0, \rho_2]^{n-1}$, we have

$$f_i(s,\mathbf{z}) \geqslant a_{ii}(s)z_i^{\mu_{ii}-1}z_i \geqslant \psi^i_{\rho_2}(s)(c\rho_2) > \psi^i_{\rho_2}(s)M_{\psi^i_{\rho_2}}(c\rho_2)$$

and $(H^0_{\geqslant})_{\psi_{\rho_2}}$ holds. The result follows from Theorem 3.16(H_1). □

Now, we consider the existence of two positive solutions of systems of fractional differential equations of the form

$$D^\alpha z_i(t) + \lambda(z_i^{\alpha_i}(t) + z_i^{\beta_i}(t))h_i(\hat{z}_i) = 0 \quad \text{for a.e. } t \in [0,1] \text{ and } i \in I_n \tag{4.12}$$

subject to (4.2), where $1 < \alpha < 2$, $\delta > 0$ and $\gamma > (2-\alpha)\delta$.

When $n = 1$ and $\alpha = 2$, we refer to [**22**, **31**, **32**] for similar equations arising from the steady flow of a power-law fluid over an impermeable, semi-infinite flat plane in boundary layer theory.

THEOREM 4.6. *Assume that the following conditions hold.*

(i) *For each $i \in I_n$, we have $1 < \alpha_i < \infty$ and $0 < \beta_i < 1$.*

(ii) *For each $i \in I_n$, we have that $h_i : \mathbb{R}^{n-1}_+ \to \mathbb{R}_+$ is continuous and*

$$\xi = \min\{h_i(\hat{z}_i) : \hat{z}_i \in \mathbb{R}^{n-1}_+ \text{ and } i \in I_n\} > 0.$$

Then there exists $\lambda_0 > 0$ such that, for each $\lambda \in (0, \lambda_0)$, (4.12) and (4.2) have two nonzero solutions in K.

Proof. Let $\rho_2 > 0$ and $\omega_i = \max\{h_i(\hat{z}_i) : \mathbf{z} \in \mathbb{R}^n_+ \text{ with } |\mathbf{z}| \in [0,\rho_2]\}$. Let m^* be the same as in Lemma 4.4 and

$$\lambda_0 := \lambda_0(\rho_2) = \min\left\{\frac{m^*}{\omega_i(\rho_2^{\alpha_i-1} + 1/\rho_2^{1-\beta_i})} : i \in I_n\right\}.$$

Let $\lambda \in (0,\lambda_0)$, $i \in I_n$ and $g_i \equiv 1$. We define a function $f_i : [0,1] \times \mathbb{R}^n_+ \to \mathbb{R}_+$ by

$$f_i(s,\mathbf{z}) = \lambda(z_i^{\alpha_i} + z_i^{\beta_i})h_i(\hat{z}_i).$$

Since $g_i \equiv 1$, we have $\gamma(0,1) = \int_0^1 \Phi(s)C(s)\,ds > 0$. Then, for $\mathbf{z} \in \mathbb{R}^n_+$ with $|\mathbf{z}| \in [0,\rho_2]$, we have that

$$f_i(s,\mathbf{z}) \leqslant \lambda(\rho_2^{\alpha_i} + \rho_2^{\beta_i})\omega_i = \lambda(\rho_2^{\alpha_i-1} + 1/\rho_2^{1-\beta_i})\omega_i\rho_2 < m^*\rho_2 \quad \text{for } s \in [0,1]$$

and $(H^1_<)_{\phi_{\rho_2}}$ with $\phi_{\rho_2} \equiv 1$ holds.

Let $\eta(x) = x^{\alpha_i-1} + 1/x^{1-\beta_i}$ for $x > 0$ and let $\rho^i = ((1-\beta_i)/(\alpha_i-1))^{1/(\alpha_i-\beta_i)}$ for $i \in I_n$. Then η is decreasing on $(0,\rho^i)$ and increasing on (ρ^i,∞) and satisfies $\lim_{x\to 0^+}\eta(x) = \lim_{x\to\infty}\eta(x) = \infty$. Let $\rho^* = \min\{\rho^i : i \in I_n\}$ and $\varepsilon > 0$. Since η is decreasing on $(0,\rho^*)$ and $\lim_{x\to 0^+}\eta(x) = \infty$, we can choose $0 < \rho_1 < \min\{\rho_2, \rho^*\}$ such that

$$\eta(\rho_1) = \rho_1^{\alpha_i-1} + 1/\rho_1^{1-\beta_i} \geqslant (\mu_1 + \varepsilon)/(\lambda\xi).$$

Then, for $i \in I_n$, $s \in [0,1]$ and $\mathbf{z} \in \mathbb{R}^n_+$ with $|\mathbf{z}| \in [0,\rho_1]$, we have

$$f_i(s,\mathbf{z}) = \lambda\eta(z_i)h_i(\hat{z}_i)z_i \geqslant \lambda\eta(\rho_1)\xi z_i \geqslant (\mu_1+\varepsilon)z_i.$$

Hence, $((f_i)_0)_{\mu_1}$ holds. Since η is increasing on (ρ^*,∞) and $\lim_{x\to\infty}\eta(x) = \infty$, we choose $\rho_3 > \rho^*/c$ satisfying $\lambda\eta(c\rho_3)\xi > M^*(a,b)$, where $M^*(a,b)$ is the same as in (4.10). Let

$\psi^i_{\rho_3}(s) \equiv \lambda\eta(c\rho_3)\xi$. Then

$$\int_a^b k(t,s)\psi^i_{\rho_3}(s)\,ds \geqslant \lambda\eta(c\rho_3)\xi/M^*(a,b) > 1 \quad \text{for } t \in [a,b]$$

and $M_{\psi^i_{\rho_3}} < 1$ for $i \in I_n$. Hence, for $s \in [a,b]$ and $\mathbf{z} = (z_i, \hat{z}_i) \in [c\rho_3, \rho_3] \times [0,\rho_3]^{n-1}$, we have

$$f_i(s,\mathbf{z}) = \lambda\eta(z_i)h_i(\hat{z}_i)z_i \geqslant \lambda\eta(c\rho_3)\xi(c\rho_3) = \psi^i_{\rho_3}(s)(c\rho_3) > \psi^i_{\rho_3}(s)M_{\psi^i_{\rho_3}}(c\rho_3)$$

and $(H^0_{\geqslant})_{\psi_{\rho_3}}$ holds. The result follows from Theorem 3.17(S_5). □

In Theorem 4.6, we proved that $((f_i)_0)_{\mu_1}$ holds. It may not be easy to show that the stronger condition (3.13) holds.

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