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Weak Hopf Algebras Corresponding to Borcherds–Cartan Matrices

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Abstract Let $\mathscr{G}$ be a generalized Kac–Moody algebra with an integral Borcherds–Cartan matrix. In this paper, we define a d-type weak quantum generalized Kac–Moody algebra $wU_q^d(\mathscr{G})$, which is a weak Hopf algebra. We also study the highest weight module over the weak quantum algebra $wU_q^d(\mathscr{G})$ and weak A-forms of $wU_q^d(\mathscr{G})$.

Keywords weak Hopf algebra, weak quantum generalized Kac–Moody algebra, highest weight module, weak A-form

MR(2000) Subject Classification 16W30, O153.3

1 Introduction

The concept of a weak Hopf algebra was first introduced by Li in [1]. A bialgebra H over a field k is called a weak Hopf algebra if there exists $T \in \mathrm{Hom}_k(H, H)$ such that $T * id * T = T$ and $id * T * id = id$, where T is called a weak antipode of H. Much work has been done on such weak Hopf algebras, see [1–6]. As is known, two typical examples of such weak Hopf algebras are the monoid algebra kS of a regular monoid S [1] and the almost quantum algebra $wsl_q(2)$ [2] (see also [5] for weak Hopf algebras corresponding to $U_q[sl_n]$). Recently, Yang has given a more nontrivial weak Hopf algebra $m_q^d(\mathscr{G})$ in [6], where $\mathscr{G}$ is a semi-simple Lie algebra. Following this idea, we will construct the more general weak Hopf algebra $wU_q^d(\mathscr{G})$, where $\mathscr{G}$ is a generalized Kac–Moody algebra. The main aim of the present paper is to study the structure and representation of $wU_q^d(\mathscr{G})$. The detailed outline of this paper is as follows.

In Section 2, we shortly review some basic concepts of the generalized Kac–Moody algebra, then we will focus on the generalization of $wU_q^d(\mathscr{G})$ by weakening the generators k_i and p_i $(i \in I)$, that is, exchanging their invertibility $k_i k_i^{-1} = p_i p_i^{-1} = 1$ to the regularity $K_i \overline{K}_i K_i = K_i$,

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$\overline{K}_i K_i \overline{K}_i = \overline{K}_i$, $D_i \overline{D}_i D_i = D_i$, $\overline{D}_i D_i \overline{D}_i = \overline{D}_i$. This leads to a weak Hopf algebra structure of $wU_q^d(\mathscr{G})$, which is studied in detail in Section 3. In Section 4 we will discuss the basis of $wU_q^d(\mathscr{G})$. In Section 5, we will define the highest weight module and Verma module over the weak quantum generalized Kac–Moody algebra $wU_q^d(\mathscr{G})$. Moreover, we study the corresponding weak A-form in Section 6. At the same time, we obtain some results, which are the natural generalization of the respective convention on the quantum enveloping algebra $U_q(\mathscr{G})$ (see [7]).

2 Weak Quantum Generalized Kac–Moody Algebra $wU_q^d(\mathscr{G})$

Throughout the paper, some notations and definitions unexplained here can be found in [7–9]. We assume the basic field is the complex number field $\mathbf{C}$. All algebras, modules and vector spaces are over $\mathbf{C}$ without being specified.

Let $I = \{1, 2, \ldots, n\}$, or $I = \mathbf{N}$, the natural number set. A real square matrix $A = (a_{ij})_{i,j \in I}$ is a Borcherds–Cartan matrix if it satisfies:

(1) $a_{ii} = 2$ or $a_{ii} \leq 0$ for all $i \in I$;

(2) $a_{ij} \leq 0$ if $i \neq j$;

(3) $a_{ij} \in \mathbf{Z}$ if $a_{ii} = 2$;

(4) $a_{ij} = 0$ if and only if $a_{ji} = 0$.

In this paper, we assume that all the entries of A are integers and the diagonal entries are even. Furthermore, we assume that A is symmetrizable, that is, there exists a diagonal matrix $D = \mathrm{diag}(s_i \in \mathbf{N}_{>0} | i \in I)$ such that DA is symmetric.

Let us introduce some useful concepts associated with generalized Kac–Moody algebras. Suppose $P^v = (\oplus_{i \in I} \mathbf{Z} h_i) \oplus (\oplus_{i \in I} \mathbf{Z} d_i)$, and let $\mathscr{H} = \mathbf{C} \otimes_{\mathbf{Z}} P^v$ be the complex vector space with basis $\{h_i, d_i\}_{i \in I}$. For $i \in I$, we define $\alpha_i \in \mathscr{H}^*$ by setting $\alpha_i(h_j) = a_{ji}$ and $\alpha_i(d_j) = \delta_{ji}$, where $\mathscr{H}^*$ is the dual space of $\mathscr{H}$. Furthermore, the weight lattice is defined to be

$$P = \{\lambda \in \mathscr{H}^* | \lambda(P^v) \subset \mathbf{Z}\}.$$

The quantum enveloping algebra $U_q(\mathscr{G})$ of a generalized Kac–Moody algebra $\mathscr{G}$ with a Borcherds–Cartan datum (A, P^v, P, Π, Π^v) is defined in [7, 9]. $U_q(\mathscr{G})$ is an associated algebra with unit 1 generated by the generators $e_i, f_i (i \in I)$ and q^h $(h \in P^v)$ with the relations:

$$q^0 = 1, q^{h_1} q^{h_2} = q^{h_1 + h_2}, \quad h_1, h_2 \in P^v, \tag{2.1}$$

$$q^h e_i q^{-h} = q^{\alpha_i(h)} e_i, \quad q^h f_i q^{-h} = q^{-\alpha_i(h)} f_i, \tag{2.2}$$

$$e_i f_j - f_j e_i = \delta_{ij} \frac{k_i - k_i^{-1}}{q_i - q_i^{-1}}, \text{ where } k_i = q^{s_i h_i}, \tag{2.3}$$

$$\sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1 - a_{ij} \\ r \end{bmatrix}_{q_i} e_i^{1-a_{ij}-r} e_j e_i^r = 0, \text{ if } a_{ii} = 2, \ i \neq j, \tag{2.4}$$

$$\sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1 - a_{ij} \\ r \end{bmatrix}_{q_i} f_i^{1-a_{ij}-r} f_j f_i^r = 0, \text{ if } a_{ii} = 2, \ i \neq j, \tag{2.5}$$

$$e_i e_j - e_j e_i = 0, \quad f_i f_j - f_j f_i = 0, \text{ if } a_{ij} = 0, \tag{2.6}$$

where $q_i = q^{s_i}$ and

$$\left[\begin{array}{c} m \\ n \end{array}\right]_{q_i} = \frac{(q_i^m - q_i^{-m})(q_i^{m-1} - q_i^{-(m-1)}) \cdots (q_i^{m-n+1} - q_i^{-(m-n+1)})}{(q_i - q_i^{-1})(q_i^2 - q_i^{-2}) \cdots (q_i^n - q_i^{-n})}, \quad m > n > 0.$$

Since the basis of $\mathscr{H}$ is $\{h_i, d_i\}_{i\in I}$, then the generators of $U_q(\mathscr{G})$ can be written as $e_i, f_i, k_i^{\pm 1}$ and $p_i^{\pm 1}$, where $k_i = q^{s_i h_i}$ and $p_i = q^{s_i d_i}$. To generalize the invertibility condition (2.1), we introduce a projector J to weaken the invertibility to regularity, replacing $\{k_i, k_i^{-1}\}$ (resp. $\{p_i, p_i^{-1}\}$) by a pair $\{K_i, \overline{K}_i\}$ (resp. $\{D_i, \overline{D}_i\}$) for all $i \in I$ subject to some relations:

$$J = K_i\overline{K}_i = \overline{K}_iK_i = D_i\overline{D}_i = \overline{D}_iD_i, \tag{2.7}$$

$$JK_i = K_iJ = K_i, \quad J\overline{K}_i = \overline{K}_iJ = \overline{K}_i, \tag{2.8}$$

$$JD_i = D_iJ = D_i, \quad J\overline{D}_i = \overline{D}_iJ = \overline{D}_i. \tag{2.9}$$

To generalize other relations of the definition of $U_q(\mathscr{G})$, we need some terminology. If E_i satisfies

$$K_jE_i = q_i^{a_{ij}}E_iK_j, \quad E_i\overline{K}_j = q_i^{a_{ij}}\overline{K}_jE_i, \quad D_jE_i = q_i^{\delta_{ij}}E_iD_j, \quad E_i\overline{D}_j = q_i^{\delta_{ij}}\overline{D}_jE_i, \tag{2.10}$$

for all $j \in I$, we say E_i is of type 1. However, if E_i satisfies

$$K_jE_i\overline{K}_j = q_i^{a_{ij}}E_i, \quad D_jE_i\overline{D}_j = q_i^{\delta_{ij}}E_i, \tag{2.11}$$

for all $j \in I$, we say E_i is of type 2. The same convention holds for F_i by replacing E_i with F_i and a_{ij} (resp. δ_{ij}) with $-a_{ij}$ (resp. $-\delta_{ij}$) in the above relations.

We borrow some notions from [5–6], E_i and $F_i (i \in I)$ are listed by starting with E_i followed by F_i, where a 1 indicates the use of a type 1 generator and a 0 indicates the use of a type 2 generator. Then write d in terms of its binary expansion,

$$d = (\{c_i\}_{i\in I} | \{\overline{c}_i\}_{i\in I}),$$

where the bar seperates the values representing the E_i and F_i, and where the c_i and $\overline{c}_i$ have values of either 0 or 1. Accordingly, we say E_i and F_i are of type d in an obvious sense.

Definition 2.1 *The algebra $wU_q^d(\mathscr{G})$ is generated by the generators $E_i, F_i, K_i, \overline{K}_i, D_i, \overline{D}_i$ and J satisfying* (2.7)–(2.9) *along with the relations: For all $i, j \in I$,*

$$K_iK_j = K_jK_i, \quad K_i\overline{K}_j = \overline{K}_jK_i, \quad \overline{K}_i\overline{K}_j = \overline{K}_j\overline{K}_i, \tag{2.12}$$

$$D_iD_j = D_jD_i, \quad D_i\overline{D}_j = \overline{D}_jD_i, \quad \overline{D}_i\overline{D}_j = \overline{D}_j\overline{D}_i, \tag{2.13}$$

$$D_iK_j = K_jD_i, \quad D_i\overline{K}_j = \overline{K}_jD_i, \quad \overline{D}_iK_j = K_j\overline{D}_i, \quad \overline{D}_i\overline{K}_j = \overline{K}_j\overline{D}_i, \tag{2.14}$$

$$E_i, F_i \text{ are type } d, \tag{2.15}$$

$$E_iF_j - F_jE_i = \delta_{ij}\frac{K_i - \overline{K}_i}{q_i - q_i^{-1}}, \tag{2.16}$$

$$\sum_{r=0}^{1-a_{ij}} (-1)^r \left[\begin{array}{c} 1-a_{ij} \\ r \end{array}\right]_{q_i} E_i^{1-a_{ij}-r}E_jE_i^r = 0, \ \text{if } a_{ii} = 2, \ i \neq j, \tag{2.17}$$

$$\sum_{r=0}^{1-a_{ij}} (-1)^r \left[\begin{array}{c} 1-a_{ij} \\ r \end{array}\right]_{q_i} F_i^{1-a_{ij}-r}F_jF_i^r = 0, \ \text{if } a_{ii} = 2, \ i \neq j, \tag{2.18}$$

$$E_iE_j = E_jE_i, \quad F_iF_j = F_jF_i, \text{ if } a_{ij} = 0. \tag{2.19}$$

The algebra $wU_q^d(\mathscr{G})$ is called a d-type weak quantum generalized Kac–Moody algebra.

There are some properties for $wU_q^d(\mathscr{G})$ which will be used later.

Proposition 2.2 *The idempotent J satisfies $aJ = Ja$, for any $a \in wU_q^d(\mathscr{G})$.*

Proof It is obvious that $aJ = Ja$ for $a = K_i, \overline{K}_i, D_i, \overline{D}_i$ or J.

If E_i is of type 1, then we obtain, from (2.10),

$$JE_i = (\overline{K}_jK_j)E_i = \overline{K}_j(q_i^{a_{ij}}E_iK_j) = (q_i^{a_{ij}}\overline{K}_jE_i)K_j = (E_i\overline{K}_j)K_j = E_iJ.$$

If E_i is of type 2, then we obtain, from (2.11),

$$JE_i = (K_j\overline{K}_j)E_i = (K_j\overline{K}_j)(q_i^{-a_{ij}}K_jE_i\overline{K}_j) = q_i^{-a_{ij}}(K_jE_i\overline{K}_j)J = E_iJ.$$

Hence, $JE_i = E_iJ$ for all $i \in I$. A similar calculation is performed for F_i.

In particular, if E_i and F_i are of type 2, then $E_iJ = JE_i = E_i$, $F_iJ = JF_i = F_i$.

Also note that, due to the relation (2.15), the following relations hold:

$$\begin{aligned} K_j^nE_i^m &= q_i^{mna_{ij}}E_i^mK_j^n, & E_i^m\overline{K}_j^n &= q_i^{mna_{ij}}\overline{K}_j^nE_i^m, \\ D_j^nE_i^m &= q_i^{mn\delta_{ij}}E_i^mD_j^n, & E_i^m\overline{D}_j^n &= q_i^{mn\delta_{ij}}\overline{D}_j^nE_i^m, \end{aligned}$$

for all $m, n \in \mathbf{Z}_{>0}$. The respective relations hold for F_i by replacing E_i with F_i and a_{ij} (resp. δ_{ij}) with $-a_{ij}$ (resp. $-\delta_{ij}$) in the above relations.

3 The Weak Hopf Algebra Structure of $wU_q^d(\mathscr{G})$

Let us define three maps

$$\begin{aligned} &\Delta : wU_q^d(\mathscr{G}) \longrightarrow wU_q^d(\mathscr{G}) \otimes wU_q^d(\mathscr{G}), \\ &\varepsilon : wU_q^d(\mathscr{G}) \longrightarrow \mathbf{C}, \\ &T : wU_q^d(\mathscr{G}) \longrightarrow wU_q^d(\mathscr{G}) \end{aligned}$$

as follows:

$$\Delta(J) = J \otimes J, \tag{3.1}$$

$$\Delta(K_i) = K_i \otimes K_i, \quad \Delta(\overline{K}_i) = \overline{K}_i \otimes \overline{K}_i, \tag{3.2}$$

$$\Delta(D_i) = D_i \otimes D_i, \quad \Delta(\overline{D}_i) = \overline{D}_i \otimes \overline{D}_i, \tag{3.3}$$

$$\Delta(E_i) = \begin{cases} E_i \otimes \overline{K}_i + 1 \otimes E_i, & \text{if } E_i \text{ is type 1;} \\ E_i \otimes \overline{K}_i + J \otimes E_i, & \text{if } E_i \text{ is type 2,} \end{cases} \tag{3.4}$$

$$\Delta(F_i) = \begin{cases} F_i \otimes 1 + K_i \otimes F_i, & \text{if } F_i \text{ is type 1;} \\ F_i \otimes J + K_i \otimes F_i, & \text{if } F_i \text{ is type 2,} \end{cases} \tag{3.5}$$

$$\varepsilon(D_i) = \varepsilon(\overline{D}_i) = \varepsilon(K_i) = \varepsilon(\overline{K}_i) = \varepsilon(J) = 1, \quad \varepsilon(E_i) = \varepsilon(F_i) = 0, \tag{3.6}$$

$$T(1) = 1,\ T(J) = J,\ T(K_i) = \overline{K}_i,\ T(\overline{K}_i) = K_i,\ T(D_i) = \overline{D}_i,\ T(\overline{D}_i) = D_i, \tag{3.7}$$

$$T(E_i) = -E_iK_i, \quad T(F_i) = -\overline{K}_iF_i,\ i \in I. \tag{3.8}$$

Let μ and η be the product and the unit of $wU_q^d(\mathscr{G})$, respectively. Then we have the following lemma:

Lemma 3.1 *$(wU_q^d(\mathscr{G}), \mu, \eta, \Delta, \varepsilon)$ is a bialgebra.*

Proof It is easy to check that $(wU_q^d(\mathscr{G}), \Delta, \varepsilon)$ is a coalgebra and ε is an algebra morphism. To show that Δ is an algebra morphism, we shall check that it preserves the relations (2.7)–(2.19). All of these are straightforward, saving the calculations involving relations (2.7)–(2.15) and (2.19). We will illustrate the arguments in these cases.

For (2.16), we should examine the identity

$$\Delta(E_i)\Delta(F_j) - \Delta(F_j)\Delta(E_i) = \delta_{ij}\frac{\Delta(K_i) - \Delta(\overline{K}_i)}{q_i - q_i^{-1}}.$$

The following cases should be considered:

(1) Both E_i and F_j are of type 1;

(2) E_i is of type 1 and F_j is of type 2;

(3) E_i is of type 2 and F_j is of type 1;

(4) Both E_i and F_j are of type 2.

For the case (3), using the facts, which are $E_iK_j = q_i^{-a_{ij}}K_jE_i$ and $F_j\overline{K}_i = q_i^{-a_{ij}}\overline{K}_iF_j$, it is obvious that

$$\begin{aligned}
\Delta(E_i)\Delta(F_j) - \Delta(F_j)\Delta(E_i) &= (E_iF_j - F_jE_i)\otimes K_i + K_iJ\otimes(E_iF_j - F_jE_i)\\
&= \delta_{ij}\frac{(K_i - \overline{K}_i)\otimes\overline{K}_i + K_i\otimes(K_i - \overline{K}_i)}{q_i - q_i^{-1}}\\
&= \delta_{ij}\frac{K_i\otimes K_i - \overline{K}_i\otimes\overline{K}_i}{q_i - q_i^{-1}}\\
&= \delta_{ij}\frac{\Delta(K_i) - \Delta(\overline{K}_i)}{q_i - q_i^{-1}}.
\end{aligned}$$

For the other cases, the proof is similar. For (2.17) and (2.18), we should consider several cases according to the type of $\{E_i, E_j\}$ or $\{F_i, F_j\}$, $i \neq j$. In fact, the argument is more or less the same as the discussion in [10, pp. 67–68].

Lemma 3.2 *T is an antimorphism from $wU_q^d(\mathscr{G})$ to $wU_q^d(\mathscr{G})$.*

Proof It is trivial that T keeps the antirelations of (2.7)–(2.16) and (2.19).

For (2.17), it can be proved as follows:

$$\begin{aligned}
&\sum_{r=0}^{1-a_{ij}}(-1)^r\begin{bmatrix}1-a_{ij}\\ r\end{bmatrix}_{q_i}T(E_i)^rT(E_j)T(E_i)^{1-a_{ij}-r}\\
&\quad= -\left(\sum_{r=0}^{1-a_{ij}}(-1)^{1-a_{ij}-r}\begin{bmatrix}1-a_{ij}\\ 1-a_{ij}-r\end{bmatrix}_{q_i}E_i^rE_jE_i^{1-a_{ij}-r}\right)K_i^{1-a_{ij}}K_j\\
&\quad= -\left(\sum_{s=0}^{1-a_{ij}}(-1)^s\begin{bmatrix}1-a_{ij}\\ s\end{bmatrix}_{q_i}E_i^{1-a_{ij}-s}E_jE_i^s\right)K_i^{1-a_{ij}}K_j\\
&\quad= 0.
\end{aligned}$$

The argument for (2.18) is similar. Hence T can be extended to an antimorphism from $wU_q^d(\mathscr{G})$ to $wU_q^d(\mathscr{G})$.

Lemma 3.3 *Let X be $E_i, F_i, K_i, \overline{K}_i, D_i, \overline{D}_i$ or J. Then*

$$(id * T * id)(X) = X, \quad (T * id * T)(X) = T(X).$$

Proof For $X = K_i, \overline{K}_i$, $D_i, \overline{D}_i$ or J, the calculations are trivial.

If E_i is of type 1, then

$$(\Delta \otimes id)\Delta(E_i) = E_i \otimes \overline{K}_i \otimes \overline{K}_i + 1 \otimes E_i \otimes \overline{K}_i + 1 \otimes 1 \otimes E_i.$$

It follows that

$$(id * T * id)(E_i) = E_i K_i \overline{K}_i + (-E_i K_i)\overline{K}_i + E_i = E_i = id(E_i),$$
$$(T * id * T)(E_i) = (-E_i K_i)\overline{K}_i K_i + E_i K_i + (-E_i K_i) = -E_i K_i = T(E_i).$$

If E_i is of type 2, then

$$(\Delta \otimes id)\Delta(E_i) = E_i \otimes \overline{K}_i \otimes \overline{K}_i + J \otimes E_i \otimes \overline{K}_i + J \otimes J \otimes E_i.$$

We also deduces that

$$(id * T * id)(E_i) = E_i K_i \overline{K}_i + J(-E_i K_i)\overline{K}_i + JE_i = JE_i = E_i = id(E_i),$$
$$(T * id * T)(E_i) = (-E_i K_i)\overline{K}_i K_i + JE_i K_i + J(-E_i K_i) = -JE_i K_i = E_i K_i = T(E_i).$$

As for F_i, the argument is similar.

Notice that the following two facts hold:

(1) The coproducts of the generators are bilinear expressions of generators;

(2) One of $id * T(X)$ and $T * id(X)$ is a central element of $wU_q^d(\mathscr{G})$ for X being E_i, F_i, $K_i, \overline{K}_i, D_i, \overline{D}_i$ or J.

From the above facts we can show that, if

$$(id * T * id)(x) = x, \quad (T * id * T)(x) = T(x),$$

$$(id * T * id)(y) = y, \quad (T * id * T)(y) = T(y),$$

for x and y being $E_i, F_i, K_i, \overline{K}_i, D_i, \overline{D}_i$ or J, then

$$(id * T * id)(xy) = xy, \quad (T * id * T)(xy) = T(xy).$$

Hence, the antipode axioms hold on arbitrary elements by induction, and T is a weak antipode.

From the above lemmas, we have the following theorem:

Theorem 3.4 *$(wU_q^d(\mathscr{G}), \mu, \eta, \Delta, \varepsilon, T)$ is a weak Hopf algebra.*

4 The Basis of $wU_q^d(\mathscr{G})$

Similarly to [2, 5–6], we can establish the relationship between $wU_q^d(\mathscr{G})$ and the quantum enveloping algebra $U_q(\mathscr{G})$ as follows.

Proposition 4.1 *$wU_q^d(\mathscr{G}) = w_q \oplus \overline{w}_q$, where $w_q = wU_q^d(\mathscr{G})J$, $\overline{w}_q = wU_q^d(\mathscr{G})(1-J)$. Moreover, $w_q \cong U_q(\mathscr{G})$ as Hopf algebras.*

Proof Due to $J^2 = J$, w_q and $\overline{w}_q$ are ideals of $wU_q^d(\mathscr{G})$. Consequently, $wU_q^d(\mathscr{G}) = w_q \oplus \overline{w}_q$ as algebras. Moreover, it is obvious to see that w_q is generated by $E_iJ,\ F_iJ,\ K_i,\ \overline{K}_i,\ D_i, \overline{D}_i$ and J subject to the relations (2.7)–(2.9) and

$$K_j(E_iJ) = q_i^{a_{ij}}(E_iJ)K_j, \quad (E_iJ)\overline{K}_j = q_i^{a_{ij}}\overline{K}_j(E_iJ), \tag{4.1}$$

$$D_j(E_iJ) = q_i^{\delta_{ij}}(E_iJ)D_j, \quad (E_iJ)\overline{D}_j = q_i^{\delta_{ij}}\overline{D}_j(E_iJ), \tag{4.2}$$

$$K_j(F_iJ) = q_i^{-a_{ij}}(F_iJ)K_j, \quad (F_iJ)\overline{K}_j = q_i^{-a_{ij}}\overline{K}_j(F_iJ), \tag{4.3}$$

$$D_j(F_iJ) = q_i^{-\delta_{ij}}(F_iJ)D_j, \quad (F_iJ)\overline{D}_j = q_i^{-\delta_{ij}}\overline{D}_j(F_iJ), \tag{4.4}$$

$$(E_iJ)(F_jJ) - (F_jJ)(E_iJ) = \delta_{ij}\frac{K_i - \overline{K}_i}{q_i - q_i^{-1}}, \tag{4.5}$$

$$\sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1-a_{ij} \\ r \end{bmatrix}_{q_i} (E_iJ)^{1-a_{ij}-r}(E_jJ)(E_iJ)^r = 0, \ \textit{if}\ a_{ii} = 2,\ i \neq j, \tag{4.6}$$

$$\sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1-a_{ij} \\ r \end{bmatrix}_{q_i} (F_iJ)^{1-a_{ij}-r}(F_jJ)(F_iJ)^r = 0, \ \textit{if}\ a_{ii} = 2,\ i \neq j, \tag{4.7}$$

$$(E_iJ)(E_jJ) = (E_jJ)(E_iJ), \quad (F_iJ)(F_jJ) = (F_jJ)(F_iJ), \ \textit{if}\ a_{ij} = 0. \tag{4.8}$$

Here J can be viewed as the identity of w_q. From this point of view w_q becomes a weak Hopf algebra, where the coproduct Δ is

$$\begin{aligned}
&\Delta(J) = J \otimes J, \quad \Delta(K_i) = K_i \otimes K_i, \quad \Delta(\overline{K}_i) = \overline{K}_i \otimes \overline{K}_i,\\
&\Delta(D_i) = D_i \otimes D_i, \quad \Delta(\overline{D}_i) = \overline{D}_i \otimes \overline{D}_i,\\
&\Delta(E_iJ) = E_iJ \otimes \overline{K}_i + J \otimes (E_iJ),\\
&\Delta(F_iJ) = F_iJ \otimes J + K_i \otimes (F_iJ), \quad i \in I.
\end{aligned}$$

The counit is

$$\begin{aligned}
&\varepsilon(D_i) = \varepsilon(\overline{D}_i) = \varepsilon(K_i) = \varepsilon(\overline{K}_i) = \varepsilon(J) = 1,\\
&\varepsilon(E_iJ) = \varepsilon(F_iJ) = 0, \quad i \in I.
\end{aligned}$$

The antipode S is

$$\begin{aligned}
&S(K_i) = \overline{K}_i, \quad S(\overline{K}_i) = K_i, \quad S(D_i) = \overline{D}_i,\\
&S(\overline{D}_i) = D_i, \quad S(E_iJ) = -(E_iJ)K_i, \quad S(F_i) = -\overline{K}_i(F_iJ), \quad i \in I.
\end{aligned}$$

Let ρ be the algebra morphism from $U_q(\mathscr{G})$ to w_q defined by

$$\begin{aligned}
&\rho(e_i) = (E_iJ), \quad \rho(f_i) = (F_iJ), \quad \rho(k_i) = K_i,\\
&\rho(k_i^{-1}) = \overline{K}_i, \quad \rho(p_i) = D_i, \quad \rho(p_i^{-1}) = \overline{D}_i, \quad i \in I.
\end{aligned}$$

Then ρ is a Hopf algebra isomorphism.

To find the basis of $wU_q^d(\mathscr{G})$, we first introduce some notions. Define

$$P_i^k = \begin{cases} K_i^k, & \text{if } k > 0,\\ J, & \text{if } k = 0,\\ \overline{K}_i^{-k}, & \text{if } k < 0, \end{cases}$$

and

$$Q_i^k = \begin{cases} D_i^k, & \text{if } k > 0, \\ J, & \text{if } k = 0, \\ \overline{D}_i^{-k}, & \text{if } k < 0. \end{cases}$$

It is easy to see that P_i^k and Q_i^k satisfy the regularity conditions:

$$P_i^k P_i^{-k} P_i^k = P_i^k, \quad Q_i^k Q_i^{-k} Q_i^k = Q_i^k.$$

Set $P^s = \prod_{i \in I} P_i^{s_i}$, $Q^t = \prod_{i \in I} Q_i^{t_i}$, where $s_i, t_i \in \mathbf{Z}$.

There exists a triangular decomposition $U_q(\mathscr{G}) \cong U_q^+ \otimes U_q^0 \otimes U_q^-$ (see [9]), where U_q^0 is the subalgebra of $U_q(\mathscr{G})$ generated by $\{q^h\}_{h \in P^v}$, and U_q^+ (resp. U_q^-) is the subalgebra of $U_q(\mathscr{G})$ generated by $\{e_i\}_{i \in I}$ (resp. $\{f_i\}_{i \in I}$). For $\alpha = \sum_{i \in I} r_i \alpha_i, r_i \in \mathbf{Z}$, we will use the notation $e_\alpha = \prod_{i \in I} e_i^{r_i}$, $f_\alpha = \prod_{i \in I} f_i^{r_i}$. Moreover, it is well known that $\{e_\alpha q^h f_\beta | \alpha, \beta \in \Omega, h \in P^v\}$ forms a basis of $U_q(\mathscr{G})$, where Ω is just a set indexing the basis elements.

Proposition 4.2 *The set $\{E_\alpha P^s Q^t F_\beta J | \alpha, \beta \in \Omega\}$ forms a basis of w_q.*

Proof Let w_q^0 be the subalgebra generated by $K_i, \overline{K}_i, D_i, \overline{D}_i, i \in I$. It is easy to see that $\{P^s Q^t\}$ forms a basis of w_q^0.

Let w_q^+ (resp. w_q^-) be the subalgebra of w_q generated by $\{E_i J\}_{i \in I}$ (resp. $\{F_i J\}_{i \in I}$). Obviously $w_q^+ \cong U_q^+$, we replace every e_i in the monomial e_α by $E_i J$, thus the set $\{E_\alpha J | \alpha \in \Omega\}$ forms a basis of w_q^+. By $w_q^- \cong U_q^-$, the set $\{F_\beta J | \beta \in \Omega\}$ forms a basis of w_q^-. Furthermore, $(E_\alpha J) P^s Q^t (F_\beta J) = E_\alpha P^s Q^t F_\beta J$, then $\{E_\alpha P^s Q^t F_\beta J | \alpha, \beta \in \Omega\}$ forms a basis of w_q.

To consider the basis of $\overline{w}_q = w U_q^d(\mathscr{G})(1-J)$, we need to recall some conventions. First note that $d = (\{c_i\}_{i \in I} | \{\overline{c}_i\}_{i \in I})$, if $c_i = 0$ (resp. $\overline{c}_i = 0$), then $E_i(1-J) = 0$ (resp. $F_i(1-J) = 0$); if $c_i \neq 0$ (resp. $\overline{c}_i \neq 0$), then $E_i(1-J) \neq 0$ (resp. $F_i(1-J) \neq 0$). Let

$$I_1 = \{i | c_i \neq 0\}, \quad I_2 = \{i | \overline{c}_i \neq 0\}$$

and

$$X_i = E_i(1-J), \quad Y_j = F_j(1-J), \quad i \in I_1, \ j \in I_2.$$

Obviously, $\{X_i, Y_j | i \in I_1, j \in I_2\} \cup \{1-J\}$ generates the ideal $\overline{w}_q$ enjoying the following relation:

$$X_i Y_j = Y_j X_i, \tag{4.9}$$

for all $i \in I_1, j \in I_2$ by (2.16).

To see what other relations X_i, Y_j enjoy, we consider the following five cases:

(1) If $I_1 = I_2 = I$, using the quantum serre relations (2.17)–(2.19), then we have

$$\sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1-a_{ij} \\ r \end{bmatrix}_{q_i} X_i^{1-a_{ij}-r} X_j X_i^r = 0, \quad \textit{if } a_{ii} = 2, \ i \neq j, \tag{4.10}$$

$$\sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1-a_{ij} \\ r \end{bmatrix}_{q_i} Y_i^{1-a_{ij}-r} Y_j Y_i^r = 0, \quad \textit{if } a_{ii} = 2, \ i \neq j, \tag{4.11}$$

$$X_i X_j = X_j X_i, \quad Y_i Y_j = Y_j Y_i, \quad \textit{if } a_{ij} = 0, \tag{4.12}$$

and other relations corresponding to (2.7)-(2.13) would vanish automatically. This means the ideal $\overline{w}_q$ can be understood as an algebra generated by $\{X_i, Y_j | i, j \in I\} \cup \{1 - J\}$ subject to the relations (4.9)–(4.12). Therefore $\{E_\alpha F_\beta(1 - J) | \alpha, \beta \in \Omega\}$ forms a basis of $\overline{w}_q$.

(2) If $I_1 = I_2 = \emptyset$, then $\{1 - J\}$ forms a basis of $\overline{w}_q$.

(3) If $I_1 \neq \emptyset, I_2 \neq \emptyset$, then $\overline{w}_q$ generated by $\{X_i, Y_j | i \in I_1, j \in I_2\} \cup \{1 - J\}$ satisfies the relations (4.9)–(4.12). For

$$\alpha' = \sum_{i \in I_1} r_i \alpha_i, r_i \in \mathbf{Z}, \quad \beta' = \sum_{i \in I_2} t_i \alpha_i, t_i \in \mathbf{Z},$$

we denote

$$E_{\alpha'} = \prod_{i \in I_1} E_i^{r_i}, \quad F_{\beta'} = \prod_{i \in I_2} F_i^{t_i}.$$

Therefore $\{E_{\alpha'} F_{\beta'}(1 - J)\}$ forms a basis of $\overline{w}_q$.

(4) If $I_1 = \emptyset, I_2 \neq \emptyset$, then $\overline{w}_q$ generated by $\{Y_j | j \in I_2\} \cup \{1 - J\}$ satisfies the relations (4.11)–(4.12), so $\{F_{\beta'}(1 - J) | \beta' \in \Omega'\}$ forms a basis of $\overline{w}_q$.

(5) If $I_1 \neq \emptyset, I_2 = \emptyset$, then $\overline{w}_q$ generated by $\{X_i | i \in I_1\} \cup \{1 - J\}$ satisfies the relations (4.10) and (4.12). Therefore $\{E_{\alpha'}(1 - J) | \alpha' \in \Omega'\}$ forms a basis of $\overline{w}_q$.

The case (1) is a special type of the case (5). For every case, we can describe the basis of $wU_q^d(\mathscr{G})$ from the above discussion. For example, for the case (1), $\{E_\alpha P^s Q^t F_\beta J | \alpha, \beta \in \Omega\} \cup \{E_\alpha F_\beta(1 - J) | \alpha, \beta \in \Omega\}$ is a basis of $wU_q^d(\mathscr{G})$. For other cases, the results are similar.

5 The Highest Weight Module

In this section, we will define some terms which are similar to the respective definitions of $U_q(\mathscr{G})$ in [9]. The following lemma is similar to [11, Lemma 1.1].

Lemma 5.1 *Let V be a $wU_q^d(\mathscr{G})-$module and $0 \neq v \in V$. For every $i \in I$, if $K_i v = \lambda_i v$ and $\overline{K}_i v = \overline{\lambda}_i v$ for $\lambda_i, \overline{\lambda}_i \in \mathbf{C}$, then $\overline{\lambda}_i = \begin{cases} \lambda_i^{-1}, & \text{if } \lambda_i \neq 0; \\ 0, & \text{if } \lambda_i = 0. \end{cases}$ Thus $Jv = v$, provided that there exists $i \in I$ such that $K_i v = \lambda_i v$ and $\lambda_i \neq 0$.*

Proof If $\lambda_i \neq 0$, we have

$$\lambda_i v = K_i v = K_i \overline{K}_i K_i v = \overline{\lambda}_i \lambda_i^2 v.$$

So $\overline{\lambda}_i \lambda_i = 1$. On the other hand, if $\lambda_i = 0$, we have

$$\overline{\lambda}_i v = \overline{K}_i v = \overline{K}_i K_i \overline{K}_i v = \lambda_i \overline{\lambda}_i^2 v = 0.$$

Hence we can conclude that if $\lambda_i \neq 0$, $\overline{K}_i v = \lambda_i^{-1} v$ and if $\lambda_i = 0$, $\overline{K}_i v = 0$. Since $J = K_i \overline{K}_i$, if there exists $i \in I$ such that $\lambda_i \neq 0$, then $Jv = \lambda_i \overline{\lambda}_i v = v$.

Similarly, we can prove the following corollary:

Corollary 5.2 *Let V be a $wU_q^d(\mathscr{G})$-module and $0 \neq v \in V$. For every $i \in I$, if $D_i v = \lambda_i v$ and $\overline{D}_i v = \overline{\lambda}_i v$ for $\lambda_i, \overline{\lambda}_i \in \mathbf{C}$, then $\overline{\lambda}_i = \begin{cases} \lambda_i^{-1} & \text{if } \lambda_i \neq 0; \\ 0 & \text{if } \lambda_i = 0. \end{cases}$ Thus $Jv = v$, provided that there exists $i \in I$ such that $D_i v = \lambda_i v$ and $\lambda_i \neq 0$.*

From the above results, we can introduce the following definition:

Definition 5.3 *A $wU_q^d(\mathscr{G})$-module V^q is called a weak quantum weight module if $V^q = \oplus_{\mu\in P} wV_\mu^q$, where*

$$wV_\mu^q = \{v \in V^q | Jv = v,\ K_i v = q_i^{\mu(h_i)}v,\ \overline{K}_i v = q_i^{-\mu(h_i)}v,\\ D_i v = q_i^{\mu(d_i)}v, \overline{D}_i v = q_i^{-\mu(d_i)}v,\ h_i, d_i \in h, i \in I\}.$$

A $wU_q^d(\mathscr{G})$-module V^q is called the highest weight module with highest weight $\lambda \in P$ if there exists a nonzero vector $v_\lambda \in V^q$ such that

(1) *$E_i v_\lambda = 0$ for every $i \in I$;*

(2) *$v_\lambda \in wV_\lambda^q$;*

(3) *$V^q = wU_q^d(\mathscr{G})v_\lambda$.*

Proposition 5.4 $V^q = w_q^- v_\lambda$.

Proof By Prop. 4.1, every $u \in wU_q^d(\mathscr{G})$ has a unique representation $u = w + \overline{w}, w \in w_q, \overline{w} \in \overline{w}_q$. Since $(1-J)v_\lambda = 0$, $\overline{w}v_\lambda = 0$, we have $uv_\lambda = wv_\lambda$. Hence $V^q = w_q v_\lambda$. Recall that every element of w of w_q can be written as a sum of elements of the form $w^- w^0 w^+$, where $w^0 \in w_q^0$ and $w^\pm \in w_q^\pm$. By Def. 5.3, $V^q = w_q^- v_\lambda$.

Since $V^q = w_q v_\lambda$ and $w_q \cong U_q(\mathscr{G})$, we have the following result:

Definition 5.5 *If $\dim_{\mathbf{C}} wV_\mu^q < \infty$ for all $\mu \in P$, then the character of V^q is*

$$ChV^q = \sum_{\mu\in P} (\dim_{\mathbf{C}} wV_\mu^q) e^\mu,$$

where e^μ is the basis of elements of the group algebra $\mathbf{C}[\mathscr{H}^*]$ with multiplication given by $e^\mu e^\nu = e^{\mu+\nu}$ for $\mu, \nu \in P$.

Definition 5.6 *A $wU_q^d(\mathscr{G})$-module $M^q(\lambda)$ with highest weight λ is called a weak Verma module if every $wU_q^d(\mathscr{G})$-module with highest weight λ is a quotient of $M^q(\lambda)$.*

Proposition 5.7 (1) *For each $\lambda \in P$, there exists a unique up to an isomorphism weak Verma module $M^q(\lambda)$;*

(2) *Viewed as a $wU_q^d(\mathscr{G})$-module, $M^q(\lambda)$ is a free module of rank 1 generated by a highest weight vector $v_\lambda = 1 + I_q(\lambda)$;*

(3) *$M^q(\lambda)$ contains a unique proper maximal submodule $J_q(\lambda)$.*

Proof (1) If $M_1^q(\lambda)$ and $M_2^q(\lambda)$ are two weak Verma modules, then by definition there exists a surjective homomorphism $\varphi : M_1^q(\lambda) \longrightarrow M_2^q(\lambda)$. In particular, $\varphi(M_1^q(\lambda)_\mu) = M_2^q(\lambda)_\mu$ for all $\mu \in P$, and hence $\dim_{\mathbf{C}} \varphi(M_1^q(\lambda)_\mu) \geq \dim_{\mathbf{C}} M_2^q(\lambda)_\mu$ for all $\mu \in P$. Exchanging $M_1^q(\lambda)$ and $M_2^q(\lambda)$ proves that φ is an isomorphism.

To prove the existence of a Verma module, consider the left ideal $I_q(\lambda)$ of $wU_q^d(g)$ generated by $\{J-1, E_i, K_i - q_i^{\lambda(h_i)}\cdot 1, \overline{K}_i - q_i^{-\lambda(h_i)}\cdot 1, D_i - q_i^{\lambda(d_i)}\cdot 1, \overline{D}_i - q_i^{-\lambda(d_i)}\cdot 1\}_{i\in I}$, and set $M^q(\lambda) = wU_q^d(\mathscr{G})/I_q(\lambda)$. Then, via the left multiplication, $M^q(\lambda)$ becomes a $wU_q^d(g)$-module. It is clear that $M^q(\lambda)$ is a $wU_q^d(\mathscr{G})$-module with the highest weight λ, the highest weight vector being the image of $1 \in wU_q^d(\mathscr{G})$.

(2) By Prop. 5.4, $\forall u \in wU_q^d(\mathscr{G})$, uv_λ can be written as a sum of elements of the form $w^- v_\lambda$. If $w^-(1 + I_q(\lambda)) = 0$, then $w^- \in I_q(\lambda)$. Hence w^- must be zero, and our assertion follows.

(3) Note that for any proper submodule M' of $M^q(\lambda)$, $M' \subseteq \oplus_{\mu\in P,\mu\neq\lambda} wV^q_\mu$, Thus the sum of proper submodules is again a submodule of $M^q(\lambda)$. Then $M^q(\lambda)$ contains a unique proper maximal submodule $J^q(\lambda)$.

The irreducible quotient $V^q(\lambda) = M^q(\lambda)/J_q(\lambda)$ is an irreducible weight module over $wU^d_q(g)$ with the highest weight λ.

6 Weak A-forms

The A-form U_A of the quantum group $U_q(\mathscr{G})$ is defined in [8, 9], where $\mathscr{G}$ is a generalized Kac–Moody algebra. In this section, we would like to define the weak A-forms of $wU^d_q(\mathscr{G})$, where $A = \mathbf{C}[q, q^{-1}, 1/[n]_{q_i},\ i \in I,\ n > 0]$.

Following [8, 9], for each $i \in I$, $c \in \mathbf{Z}$, $n \in \mathbf{Z}_{\geq 0}$, we define

$$\begin{bmatrix} K_i; c \\ n \end{bmatrix}_w = \prod_{r=1}^{n} \frac{K_i q_i^{c-r+1} - \overline{K}_i q_i^{-(c-r+1)}}{q_i^r - q_i^{-r}}, \tag{6.1}$$

$$\begin{bmatrix} \overline{K}_i; c \\ n \end{bmatrix}_w = \prod_{r=1}^{n} \frac{\overline{K}_i q_i^{c-r+1} - K_i q_i^{-(c-r+1)}}{q_i^r - q_i^{-r}}, \tag{6.2}$$

$$\begin{bmatrix} D_i; c \\ n \end{bmatrix}_w = \prod_{r=1}^{n} \frac{D_i q_i^{c-r+1} - \overline{D}_i q_i^{-(c-r+1)}}{q_i^r - q_i^{-r}}, \tag{6.3}$$

$$\begin{bmatrix} \overline{D}_i; c \\ n \end{bmatrix}_w = \prod_{r=1}^{n} \frac{\overline{D}_i q_i^{c-r+1} - D_i q_i^{-(c-r+1)}}{q_i^r - q_i^{-r}}. \tag{6.4}$$

From the above definition, we have

$$\begin{aligned}
&\frac{K_i q_i^{c-r+1} - \overline{K}_i q_i^{-(c-r+1)}}{q_i^r - q_i^{-r}} \\
&= \frac{K_i q_i^{c-r+1} - \overline{K}_i q_i^{c-r+1} + \overline{K}_i q_i^{c-r+1} - \overline{K}_i q_i^{-(c-r+1)}}{q_i^r - q_i^{-r}} \\
&= q_i^{c-r+1} \frac{K_i - \overline{K}_i}{q_i^r - q_i^{-r}} + \overline{K}_i \frac{q_i^{c-r+1} - q_i^{-(c-r+1)}}{q_i^r - q_i^{-r}} \\
&= q_i^{c-r+1} \frac{q_i - q_i^{-1}}{q_i^r - q_i^{-r}} \frac{K_i - \overline{K}_i}{q_i - q_i^{-1}} + \overline{K}_i \frac{q_i^{c-r+1} - q_i^{-(c-r+1)}}{q_i - q_i^{-1}} \frac{q_i - q_i^{-1}}{q_i^r - q_i^{-r}} \\
&= \frac{1}{[r]_{q_i}} \left(q_i^{c-r+1} \begin{bmatrix} K_i; 0 \\ 1 \end{bmatrix}_w + [c-r+1]_{q_i} \overline{K}_i \right).
\end{aligned}$$

Then the following identity holds:

$$\begin{bmatrix} K_i; c \\ n \end{bmatrix}_w = \prod_{r=1}^{n} \frac{1}{[r]_{q_i}} \left(q_i^{c-r+1} \begin{bmatrix} K_i; 0 \\ 1 \end{bmatrix}_w + [c-r+1]_{q_i} \overline{K}_i \right), \tag{6.5}$$

for all $c \in \mathbf{Z}$.

Similarly,

$$\begin{bmatrix} \overline{K}_i; c \\ n \end{bmatrix}_w = \prod_{r=1}^{n} \frac{1}{[r]_{q_i}} \left(q_i^{c-r+1} \begin{bmatrix} \overline{K}_i; 0 \\ 1 \end{bmatrix}_w + [c-r+1]_{q_i} K_i \right), \tag{6.6}$$

for all $c \in \mathbf{Z}$, and the respective relations hold with D_i (resp. $\overline{D}_i$) in place of K_i (resp. $\overline{K}_i$).

Note that

$$\left[\begin{array}{c} \overline{K}_i;0 \\ 1 \end{array}\right]_w = -\left[\begin{array}{c} K_i;0 \\ 1 \end{array}\right]_w, \quad \left[\begin{array}{c} \overline{D}_i;0 \\ 1 \end{array}\right]_w = -\left[\begin{array}{c} D_i;0 \\ 1 \end{array}\right]_w, \tag{6.7}$$

for all $i \in I$.

We define the d-type weak A-form wU_A^d of $wU_q^d(\mathscr{G})$ to be the A-subalgebra of $wU_q^d(\mathscr{G})$ with unit 1 generated by the elements E_i, F_i, K_i, $\overline{K}_i$, D_i, $\overline{D}_i$, J, $\left[\begin{smallmatrix} K_i;0 \\ 1 \end{smallmatrix}\right]_w$ and $\left[\begin{smallmatrix} D_i;0 \\ 1 \end{smallmatrix}\right]_w$ $(i \in I)$. Obviously, $(wU_A^d, \mu, \eta, \Delta, \epsilon)$ is a d-type weak Hopf subalgebra.

Lemma 6.1 *For $i, j \in I$, $c \in \mathbf{Z}$, and $n \in \mathbf{Z}_{>0}$, we have*

$$\left[\begin{array}{c} K_i;c \\ n \end{array}\right]_w E_j = E_j \left[\begin{array}{c} K_i;c+a_{ij} \\ n \end{array}\right]_w, \tag{6.8}$$

$$E_j \left[\begin{array}{c} \overline{K}_i;c \\ n \end{array}\right]_w = \left[\begin{array}{c} \overline{K}_i;c+a_{ij} \\ n \end{array}\right]_w E_j, \tag{6.9}$$

$$\left[\begin{array}{c} K_i;c \\ n \end{array}\right]_w F_j = F_j \left[\begin{array}{c} K_i;c-a_{ij} \\ n \end{array}\right]_w, \tag{6.10}$$

$$F_j \left[\begin{array}{c} \overline{K}_i;c \\ n \end{array}\right]_w = \left[\begin{array}{c} \overline{K}_i;c-a_{ij} \\ n \end{array}\right]_w F_j, \tag{6.11}$$

$$\left[\begin{array}{c} D_i;c \\ n \end{array}\right]_w E_j = E_j \left[\begin{array}{c} D_i;c+\delta_{ij} \\ n \end{array}\right]_w, \tag{6.12}$$

$$E_j \left[\begin{array}{c} \overline{D}_i;c \\ n \end{array}\right]_w = \left[\begin{array}{c} \overline{D}_i;c+\delta_{ij} \\ n \end{array}\right]_w E_j, \tag{6.13}$$

$$\left[\begin{array}{c} D_i;c \\ n \end{array}\right]_w F_j = F_j \left[\begin{array}{c} D_i;c-\delta_{ij} \\ n \end{array}\right]_w, \tag{6.14}$$

$$F_j \left[\begin{array}{c} \overline{D}_i;c \\ n \end{array}\right]_w = \left[\begin{array}{c} \overline{D}_i;c-\delta_{ij} \\ n \end{array}\right]_w F_j, \tag{6.15}$$

$$E_iF_j - F_jE_i = \delta_{ij} \left[\begin{array}{c} K_i;0 \\ 1 \end{array}\right]_w, \tag{6.16}$$

$$E_iF_j^n = \begin{cases} F_j^nE_i + F_i^{n-1}\sum_{r=0}^{n-1} \left[\begin{array}{c} K_i;-ra_{ii} \\ 1 \end{array}\right]_w, & \text{if } i=j; \\ F_j^nE_i, & \text{if } \ i \neq j. \end{cases} \tag{6.17}$$

Proof The first nine equalities follow directly from the defining relations of $wU_q^d(\mathscr{G})$ and (6.1)–(6.4), while (6.17) is proved by induction.

Let $w_A = wU_A^dJ$, $\overline{w}_A = wU_A^d(1-J)$. Then

Proposition 6.2 *As algebras, $wU_A^d = w_A \oplus \overline{w}_A$. Moreover, $w_A \cong U_A$ as Hopf algebras, where U_A is the A-form of $U_q(\mathscr{G})$ (see [9]).*

Proof Since $J^2 = J$, w_A and $\overline{w}_A$ are ideals of wU_A^d. Consequently, $wU_A^d = w_A \oplus \overline{w}_A$ as algebras. Moreover, w_A is generated by JE_i, JF_i, K_i, $\overline{K}_i$, D_i, $\overline{D}_i$, J, $\left[\begin{smallmatrix} K_i;0 \\ 1 \end{smallmatrix}\right]_w$ and $\left[\begin{smallmatrix} D_i;0 \\ 1 \end{smallmatrix}\right]_w$ $(i \in I)$. The respective relations of Lemma 6.1 hold by replacing E_i (resp. F_i) with JE_i (resp. JF_i). Let $\rho : w_A \to U_A$ satisfy

$$\rho(e_i) = E_iJ,\ \rho(f_i) = F_iJ,\ \rho(k_i) = K_i,$$
$$\rho(k_i^{-1}) = \overline{K}_i,\ \rho(p_i) = D_i,\ \rho(p_i^{-1}) = \overline{D}_i,$$

$$\rho\left(\begin{bmatrix} k_i;0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} K_i;0 \\ 1 \end{bmatrix}_w, \quad \rho\left(\begin{bmatrix} p_i;0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} D_i;0 \\ 1 \end{bmatrix}_w.$$

It is easy to check ρ is a Hopf algebra isomorphism.

As an immediate consequence of Lemma 6.2, we have the triangular decomposition of the algebra w_A:

$$w_A \cong w_A^+ \otimes w_A^0 \otimes w_A^-,$$

where w_A^0 is a subalgebra of w_A generated by

$$\left\{K_i, \overline{K}_i, D_i, \overline{D}_i, J, \begin{bmatrix} \overline{K}_i;0 \\ 1 \end{bmatrix}_w, \begin{bmatrix} \overline{D}_i;0 \\ 1 \end{bmatrix}_w\right\}_{i\in I}$$

and w_A^+ (resp. w_A^-) is a subalgebra of w_A generated by $\{JE_i\}_{i\in I}$ (resp. $\{JF_i\}_{i\in I}$).

Corollary 6.3 *Let $V^q(\lambda)$ be the irreducible highest weight module with the highest weight $\lambda \in P^+$ and the highest weight vector v_λ.*

(1) *If $\lambda(h_i) = 0$, then $F_iv_\lambda = 0$ for $i \in I$;*

(2) *If $a_{ii} = 2$, then $F_i^{\lambda(h_i)+1}v_\lambda = 0$ for $i \in I$.*

Proof (1) Obviously, the equality $E_iF_iv_\lambda = F_iE_iv_\lambda = 0$ holds by (6.16). If $i \neq j$, from $\lambda(h_i) = 0$ we can obtain that

$$E_iF_jv_\lambda = F_jE_iv_\lambda + \frac{K_i - \overline{K}_i}{q_i - q_i^{-1}}v_\lambda = \frac{q_i^{\lambda(h_i)} - q_i^{-\lambda(h_i)}}{q_i - q_i^{-1}}v_\lambda = 0.$$

Hence F_jv_λ is a primitive vector of $V_q(\lambda)$. Note that $V_q(\lambda)$ is irreducible, so $F_jv_\lambda = 0$, for otherwise F_jv_λ would generate a proper submodule of $V_q(\lambda)$ with highest weight $\lambda - \alpha_i(\neq \lambda)$, which is a contradiction.

(2) Applying (6.17), for $i \neq j$, we can conclude that

$$E_iF_j^{\lambda(h_i)+1}v_\lambda = F_j^{\lambda(h_i)+1}E_iv_\lambda = 0.$$

For $i = j$ and $a_{ii} = 2$, we have

$$\begin{aligned} E_iF_i^{\lambda(h_i)+1}v_\lambda &= F_i^{\lambda(h_i)+1}E_iv_\lambda + F_i^{\lambda(h_i)}\sum_{r=0}^{\lambda(h_i)}\begin{bmatrix} K_i;-2r \\ 1 \end{bmatrix}_w v_\lambda \\ &= F_i^{\lambda(h_i)}\sum_{r=0}^{\lambda(h_i)}\frac{q_i^{\lambda(h_i)-2r} - q_i^{-\lambda(h_i)+2r}}{q_i - q_i^{-1}}v_\lambda \end{aligned}$$

$$= (q_i - q_i^{-1})((q_i^{\lambda(h_i)} - q_i^{-\lambda(h_i)}) + (q_i^{\lambda(h_i)-2} - q_i^{-\lambda(h_i)+2})$$
$$+ \cdots + (q_i^{2-\lambda(h_i)} - q_i^{\lambda(h_i)-2}) + (q_i^{-\lambda(h_i)} - q_i^{\lambda(h_i)}))F_i^{\lambda(h_i)}v_\lambda = 0.$$

Therefore, $F_j^{\lambda(h_i)+1}v_\lambda$ is a primitive vector of weight $\lambda - (\lambda(h_i)+1)\alpha_i \neq \lambda$, and hence $F_j^{\lambda(h_i)+1}v_\lambda = 0$.

Assume $\lambda \in P$, and let V^q be a highest weight module over $wU_q^d(\mathscr{G})$ with highest weight λ and highest weight vector v_λ. We define the weak A-form wV_A^q to be the wU_A^d-submodule of V^q generated by v_λ. That is, $wV_A^q = wU_A^d v_\lambda$.

Proposition 6.4 $wV_A^q = w_A^- v_\lambda$.

Proof By Lemma 6.2, every $u \in wU_A^d$ has a unique representation $u = w + \overline{w}, w \in w_A, \overline{w} \in \overline{w}_A$. Since $(1-J)v_\lambda = 0$, $\overline{w}v_\lambda = 0$, we have $uv_\lambda = wv_\lambda$. Recall that every element of w of w_q can be written as a sum of elements of the form $w^-w^0w^+$, where $w^0 \in w_A^0$ and $w^\pm \in w_A^\pm$. By definition, $w^+v_\lambda = 0$, unless $w^+ \in A$, and $K_iv_\lambda = q_i^{\mu(h_i)}v_\lambda \in A_\lambda, D_iv_\lambda = q_i^{\mu(d_i)}v_\lambda \in A_\lambda$. For $i \in I$, $c \in \mathbf{Z}$ and $n \in \mathbf{Z}_{\geq 0}$, we have

$$\left[\begin{array}{c} K_i; c \\ n \end{array}\right]_w v_\lambda = \left[\begin{array}{c} \lambda(h_i) + c \\ n \end{array}\right]_{q_i} v_\lambda,$$

where

$$\left[\begin{array}{c} \lambda(h_i) + c \\ n \end{array}\right]_{q_i} = \prod_{r=1}^{n} \frac{q_i^{\lambda(h_i)+c-r+1} - q_i^{-(\lambda(h_i)+c-r+1)}}{q_i^r - q_i^{-r}}$$
$$= \frac{[\lambda(h_i)+c]_{q_i}!}{[n]_{q_i}!\,[\lambda(h_i)+c-n]_{q_i}!} \in A.$$

Hence, $\left[\begin{smallmatrix} K_i;c \\ n \end{smallmatrix}\right]_w v_\lambda \in Av_\lambda$. Similarly, $\left[\begin{smallmatrix} D_i;c \\ n \end{smallmatrix}\right]_w v_\lambda \in Av_\lambda$. Then $w^-w^0w^+v_\lambda \in w_A^- v_\lambda$. That is, $wV_A^q \subseteq w_A^- v_\lambda$. It follows that $wV_A^q = w_A^- v_\lambda$.

Proposition 6.5 *The map $\varphi : \mathbf{C}[q] \otimes wV_A^q \to V^q$ given by $f \otimes v \to fv$ ($f \in \mathbf{C}[q]$, $v \in wV_A^q$) is a $\mathbf{C}[q]$-linear isomorphism.*

Proof It is clear that the $C[q]$-linear map given above is surjective. Let $\{F_\eta Jv_\lambda | \eta \in \Omega\}$ be a basis of V^q, where F_η is a monomial in $F_i's$. Define a $C[q]$-linear map $\psi : V^q \to C[q] \otimes wV_A^q$ by

$$\psi(F_\eta Jv_\lambda) = 1 \otimes F_\eta Jv_\lambda.$$

Then it is easy to see that ψ and φ are inverse to each other, which proves our assertion.

Proposition 6.6 *For $\mu \in P$, let $(wV_A^q)_\mu = wV_A^q \cap wV_\mu^q$. Then wV_A^q has the weight space decomposition $wV_A^q = \oplus_{\mu \in P}(wV_A^q)_\mu$.*

Proof Let $v = v_1 + v_2 + \cdots + v_p \in wV_A^q$, where $v_j \in wV_{\mu_j}^q$ ($\mu_j \in P, j = 1, 2, \ldots, p$). We would like to show $v_j \in wV_A^q$ for all $j = 1, 2, \ldots, p$. We will prove that $v_1 \in wV_A^q$. The other cases can be proved in a similar way.

For $j = 1, 2, \ldots, p$ and $i \in I$, write $\mu_j(h_i) = S_{ij}$ and $\mu_j(d_i) = T_{ij}$. Since $\mu_j \neq \mu_1$ for $j = 2, \ldots, p$, we can choose an index $i_j \in I$ such that $S_{i_j,j} \neq S_{i_j,1}$ or $T_{i_j,j} \neq T_{i_j,1}$. Let $I_0 = \{i_2, i_3, \ldots, i_p\}$, and take $s = \max\{|S_{ij} - S_{i1}|, |T_{ij} - T_{i1}|\}$ for all $i \in I_0, j = 1, \ldots, p$. We define an element u of wU_A^d to be

$$u = \prod_{i\in I_0} \left[\begin{array}{c} K_i; -S_{i1}+s \\ s \end{array} \right]_w \left[\begin{array}{c} K_i; -S_{i1}-1 \\ s \end{array} \right]_w \left[\begin{array}{c} D_i; -T_{i1}+s \\ s \end{array} \right]_w \left[\begin{array}{c} D_i; -T_{i1}-1 \\ s \end{array} \right]_w .$$

Then we have

$$\begin{aligned} \left[\begin{array}{c} K_i; -S_{i1}-1 \\ s \end{array} \right]_w v_1 &= \prod_{r=1}^{s} \frac{K_i q_i^{-S_{i1}-r} - \overline{K}_i q_i^{S_{i1}+r}}{q_i^r - q_i^{-r}} v_1 \\ &= \prod_{r=1}^{s} \frac{q_i^{-r} - q_i^r}{q_i^r - q_i^{-r}} v_1 = (-1)^s v_1 \end{aligned}$$

and

$$\begin{aligned} \left[\begin{array}{c} K_i; -S_{i1}+s \\ s \end{array} \right]_w v_1 &= \prod_{r=1}^{s} \frac{K_i q_i^{-S_{i1}+s-r+1} - \overline{K}_i q_i^{-(-S_{i1}+s-r+1)}}{q_i^r - q_i^{-r}} v_1 \\ &= \prod_{r=1}^{s} \frac{q_i^{s-r+1} - q_i^{-s+r-1}}{q_i^r - q_i^{-r}} v_1 = v_1 . \end{aligned}$$

Similarly,

$$\left[\begin{array}{c} D_i; -T_{i1}-1 \\ s \end{array} \right]_w v_1 = (-1)^s v_1$$

and

$$\left[\begin{array}{c} D_i; -T_{i1}+s \\ s \end{array} \right]_w v_1 = v_1 .$$

Therefore, $uv_1 = (-1)^{2s(p-1)} v_1 = v_1$.

If $j \neq 1$, then

$$\left[\begin{array}{c} K_i; -S_{i1}-1 \\ s \end{array} \right]_w v_j = \prod_{r=1}^{s} \frac{q_i^{S_{ij}-S_{i1}-r} - q_i^{-(S_{ij}-S_{i1}-r)}}{q_i^r - q_i^{-r}} v_j$$

and

$$\left[\begin{array}{c} K_i; -S_{i1}+s \\ s \end{array} \right]_w v_j = \prod_{r=1}^{s} \frac{q_i^{S_{ij}-S_{i1}+s-r+1} - q_i^{-(S_{ij}-S_{i1}+s-r+1)}}{q_i^r - q_i^{-r}} v_j .$$

Thus,

$$\begin{aligned} &\prod_{i\in I_0} \left[\begin{array}{c} K_i; -S_{i1}-1 \\ s \end{array} \right]_w \left[\begin{array}{c} K_i; -S_{i1}+s \\ s \end{array} \right]_w v_j \\ &\quad = \prod_{i\in I_0} \prod_{r,t=1}^{s} \frac{(q_i^{S_{ij}-S_{i1}-r} - q_i^{-(S_{ij}-S_{i1}-r)})(q_i^{S_{ij}-S_{i1}+s-t+1} - q_i^{-(S_{ij}-S_{i1}+s-t+1)})}{(q_i^r - q_i^{-r})(q_i^t - q_i^{-t})} v_j . \end{aligned}$$

Similarly,

$$\begin{aligned} &\prod_{i\in I_0} \left[\begin{array}{c} D_i; -T_{i1}-1 \\ s \end{array} \right]_w \left[\begin{array}{c} D_i; -T_{i1}+s \\ s \end{array} \right]_w v_j \\ &\quad = \prod_{i\in I_0} \prod_{r,t=1}^{s} \frac{(q_i^{T_{ij}-T_{i1}-r} - q_i^{-(T_{ij}-T_{i1}-r)})(q_i^{T_{ij}-T_{i1}+s-t+1} - q_i^{-(T_{ij}-T_{i1}+s-t+1)})}{(q_i^r - q_i^{-r})(q_i^t - q_i^{-t})} v_j . \end{aligned}$$

The terms where $r + t = s + 1$ are

$$\begin{aligned}
&(q_i^{S_{ij}-S_{i1}-r} - q_i^{-(S_{ij}-S_{i1}-r)})(q_i^{S_{ij}-S_{i1}+s-t+1} - q_i^{-(S_{ij}-S_{i1}+s-t+1)}) \\
&\quad = q_i^{2(S_{ij}-S_{i1})} - q_i^{2r} - q_i^{-2r} + q_i^{-2(S_{ij}-S_{i1})}, \\
&(q_i^{T_{ij}-T_{i1}-r} - q_i^{-(T_{ij}-T_{i1}-r)})(q_i^{T_{ij}-T_{i1}+s-t+1} - q_i^{-(T_{ij}-T_{i1}+s-t+1)}) \\
&\quad = q_i^{2(T_{ij}-T_{i1})} - q_i^{2r} - q_i^{-2r} + q_i^{-2(T_{ij}-T_{i1})}.
\end{aligned}$$

By the definition of I_0, we have $S_{i,j} - S_{i,1} \neq 0$ or $T_{i,j} - T_{i,1} \neq 0$ for $i = i_j \in I_0$. Since r runs from 1 to s, there exists some value of r such that $r = |S_{i,j} - S_{i,1}|$ or $r = |T_{i,j} - T_{i,1}|$ for $i = i_j \in I_0$, which implies $uv_j = 0$. It follows that $uv = uv_1$, and hence $v_1 \in wV_A^q$.

Corollary 6.7 *For all $\mu \in P$, $(wV_A^q)_\mu$ is a free A-module, and* $\mathrm{rank}(wV_A^q)_\mu = \dim_{\mathbf{C}(q)}(wV_\mu^q)$.

Proof By Prop. 6.5 and Prop. 6.6, we get a $\mathbf{C}(q)$-linear isomorphism $\mathbf{C}(q) \otimes (wV_A^q)_\mu \cong wV_\mu^q$ for all $\mu \in P$, and our assertion follows.

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