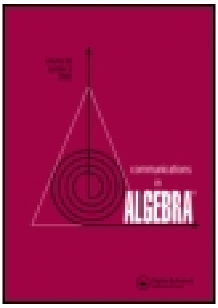

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A NOTE ON RELATIVE FLATNESS AND COHERENCE

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Let R be a ring and M a fixed right R-module. A new characterization of M-flatness is given by certain linear equations. For a left R-module F such that the canonical map $M \otimes_R F \to Hom_R(M^*, F)$ is injective, where $M^* = Hom_R(M, R)$, the M-flatness of F is characterized via certain matrix subgroups. An example is given to show that R need not be M-coherent even if every left R-module is M-flat. Moreover, some properties of M-coherent rings are discussed.

Key Words: Matrix subgroup; M-coherent ring; M-flat module.

2000 Mathematics Subject Classification: 16D50; 16P70.

Recently, Dauns introduced the notion of coherence of a ring R relative to a right R-module M (Dauns, 2006). Let R be a ring, M a fixed right R-module and $\sigma[M]$ the full subcategory of the category of right R-modules subgenerated by M (see Wisbauer, 1991, p. 118). Recall from Dauns (2006) that a left R-module F is *$\sigma[M]$-flat* if for any exact sequence $0 \to X \to Y$ in $\sigma[M]$, the sequence $0 \to X \otimes_R F \to Y \otimes_R F$ is exact. Following Wisbauer (1991, 12.13), F is called *M-flat* if the sequence $0 \to K_R \otimes F \to M_R \otimes F$ is exact for every submodule $0 \leq K < M$. It is a trivial consequence of Wisbauer (1991, 12.15) that ${}_RF$ is M-flat if and only if it is $\sigma[M]$-flat (see also Dauns, 2006, Proposition 1.6).

Following Dauns (2006), a right R-module N is *M-coherent* if for any $0 \leq A < B \leq N$ such that $B/A \hookrightarrow mR$ for some $m \in M$, if B/A is finitely generated, then B/A is finitely presented. R is defined to be *M-coherent* if the right R-module R_R is M-coherent.

The purpose of this note is to investigate M-flat modules and M-coherent rings from some new aspects.

Throughout R is an associative ring with identity and all modules are unitary. For a positive integer n, R^n (resp. R_n) denotes the direct sum of n copies of ${}_RR$ (resp. R_R) whose elements are written as "row (resp. column) vectors." Similarly, M^n stands for the direct sum of n copies of M_R. For each $\underline{m} = (m_1, m_2, \ldots, m_n) \in M^n$,

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the right annihilator of $\underline{m}$ in R_n is symbolized by $\mathbf{r}_{R_n}(\underline{m})$, that is,

$$\mathbf{r}_{R_n}(\underline{m}) = \left\{(r_1, r_2, \ldots, r_n)^{\mathrm{T}} \in R_n \mid (m_1, m_2, \ldots, m_n)(r_1, r_2, \ldots, r_n)^{\mathrm{T}} = \sum_{i=1}^{n} m_i r_i = 0\right\}.$$

Note that $\mathbf{r}_R(m)$ is nothing more than $m^{\perp}$ in Dauns (2006) for every $m \in M$.

Let us start with the following result.

Theorem 1. *Let M be a fixed right R-module. The following are equivalent for a left R-module F.*

(1) *F is M-flat.*
(2) *For any $a_1, \ldots, a_n \in R$, and $x_1, \ldots, x_n \in F$ such that $\sum_{i=1}^{n} a_i x_i \in \mathbf{r}_R(m)F$ for some $m \in M$, there exist $y_1, \ldots, y_k \in F$ and $n \times k$ matrix $B = (b_{ij})_{n\times k}$ over R such that $x_i = \sum_{j=1}^{k} b_{ij} y_j$ for each $1 \leq i \leq n$ and $m \sum_{i=1}^{n} a_i b_{ij} = 0$ for each $1 \leq j \leq k$.*

Proof. (1) $\Rightarrow$ (2) For any $a_1, \ldots, a_n \in R$, and $x_1, \ldots, x_n \in F$, if $\sum_{i=1}^{n} a_i x_i \in \mathbf{r}_R(m)F$, we have

$$\mathbf{r}_R(m) \subseteq L = a_1 R + \cdots + a_n R + \mathbf{r}_R(m) \leq R_R$$

with $L/\mathbf{r}_R(m)$ finitely generated and

$$\left(\sum_{i=1}^{n} a_i x_i\right) + \mathbf{r}_R(m)F = 0 + \mathbf{r}_R(m)F \in LF/\mathbf{r}_R(m)F.$$

It follows by Dauns (2006, Theorem 1.8(d)) that

$$\sum_{i=1}^{n} (a_i + \mathbf{r}_R(m)) \otimes x_i = 0 \in (L/\mathbf{r}_R(m)) \otimes F.$$

Let $\underline{a} = (a_1 + \mathbf{r}_R(m), \ldots, a_n + \mathbf{r}_R(m))$ and let $\{e_1, \ldots, e_n\}$ be the canonical basis of the free right R-module R_n. There is an exact sequence

$$0 \longrightarrow \mathbf{r}_{R_n}(\underline{a}) \xrightarrow{\alpha} R_n \xrightarrow{\eta} L/\mathbf{r}_R(m) \longrightarrow 0$$

where α is the inclusion map and η is given by $\eta(e_i) = a_i + \mathbf{r}_R(m)$ $(i = 1, \ldots, n)$. Tensoring by F yields the following exact sequence

$$\mathbf{r}_{R_n}(\underline{a}) \otimes F \xrightarrow{\alpha\otimes \mathrm{id}_F} R_n \otimes F \xrightarrow{\eta\otimes \mathrm{id}_F} [L/\mathbf{r}_R(m)] \otimes F \longrightarrow 0$$

where $\mathrm{id}_F : F \to F$ is the identity map. Since

$$(\eta \otimes \mathrm{id}_F)\left(\sum_{i=1}^{n} e_i \otimes x_i\right) = \sum_{i=1}^{n} (a_i + \mathbf{r}_R(m)) \otimes x_i = 0 \in (L/\mathbf{r}_R(m)) \otimes F,$$

$\sum_{i=1}^{n} e_i \otimes x_i \in \mathrm{Ker}(\eta \otimes \mathrm{id}_F) = \mathrm{Im}(\alpha \otimes \mathrm{id}_F)$. So we have $b_1 = \begin{bmatrix} b_{11} \\ \vdots \\ b_{n1} \end{bmatrix}, \ldots, b_k = \begin{bmatrix} b_{1k} \\ \vdots \\ b_{nk} \end{bmatrix} \in$ $\mathbf{r}_{R_n}(\underline{a})$ and $y_1, \ldots, y_k \in F$ such that $\sum_{j=1}^{k} b_j \otimes y_j = \sum_{i=1}^{n} e_i \otimes x_i \in R_n \otimes F$. But

$$\begin{aligned}\sum_{j=1}^{k} b_j \otimes y_j &= \sum_{j=1}^{k} \left[\left(\sum_{i=1}^{n} e_i b_{ij}\right) \otimes y_j\right] = \sum_{j=1}^{k} \left(\sum_{i=1}^{n} e_i b_{ij} \otimes y_j\right) \\ &= \sum_{j=1}^{k} \left(\sum_{i=1}^{n} e_i \otimes b_{ij} y_j\right) = \sum_{i=1}^{n} \left(\sum_{j=1}^{k} e_i \otimes b_{ij} y_j\right) = \sum_{i=1}^{n} \left[e_i \otimes \left(\sum_{j=1}^{k} b_{ij} y_j\right)\right].\end{aligned}$$

Hence $x_i = \sum_{j=1}^{k} b_{ij} y_j$ for each i. Finally, it is easy to see that $m \sum_{i=1}^{n} a_i b_{ij} = 0$ for each $1 \leq j \leq k$.

(2) $\Rightarrow$ (1) Suppose that $L/\mathbf{r}_R(m)$ is finitely generated with $m \in M$ and

$$\mathbf{r}_R(m) \subseteq L = a_1 R + \cdots + a_n R + \mathbf{r}_R(m) \leq R.$$

Then each element in $(L/\mathbf{r}_R(m)) \otimes F$ is of the form

$$\sum_{i=1}^{n} (a_i + \mathbf{r}_R(m)) \otimes x_i$$

with $x_i \in F (i = 1, \ldots, n)$. If $\sum_{i=1}^{n} (a_i + \mathbf{r}_R(m)) \otimes x_i$ is contained in the kernel of the natural map

$$\mu : (L/\mathbf{r}_R(m)) \otimes F \to LF/\mathbf{r}_R(m)F,$$

i.e., $\sum_{i=1}^{n} a_i x_i \in \mathbf{r}_R(m)F$, we have $y_1, \ldots, y_k \in F$ and $n \times k$ matrix $B = (b_{ij})_{n \times k}$ over R such that $x_i = \sum_{j=1}^{k} b_{ij} y_j$ for each $1 \leq i \leq n$ and $m \sum_{i=1}^{n} a_i b_{ij} = 0$ for each $1 \leq j \leq k$ by (2). Consequently,

$$\begin{aligned}\sum_{i=1}^{n} (a_i + \mathbf{r}_R(m)) \otimes x_i &= \sum_{i=1}^{n} (a_i + \mathbf{r}_R(m)) \otimes \left(\sum_{j=1}^{k} b_{ij} y_j\right) \\ &= \sum_{i=1}^{n} \left(\sum_{j=1}^{k} (a_i b_{ij} + \mathbf{r}_R(m)) \otimes y_j\right) = \sum_{j=1}^{k} \left[\left(\sum_{i=1}^{n} a_i b_{ij} + \mathbf{r}_R(m)\right) \otimes y_j\right] \\ &= \sum_{j=1}^{k} (0 + \mathbf{r}_R(m)) \otimes y_j = 0 \in (L/\mathbf{r}_R(m)) \otimes F.\end{aligned}$$

This shows μ is monic and hence F is M-flat by Dauns (2006, Theorem 1.8 (d)). $\square$

Note that a left R-module F is M-flat if and only if $0 \to K \otimes_R F \to M \otimes_R F$ is exact for every finitely generated submodule K of M (see Wisbauer, 1991, 12.15(1)). Let $K = \sum_{i=1}^{n} m_i R$ and $f : R_n \to M$ be the composition of the canonical epimorphism $\pi : R_n \to K$ and the inclusion map $\alpha : K \to M$. Applying $\mathrm{Hom}_R(-, R)$ to f, we obtain $f^* : M^* \to R^n$ with image

$$\mathrm{Im} f^* = \left\{(g(m_1), g(m_2), \ldots, g(m_n)) \mid g \in M^* = \mathrm{Hom}_R(M, R)\right\},$$

which is a matrix subgroup of R^n in the sense of Zimmermann (1997). We refer the reader to Zimmermann (1997) for the general case. Following Zimmermann (1997), we write $\underline{m} = (m_1, \ldots, m_n) \in M^n$ and

$$H_{M,\underline{m}}(R) = \big\{(g(m_1), g(m_2), \ldots, g(m_n)) \mid g \in M^*\big\}.$$

Then we have the following commutative diagram with exact bottom row

$$\begin{array}{ccccc} & & \left(\sum_{i=1}^n m_iR\right) \otimes_R F & \xrightarrow{\alpha\otimes \mathrm{id}_F} & M \otimes_R F \\ & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\psi} \\ 0 & \longrightarrow & \mathrm{Hom}_R\big(H_{M,\underline{m}}(R), F\big) & \longrightarrow & \mathrm{Hom}_R(M^*, F) \end{array}$$

where $\psi : M \otimes_R F \to \mathrm{Hom}_R(M^*, F)$ is given by

$$\psi(m \otimes y) : g \mapsto g(m)y$$

for all $m \in M$, $y \in F$ and $g \in M^*$, and $\varphi : \left(\sum_{i=1}^n m_iR\right) \otimes_R F \to \mathrm{Hom}_R\big(H_{M,\underline{m}}(R), F\big)$ is defined such that

$$\varphi\left(\sum_{i=1}^n m_i \otimes y_i\right) : (g(m_1), g(m_2), \ldots, g(m_n)) \mapsto \sum_{i=1}^n g(m_i)y_i,$$

for all $y_i \in F$ $(i = 1, \ldots, n)$ and $g \in M^*$. If φ is injective then so is $\alpha \otimes \mathrm{id}_F$. And conversely if ψ and $\alpha \otimes \mathrm{id}_F$ are both injective, so is φ. Therefore we have the following proposition.

Proposition 2. *Let M be a right R-module and F a left R-module such that the canonical map $\psi : M \otimes_R F \to \mathrm{Hom}_R(M^*, F)$ is injective, then the following are equivalent for a left R-module F.*

(1) *F is M-flat.*
(2) *The canonical map $\varphi : \left(\sum_{i=1}^n m_iR\right) \otimes_R F \to \mathrm{Hom}_R(H_{M,\underline{m}}(R), F)$ is injective for all positive integers n and all $\underline{m} = (m_1, \ldots, m_n) \in M^n$.*

Remark 3. (1) It should be pointed out that the canonical map $\psi : M \otimes_R F \to \mathrm{Hom}_R(M^*, F)$ plays a prominent role in the theory of comodules. In particular, modules M such that ψ is injective for all (cyclic) ${}_RF$ have drawn the attention of many authors. We refer the readers to Brzeziński and Wisbauer (2003, pp. 438–440) for details.

(2) The hypothesis "$M \otimes_R F \to \mathrm{Hom}_R(M^*, F)$ is injective" is essential for (1) $\Rightarrow$ (2) in Proposition 2. To see this we need only take a finitely generated right R-module $M = \sum_{i=1}^n m_iR$ which is not torsionless. Then $F = {}_RR$ is M-flat but it does not satisfy (2) in case $\underline{m} = (m_1, \ldots, m_n) \in M^n$. In fact, there exist $0 \neq m = \sum_{i=1}^n m_ir_i \in M$ and for all $g \in M^*$, $g(m) = 0$ since M is not torsionless. It follows that $\varphi(\sum_{i=1}^n m_i \otimes r_i) = 0$ but $\sum_{i=1}^n m_i \otimes r_i = (\sum_{i=1}^n m_ir_i) \otimes 1 = m \otimes 1 \neq 0$ in $M \otimes R$.

Observing the following commutative diagram with exact rows

$$\begin{array}{ccccccc}
 & \mathbf{r}_{R_n}(\underline{m}) \otimes_R F & \longrightarrow & R_n \otimes_R F & \longrightarrow & \left(\sum_{i=1}^n m_i R\right) \otimes_R F & \longrightarrow 0 \\
 & \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \varphi} & \\
0 \longrightarrow & \mathrm{Hom}_R(R^n/H_{M,\underline{m}}(R), F) & \longrightarrow & \mathrm{Hom}_R(R^n, F) & \longrightarrow & \mathrm{Hom}_R(H_{M,\underline{m}}(R), F) &
\end{array}$$

where σ is the canonical map such that $\sigma(a \otimes y) : b + H_{M,\underline{m}}(R) \mapsto bay$, for all $a \in \mathbf{r}_{R_n}(\underline{m})$, $y \in F$, and $b \in R^n$, we have the following proposition.

Proposition 4. *Let M be a right R-module and F a left R-module. Then the following are equivalent for every $\underline{m} = (m_1, m_2, \ldots, m_n) \in M^n$:*

(1) *The canonical map $\varphi : \left(\sum_{i=1}^n m_i R\right) \otimes_R F \to \mathrm{Hom}_R(H_{M,\underline{m}}(R), F)$ is injective;*
(2) *The canonical map $\sigma : \mathbf{r}_{R_n}(\underline{m}) \otimes_R F \to \mathrm{Hom}_R(R^n/H_{M,\underline{m}}(R), F)$ is surjective;*
(3) *Every $f \in \mathrm{Hom}_R(R^n/H_{M,\underline{m}}(R), F)$ factors through a finitely generated free R-module, i.e., there exists ${}_RR^k$ $(1 \le k \in \mathbb{N})$, $g \in \mathrm{Hom}_R(R^n/H_{M,\underline{m}}(R), R^k)$ and $h \in \mathrm{Hom}_R(R^k, F)$ such that $f = hg$;*
(4) *$R^n/H_{M,\underline{m}}(R)$ is projective with respect to every short exact sequence of the form*

$$0 \to {}_RL \to {}_RP \to {}_RF \to 0, \qquad (*)$$

i.e., $(*)$ is $R^n/H_{M,\underline{m}}(R)$-pure in sense of Rothmaler (1994).

Proof. (1) $\Leftrightarrow$ (2) follows from the commutative diagram mentioned above.

(2) $\Rightarrow$ (3) For each $f \in \mathrm{Hom}_R(R^n/H_{M,\underline{m}}(R), F)$, by (2), there exist $b_j = \begin{bmatrix} b_{1j} \\ \vdots \\ b_{nj} \end{bmatrix} \in \mathbf{r}_{R_n}(\underline{m})(1 \le j \le k)$ and $y_1, \ldots, y_k \in F$ such that $\sigma\left(\sum_{j=1}^k b_j \otimes y_j\right) = f$. Now, define $g \in \mathrm{Hom}_R(R^n/H_{M,\underline{m}}(R), R^k)$ via $g(a + H_{M,\underline{m}}(R)) = aB$ for all $a \in R^n$, where $B = (b_1, \ldots, b_k)$, and define $h \in \mathrm{Hom}_R(R^k, F)$ via $h(e_j) = y_j$ for all $1 \le j \le k$, where $\{e_1, \ldots, e_k\}$ is the canonical basis of R^k. It is easy to verify that h and g are as desired.

(3) $\Rightarrow$ (2) Suppose (3), then for each $f \in \mathrm{Hom}_R(R^n/H_{M,\underline{m}}(R), F)$, we have $f = hg$ for some $h \in \mathrm{Hom}_R(R^k, F)$ and $g \in \mathrm{Hom}_R(R^n/H_{M,\underline{m}}(R), R^k)$. Let $B = \begin{bmatrix} g(\varepsilon_1 + H_{M,\underline{m}}(R)) \\ \vdots \\ g(\varepsilon_n + H_{M,\underline{m}}(R)) \end{bmatrix}$, where $\{\varepsilon_1, \ldots, \varepsilon_n\}$ is the canonical basis of R^n, and let $y_j = h(e_j)$, where $\{e_1, \ldots, e_k\}$ is the canonical basis of R^k. It follows that $\sigma\left(\sum_{j=1}^k b_j \otimes y_j\right) = f$, where b_j is the jth column of B $(1 \le j \le k)$. Therefore σ is surjective.

(3) $\Rightarrow$ (4) follows by the following commutative diagram

$$\begin{tikzcd}
 & R^k \arrow[d, dotted] \arrow[drr, dotted, "h"] & & R^n/H_{M,\underline{m}}(R) \arrow[ll, dotted, "g"'] \arrow[d, "f"] & \\
0 \arrow[r] & {}_RL \arrow[r] & {}_RP \arrow[r] & {}_RF \arrow[r] & 0\,.
\end{tikzcd}$$

(4) $\Rightarrow$ (3) Applying (4) to the exact sequence $0 \to {}_RL \to {}_RR^{(I)} \to {}_RF \to 0$, where I is a suitable index set, we have the following commutative diagram

$$\begin{tikzcd} & & R^n/H_{M,\underline{m}}(R) \arrow[dl, dotted, "\overline{f}"'] \arrow[d, "f"] & \\ 0 \arrow[r] & {}_RL \arrow[r] & {}_RR^{(I)} \arrow[r] & {}_RF \arrow[r] & 0. \end{tikzcd}$$

Since $R^n/H_{M,\underline{m}}(R)$ is finitely generated, the image of $\overline{f}$ is contained in a finitely generated free submodule R^k of ${}_RR^{(I)}$. Then (3) follows easily. $\square$

Corollary 5. *Let M be a right R-module and F a left R module such that the canonical map $\psi : M \otimes_R F \to \mathrm{Hom}_R(M^*, F)$ is injective, then the following are equivalent:*

(1) *F is M-flat;*
(2) *The canonical map $\sigma : \mathbf{r}_{R_n}(\underline{m}) \otimes_R F \to \mathrm{Hom}_R(R^n/H_{M,\underline{m}}(R), F)$ is surjective for all positive integers n and all $\underline{m} = (m_1, \ldots, m_n) \in M^n$;*
(3) *For all positive integers n and all $\underline{m} = (m_1, \ldots, m_n) \in M^n$, every R-homomorphism from $R^n/H_{M,\underline{m}}(R)$ to F factors through a finitely generated free R-module;*
(4) *Every exact sequence of the form $0 \to {}_RL \to {}_RP \to {}_RF \to 0$ is $R^n/H_{M,\underline{m}}(R)$-pure for all positive integers n and all $\underline{m} = (m_1, \ldots, m_n) \in M^n$.*

Next, we consider M-flatness for factor modules of M-flat modules.

Proposition 6. *Suppose that M is a fixed right R-module and L is a submodule of an M-flat left R-module F. If the canonical map $(M/K) \otimes L \to (M/K) \otimes F$ is injective for each (finitely generated) $K_R < M_R$ then F/L is M-flat. The converse holds if the canonical map $M \otimes L \to M \otimes F$ is injective.*

Proof. Let us denote M/K and F/L by $\overline{M}$ and $\overline{F}$, respectively. Consider the following exact commutative diagram for each (finitely generated) $K_R < M_R$

$$\begin{tikzcd} \mathrm{Tor}_1^R(M, F) \arrow[r, "d_7"] \arrow[d, "d_1"] & \mathrm{Tor}_1^R(M, \overline{F}) \arrow[r] \arrow[d, "d_5"] & M \otimes L \arrow[r, "d_8"] \arrow[d] & M \otimes F \arrow[d] \\ \mathrm{Tor}_1^R(\overline{M}, F) \arrow[r, "d_3"] \arrow[d] & \mathrm{Tor}_1^R(\overline{M}, \overline{F}) \arrow[r] \arrow[d] & \overline{M} \otimes L \arrow[r, "d_4"] & \overline{M} \otimes F \\ K \otimes F \arrow[r] \arrow[d, "d_2"] & K \otimes \overline{F} \arrow[d, "d_6"] & & \\ M \otimes F \arrow[r] & M \otimes \overline{F} & & \end{tikzcd}$$

where d_2 is injective since F is M-flat by hypotheses and hence d_1 is surjective. If d_4 is injective then d_3 is surjective and so is d_5. It follows that d_6 is injective. This shows that $\overline{F}$ is M-flat. Conversely, if d_6 and d_8 are injective then d_5 and d_7 are surjective. Consequently, d_3 is surjective and hence d_4 is injective. $\square$

Remark 7. Note that the canonical map $M \otimes L \to M \otimes F$ in the above proposition need not be injective in general even if every left R-module is M-flat. Recall that a ring R is *right FS* (see Liu, 1995) if the socle of R_R is flat as a right R-module. Now, let us take a ring R which is not right FS (e.g., the ring R in Example 10 below) and $M = \mathrm{Soc}(R_R)$, the socle of R_R. Obviously, every left R-module is M-flat since M is semisimple. But M is not flat hence $M \otimes L \to M \otimes F$ need not be injective in general.

Corollary 8. *Let L be a pure submodule of an M-flat left R-module F. Then both L and F/L are M-flat.*

Proof. Let L be a pure submodule of an M-flat left R-module F. Then for every submodule K of M, we have the following commutative diagram

$$\begin{array}{ccc} K \otimes L & \xrightarrow{f} & K \otimes F \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\ M \otimes L & \longrightarrow & M \otimes F \end{array}$$

where f and g are monic and hence so is h. Therefore L is M-flat. Finally, F/L is M-flat by Proposition 6. □

Given a ring R and a fixed right R-module M. Let M-$\mathcal{F}$ (respectively, $\mathcal{F}$) be the class of all M-flat (respectively, flat) left R-modules. Then we have the following proposition.

Proposition 9.

(1) *R is von Neumann regular if and only if M-$\mathcal{F} = \mathcal{F}$ for every right R-module M.*
(2) *R is right noetherian if and only if R is M-coherent for every (cyclic) right R-module M.*

Proof. (1) is obvious.

(2) If R is right noetherian, then R is right coherent and $\mathbf{r}_R(m)$ is finitely generated for all $m \in M$. Hence R is M-coherent by Dauns (2006, Observations 2.5(i)). Conversely, for each right ideal I of R, R is R/I-coherent and hence $I = \mathbf{r}_R(1 + I)$ is finitely generated. Therefore R is right noetherian. □

It is well known that R is right coherent if and only if every direct product of flat left R-modules is flat. The following example shows that the analogous statement for M-flatness and M-coherence fails. It also shows that the hypothesis "$\mathbf{r}_R(m)$ is finitely generated for all $m \in M$" is essential for Dauns (2006, Theorem 2.6) and answers the question in Dauns (2006, p. 308).

Example 10. There is a ring R with a module M_R such that every left R-module is M-flat but R is not M-coherent.

Proof. Note that every left R-module is M-flat if and only if every submodule of M is pure in M, i.e., M is regular in Mod-R in sense of Wisbauer (1991). Let $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ and $A = \bigoplus_{i=1}^{\infty} \mathbb{Z}_2$ be the direct sum of countably infinite copies of $\mathbb{Z}_2$. Then,

$$R = \mathbb{Z}_2 \propto A = \left\{ \begin{bmatrix} a & \alpha \\ 0 & a \end{bmatrix} \middle| a \in \mathbb{Z}_2, \alpha \in A \right\},$$

the trivial extension of $\mathbb{Z}_2$ by A is a commutative ring with $\mathrm{Soc}(R) = 0 \propto A$. Now, let $M = \mathrm{Soc}(R) = 0 \propto A$. It follows that every left R-module is M-flat since M is semisimple and hence regular in Mod-R. But the right annihilator of

$$m = \begin{bmatrix} 0 & (1, 0, 0, \dots) \\ 0 & 0 \end{bmatrix}$$

is not finitely generated. Therefore R is not M-coherent by Dauns (2006, Consequences 2.3(1)). □

Note that $M = \mathrm{Soc}(R)$ is the unique maximal (right) ideal of the ring R in Example 10. Moreover, the left annihilator of M in R is M itself. By Liu (1995, Theorem 2.4), R is not (right) FS as we claimed in Remark 7.

Let L be a left R-module and $\mathscr{C}$ a class of left R-modules. Recall from Enochs and Jenda (2000) that an R-homomorphism $\varphi : L \to C$ with $C \in \mathscr{C}$ is called a *$\mathscr{C}$-preenvelope* of L if every $\psi \in \mathrm{Hom}_R(L, C')$ with $C' \in \mathscr{C}$ factors through φ. The class $\mathscr{C}$ is said to be *preenveloping* if every left R-module has a *$\mathscr{C}$-preenvelope*.

Theorem 11. *Let M be a right R-module. Then*

(1) *M-$\mathscr{F}$ is closed under direct products if and only if M-$\mathscr{F}$ is preenveloping.*
(2) *R^{Γ} is M-flat for any set Γ if and only if every projective right R-module P has an M-flat dual module P^* if and only if every direct product of flat left R-modules is M-flat.*

Proof. (1) By Corollary 8, M-$\mathscr{F}$ is closed under pure submodules. Thus (1) follows by a slight modification of the proof of Enochs and Jenda (2000, Proposition 6.5.1).

(2) For every projective right R-module P, we have $P \oplus Q = R^{(I)}$ for some Q_R and indexed set I. It follows that $P^* \oplus Q^* = (R^{(I)})^* = R^I$. If R^I is M-flat then so is P^*. Conversely, suppose every projective right R-module P has an M-flat dual module P^*. Then for every index set I, the free right R-module $R^{(I)}$ has an M-flat dual $(R^{(I)})^* = R^I$.

To complete the proof, it remains only to show that every direct product of flat left R-modules is M-flat provided R^{Γ} is M-flat for any set Γ.

For any set $\{F_\gamma \mid \gamma \in \Gamma\}$ of flat left R-modules, we have pure exact sequences

$$0 \to K_\gamma \to R^{(I_\gamma)} \to F_\gamma \to 0 \qquad (\gamma \in \Gamma)$$

and

$$0 \to \prod_{\gamma\in\Gamma} K_\gamma \to \prod_{\gamma\in\Gamma} R^{(I_\gamma)} \to \prod_{\gamma\in\Gamma} F_\gamma \to 0$$

where $\prod_{\gamma\in\Gamma} R^{(I_\gamma)}$ is a pure submodule of $\prod_{\gamma\in\Gamma} R^{I_\gamma}$. But $\prod_{\gamma\in\Gamma} R^{I_\gamma}$ is M-flat by hypothesis. Therefore $\prod_{\gamma\in\Gamma} R^{(I_\gamma)}$ is M-flat by Corollary 8. Since $0 \to \prod_{\gamma\in\Gamma} K_\gamma \to \prod_{\gamma\in\Gamma} R^{(I_\gamma)}$ is pure, $\prod_\Gamma F_\gamma$ is M-flat by Corollary 8 again. □

Remark 12. So far it is unknown to us whether M-$\mathcal{F}$ is closed under products whenever R^Γ is M-flat for any set Γ. To find a ring R with a right R-module M such that R^Γ is M-flat for any set Γ but M-$\mathcal{F}$ is not closed under products, we have to consider those rings which are neither right noetherian nor von Neumann regular. Moreover the module M should not be regular in Mod-R.

But by Dauns (2006, Theorem 2.6), we have the following corollary.

Corollary 13. *The following are equivalent for a fixed right R-module M such that $\mathbf{r}_R(m)$ is finitely generated for all $m \in M$:*

(1) *R is M-coherent*;
(2) *M-$\mathcal{F}$ is preenveloping*;
(3) *$P^* \in M$-$\mathcal{F}$ for every projective right R-module P.*

Recall from Wisbauer (1991) that a right R module U is *M-injective* if every diagram of right R-modules with exact row

$$\begin{array}{ccccc} 0 & \longrightarrow & K & \xrightarrow{f} & M \\ & & \downarrow & & \\ & & U & & \end{array}$$

can be extended commutatively by a morphism $M \to U$. It is proved by Dauns (2006, Theorem 1.7) that a left R-module F is M-flat if and only if the character module $F^+ = \mathrm{Hom}_{\mathbb{Z}}(F, \mathbb{Q}/\mathbb{Z})$ is M-injective. Note that this is in fact a special case of Wisbauer (1991, 17.14). The following result is motivated by Cheatham and Strone (1981, Theorem 1).

Proposition 14. *Consider the following conditions for a fixed right R-module M:*

(1) *A right R-module N is M-injective if and only if N^+ is M-flat*;
(2) *A right R-module N is M-injective if and only if N^{++} is M-injective*;
(3) *A left R-module F is M-flat if and only if F^{++} is M-flat*;
(4) *M-$\mathcal{F}$ is closed under direct products.*

We have (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4).

Proof. (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3) follows from Dauns (2006, Theorem 1.7).

(3) $\Rightarrow$ (4) We adopt the method in Cheatham and Strone (1981, Theorem 1). For any index set I and $F_i \in M\text{-}\mathscr{F}(i \in I)$, we have

$$\begin{tikzcd}
0 \arrow[r] & \prod_{i\in I} F_i \arrow[r, "f"] & \prod_{i\in I} F_i^{++} \arrow[d, "\cong"] & \\
\left(\bigoplus_{i\in I} F_i\right)^{++} \arrow[r, "\cong"] & \left(\prod_{i\in I} F_i^{+}\right)^{+} \arrow[r, "g"] & \left(\bigoplus_{i\in I} F_i^{+}\right)^{+} \arrow[r] & 0
\end{tikzcd}$$

where $\bigoplus_{i\in I} F_i$ is an M-flat left R-module. Then $(\bigoplus_{i\in I} F_i)^{++}$ is M-flat by (3). Since $\bigoplus_{i\in I} F_i^{+}$ is a pure submodule of $\prod_{i\in I} F_i^{+}$ by Cheatham and Strone (1981, Lemma 1(1)), it follows that g is split epic. Consequently, $\left(\bigoplus_{i\in I} F_i^{+}\right)^{+}$ is M-flat and so is $\prod_{i\in I} F_i^{++}$. By Cheatham and Strone (1981, Lemma 1(2)), f is pure monic. This guarantees that $\prod_{i\in I} F_i$ is an M-flat left R-module. $\square$

Remark 15. Note that (3) of Proposition 14 does not imply (1) even if R is M-coherent. For instance, the endomorphism ring R of a countably infinite dimensional vector space V over a field is von Neumann regular. Let $M = R_R$ then (3) of Proposition 14 holds since every left R-module F is (M-)flat. By Cheatham and Strone (1981, Theorem 2), (1) of Proposition 14 does not hold since R is not right noetherian.

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REFERENCES

Brzeziński, T., Wisbauer, R. (2003). *Corings and Comodules*. London Math. Soc. Lecture Note Series 309. Cambridge: Cambridge University Press.

Cheatham, T. J., Strone, D. R. (1981). Flat and projective character modules. *Proc. Amer. Math. Soc.* 81:175–177.

Dauns, J. (2006). Relative flatness and coherence. *Comm. Algebra* 34:303–312.

Enochs, E. E., Jenda, O. M. G. (2000). *Relative Homological Algebra*. de Gruyter Expositions in Math. 30. Berlin: de Gruyter.

Liu, Z. K. (1995). Rings with flat left socle. *Comm. Algebra* 23:1645–1656.

Rothmaler, P. (1994). *Mittag-Leffler Modules and Positive Atomicity*. Kiel: Habilitationsschrift.

Wisbauer, R. (1991). *Foundations of Module and Ring Theory*. Philadelphia: Gordon and Breach Sience Publishers.

Zimmermann, W. (1997). Modules with chain conditions for finite matrix subgroups. *J. Algebra* 190:68–87.