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Minimax optimal control of uncertain quasi-integrable Hamiltonian systems with time-delayed bounded feedback

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ABSTRACT

A minimax optimal control strategy for uncertain quasi-integrable Hamiltonian systems with time-delayed bounded feedback control is proposed. First, a quasi-integrable Hamiltonian system with time-delayed bounded control forces and uncertain excitation and system parameters is converted into a set of Itô stochastic differential equations without time delay. Then, the partially averaged Itô stochastic differential equations for the energy processes are derived by using the stochastic averaging method for quasi-integrable Hamiltonian systems. For these equations together with an appropriate performance index, a worst-case optimal control strategy is derived via solving a stochastic differential game problem. The worst-case disturbances and the optimal bounded controls are obtained by solving a Hamilton–Jacobi–Isaacs (HJI) equation. Finally, two examples are worked out in detail to illustrate the application and effectiveness of the proposed method.

1. Introduction

Stochastic optimal control is a significant subject in the field of structural engineering since engineering systems are subjected to severe random vibration due to wind, wave, earthquake, etc. The basic mathematical theory of stochastic optimal control has been well developed [1–4]. However, a lot of problems have to be solved before the theory can be applied to engineering systems, such as the boundness and time delay of control forces, and the uncertainty of system parameters and excitations, etc.

In engineering applications, the capacity of control actuators is always bounded. In view of the stochastic nature of the excitations, it is conceivable that the required control force may well exceed the capacity which will result in actuator saturation. The actuator saturation may cause deterioration of the control performance. Consequently, special attention should be paid to the actuator saturation problem. Zhu and Ying have proposed a modified control strategy for quasi-Hamiltonian systems with actuator saturation [5–7]. Huan extended this control strategy to a partially observable and hysteretic quasi-Hamiltonian system with actuator saturation [8,9]. It has been proved that this control strategy has high control effectiveness and control efficiency and chattering is reduced significantly compared with the bang–bang control strategy.

In the implementation of feedback control, time delay is usually unavoidable due to the time spent in measuring and estimating the system state, calculating and executing the control forces. Time delay may deteriorate the control performance and even render dynamical systems unstable. The effects of time delay on the controlled linear systems under Gaussian random excitation have been studied by Di Paola and Pirrotta [10] using an approach based on the Taylor expansion of the control force. The response, stability and bifurcation of quasi-integrable Hamiltonian systems with time-delayed feedback control under Gaussian white noise excitation has been studied by Liu and Zhu [11–13] using the stochastic averaging method. Two time delay compensation methods are also proposed in Ref. [14].

For practical engineering systems, parametric and external disturbances are usually uncertain, which may also degenerate the performance of the controller. In the past several decades, the robust control of the deterministic linear and nonlinear systems with uncertain disturbances has been studied extensively [15]. The robust control of linear and nonlinear stochastic systems with uncertain parameters and external disturbances has also been investigated [16,17].

In the present paper, a minimax optimal bounded control strategy for uncertain quasi-integrable Hamiltonian systems with time-delayed feedback control is proposed based on stochastic averaging method [18–20] and stochastic differential game, and the boundness and time delay of control forces, and the uncertainty of system parameters and excitation are all considered. The application and effectiveness of the proposed control strategy are illustrated by using two examples.

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2. Formulation of the problem

Consider an n-degree-of-freedom (nDOF) uncertain quasi-Hamiltonian system with time-delayed bounded feedback control. The differential equations of motion of the system is

$$
\begin{aligned}
\dot{Q}_i &= \frac{\partial H'(\boldsymbol{Q},\boldsymbol{P})}{\partial P_i}\\
\dot{P}_i &= -\frac{\partial H'(\boldsymbol{Q},\boldsymbol{P})}{\partial Q_i} - [\bar{c}_{ij}(\boldsymbol{Q},\boldsymbol{P}) + \tilde{c}_{ij}(t)]\frac{\partial H'(\boldsymbol{Q},\boldsymbol{P})}{\partial P_j}\\
&\quad + u_i(\boldsymbol{Q}_\tau,\boldsymbol{P}_\tau) + f_{ik}(\boldsymbol{Q},\boldsymbol{P})\xi_k(t) - \tilde{s}_l(t)g_{il}(\boldsymbol{Q}) + \tilde{w}_i(t)\\
|u_i| &\le U_i^0, \quad i,j = 1,2,\ldots,n;\ k = 1,2,\ldots,m;\ l = 1,2,\ldots,p
\end{aligned}
\tag{1}
$$

where Q_i and P_i are generalized displacements and momenta, respectively, $\boldsymbol{Q} = [Q_1, Q_2, \ldots, Q_n]^T$, $\boldsymbol{P} = [P_1, P_2, \ldots, P_n]^T$; $H'(\boldsymbol{Q}, \boldsymbol{P}, \bar{\boldsymbol{s}})$ is the Hamiltonian generally representing total system energy; $\bar{c}_{ij}$ are the nominal values of damping coefficients; f_{ik} are the amplitudes of excitations; $\xi_k(t)$ are Gaussian white noises with zero mean and correlation function $2D_{kl}\delta(s)$; $u_i(\boldsymbol{Q}_\tau, \boldsymbol{P}_\tau)$ with $\boldsymbol{Q}_\tau = Q(t-\tau)$ and $\boldsymbol{P}_\tau = P(t-\tau)$ represent time-delayed feedback control, which is usually bounded due to actuator saturation; $\tilde{s}_l(t)$, $\tilde{c}_{ij}(t)$ and $\tilde{w}_i(t)$ represent the uncertain disturbances, respectively. It is assumed that $\tilde{c}_{ij}(t)$, $\tilde{s}_l(t)$ and $\tilde{w}_i(t)$ are all bounded, i.e. $\tilde{c}_{ij}(t) \in [-c_{ij}^0, c_{ij}^0]$, $\tilde{s}_l(t) \in [-s_l^0, s_l^0]$ and $\tilde{w}_i(t) \in [-w_i^0, w_i^0]$. In practice, the bounds of all disturbances can be determined by the confidence interval with higher degree of confidence, for example, 3σ criteria for a Gaussian distribution.

The objective of control is to minimize the response of system (1) which can be expressed in terms of minimizing a performance index depending on control time interval. For finite time-interval control, the performance index is of the form

$$
J_1(\tilde{s}_l, \tilde{c}_{ij}, \tilde{w}_i, u_i) = E\left[\int_0^{t_f} f_1(\boldsymbol{Q}(t), \boldsymbol{P}(t), \boldsymbol{u}(\boldsymbol{Q}_\tau, \boldsymbol{P}_\tau))\mathrm{d}t + g(\boldsymbol{Q}(t_f), \boldsymbol{P}(t_f))\right]
\tag{2}
$$

and for infinite time-interval ergodic control, the index is of the form

$$
J_2(\tilde{s}_l, \tilde{c}_{ij}, \tilde{w}_i, u_i) = \lim_{T\to\infty}\frac{1}{T}\int_0^T f_2(\boldsymbol{Q}(t), \boldsymbol{P}(t), \boldsymbol{u}(\boldsymbol{Q}_\tau, \boldsymbol{P}_\tau))\mathrm{d}t
\tag{3}
$$

where $E[\cdot]$ denotes an expectation operation; t_f is the terminal time of control; f_1, f_2 are called cost functions and g is final cost. Eqs. (1) and (2) or (3) constitute the mathematical formulation of the optimal control problem for an nDOF uncertain quasi-Hamiltonian system with time-delayed bounded feedback control.

3. Converted partially averaged system without time delay

Assume that the Hamiltonian system associated with system (1) is integrable and nonresonant. That is, the Hamiltonian system has n independent first integrals $H_1, H_2, \ldots, H_n$. In this case, Liu and Zhu [11] has proved that for small delay time τ, the time-delayed feedback control forces can be expressed approximately in terms of system state variables without time delay by using the following approximate expressions [11]

$$
\begin{aligned}
Q_{\tau i} &= Q_i(t-\tau_i) \approx Q_i\cos\omega_i\tau_i - \frac{P_i}{\omega_i}\sin\omega_i\tau_i\\
P_{\tau i} &= P_i(t-\tau_i) \approx P_i\cos\omega_i\tau_i + Q_i\omega_i\sin\omega_i\tau_i
\end{aligned}
\tag{4}
$$

where ω_i denote the averaging frequencies of system (1). In this case, the time-delayed feedback control forces $u_i(\boldsymbol{Q}_\tau, \boldsymbol{P}_\tau)$ can be replaced by the control forces without time delay, i.e., $\bar{u}_i(Q_i, P_i, \tau_i) = \bar{u}_i(Q_i\cos\omega_i\tau_i - \frac{P_i}{\omega_i}\sin\omega_i\tau_i, P_i\cos\omega_i\tau_i + Q_i\omega_i\sin\omega_i\tau_i)$. Then, by adding Wong–Zakai correction terms the Itô differential equations of system (1) without time delay can be obtained:

$$
\begin{aligned}
\mathrm{d}Q_i &= \frac{\partial H(\boldsymbol{Q},\boldsymbol{P})}{\partial P_i}\mathrm{d}t\\
\mathrm{d}P_i &= -\left[\frac{\partial H(\boldsymbol{Q},\boldsymbol{P})}{\partial Q_i} + (m_{ij}(\boldsymbol{Q},\boldsymbol{P}) + \tilde{c}_{ij}(t))\frac{\partial H(\boldsymbol{Q},\boldsymbol{P})}{\partial P_j} - \bar{u}_i(\boldsymbol{Q},\boldsymbol{P},\boldsymbol{\tau}) + \tilde{s}_l(t)g_{il}(\boldsymbol{Q}) - \tilde{w}_i(t)\right]\mathrm{d}t\\
&\quad + \sigma_{ik}(\boldsymbol{Q},\boldsymbol{P})\mathrm{d}B_k(t)\\
|\bar{u}_i| &\le U_i^0
\end{aligned}
\tag{5}
$$

where H and m_{ij} represent the new Hamiltonian and damping coefficients possibly modified by the Wong–Zakai correction terms, respectively; σ_{ik} are the elements of matrix $\boldsymbol{\sigma}$ with $\boldsymbol{\sigma}\boldsymbol{\sigma}^T = 2\boldsymbol{f}\boldsymbol{D}\boldsymbol{f}^T$; $B_k(t)$ are the standard Wiener processes; $\bar{u}_i$ are bounded feedback control without time delay.

The stochastic averaging method for quasi-Hamiltonian systems has been well developed [18–20]. The dimension and form of the averaged Itô equations depend upon the integrability and resonance of the associated Hamiltonian system. For integrable and nonresonant case, the partially averaged Itô differential equations of system (5) can be obtained by using the stochastic averaging method for quasi-integrable Hamiltonian systems [19] as follows

$$
\begin{aligned}
\mathrm{d}H_r &= \left[\bar{m}_r(\boldsymbol{H}) + \left\langle\frac{\partial H_r}{\partial P_i}(\bar{u}_i(\boldsymbol{Q},\boldsymbol{P},\boldsymbol{\tau}) - \tilde{s}_l(t)g_{il} - \tilde{c}_{ij}(t)\frac{\partial H}{\partial P_j} + \tilde{w}_i(t))\right\rangle\right]\mathrm{d}t + \bar{\sigma}_{rk}(\boldsymbol{H})\mathrm{d}B_k(t)\\
|\bar{u}_i| &\le U_i^0, \quad r = 1,2,\ldots,n
\end{aligned}
\tag{6}
$$

where H_r are the independent first integrals; $\bar{m}_r(\boldsymbol{H})$ and $\bar{\sigma}_{rk}(\boldsymbol{H})$ are the drift and diffusion coefficients determined by

$$
\begin{aligned}
\bar{m}_r(\boldsymbol{H}) &= \frac{1}{T(\boldsymbol{H})}\oint\left[\left(-m_{ij}\frac{\partial H}{\partial p_j}\frac{\partial H_r}{\partial p_i} + D_{kl}f_{ik}f_{jl}\frac{\partial^2 H_r}{\partial p_i\partial p_j}\right)\Big/\prod_{u=1}^n\left(\frac{\partial H_u}{\partial p_u}\right)\right]\mathrm{d}q_1\mathrm{d}q_2\cdots\mathrm{d}q_n\\
\bar{\sigma}_{ru}(\boldsymbol{H})\bar{\sigma}_{su}(\boldsymbol{H}) &= \frac{1}{T(\boldsymbol{H})}\oint\left[\left(2D_{kl}f_{ik}f_{jl}\frac{\partial H_r}{\partial p_i}\frac{\partial H_s}{\partial p_j}\right)\Big/\prod_{u=1}^n\left(\frac{\partial H_u}{\partial p_u}\right)\right]\mathrm{d}q_1\mathrm{d}q_2\cdots\mathrm{d}q_n\\
T(\boldsymbol{H}) &= \oint\left[1\Big/\prod_{u=1}^n\left(\frac{\partial H_u}{\partial p_u}\right)\right]\mathrm{d}q_1\mathrm{d}q_2\cdots\mathrm{d}q_n\\
\langle\cdot\rangle &= \frac{1}{T(\boldsymbol{H})}\oint\left[\cdot\Big/\prod_{u=1}^n\left(\frac{\partial H_u}{\partial p_u}\right)\right]\mathrm{d}q_1\mathrm{d}q_2\cdots\mathrm{d}q_n.
\end{aligned}
\tag{7}
$$

Note that the second terms on the right-hand side of Eq. (6) has not been averaged since $\bar{u}_i$, $\tilde{s}_l$, $\tilde{c}_{ij}$ and $\tilde{w}_i$ are unknown at this stage. It is seen from Eq. (6) that the dimension of system (5) is reduced from $2n$ to n after using the stochastic averaging method.

To be consistent with partially averaged Eq. (6), performance index (2) and (3) are also partially averaged, i.e., Eqs. (2) and (3) are replaced with

$$
J_3(\tilde{s}_l, \tilde{c}_{ij}, \tilde{w}_i, u_i) = E\left[\int_0^{t_f} f_3(H(s), \langle\bar{\boldsymbol{u}}(\boldsymbol{Q},\boldsymbol{P},\boldsymbol{\tau})\rangle)\mathrm{d}s + g_1(H(t_f))\right]
\tag{8}
$$

and

$$J_4(\tilde{s}_l, \tilde{c}_{ij}, \tilde{w}_i, u_i) = \lim_{T\to\infty} \frac{1}{T}\int_0^T f_4(H(s), \langle \bar{u}(\boldsymbol{Q}, \boldsymbol{P}, \boldsymbol{\tau})\rangle)\mathrm{d}s \tag{9}$$

respectively. f_3, f_4 are partially averaged cost functions and g_1 is averaged final cost.

4. Minimax optimal bounded control

Eqs. (6) and (8) or (9) constitute the mathematical formulation of the optimal bounded control problem for an uncertain partially averaged quasi-integrable Hamiltonian system without time delay. The proposed control strategy is based on searching for a worst-case optimal control law by solving the following stochastic differential game problem

$$\inf_{\bar{u}_i\in U^0} \sup_{\tilde{s}_l,\tilde{c}_{ij},\tilde{w}_i} J(\tilde{s}_l, \tilde{c}_{ij}, \tilde{w}_i, \bar{u}_i) \tag{10}$$

i.e., selecting $\tilde{s}_l^*, \tilde{c}_{ij}^*, \tilde{w}_i^*$ and $\bar{u}_i^*$ so that

$$J(\tilde{s}_l, \tilde{c}_{ij}, \tilde{w}_i, \bar{u}_i^*) \le J(\tilde{s}_l^*, \tilde{c}_{ij}^*, \tilde{w}_i^*, \bar{u}_i^*) \le J(\tilde{s}_l^*, \tilde{c}_{ij}^*, \tilde{w}_i^*, \bar{u}_i). \tag{11}$$

By applying the principle of optimality to system (6) with performance index (8), the following Hamilton–Jacobi–Isaacs (HJI) equation is established

$$\begin{aligned}\frac{\partial V_1}{\partial t} = \inf_{\bar{\boldsymbol{u}}\in U^0}\sup_{\tilde{s}_l,\tilde{c}_{ij},\tilde{w}_i} \Bigg\{ & f_3(\boldsymbol{H}, \langle\bar{\boldsymbol{u}}\rangle) + \Bigg[\bar{m}_r(\boldsymbol{H}) \\ &+ \left\langle \frac{\partial H_r}{\partial P_i}\left(\bar{u}_i - \tilde{s}_l(t)g_{il}(\boldsymbol{Q}) - \tilde{c}_{ij}(t)\frac{\partial H}{\partial P_j} + \tilde{w}_i(t)\right)\right\rangle\Bigg]\frac{\partial V_1}{\partial H_r} \\ &+ \frac{1}{2}\bar{\sigma}_{rk}(\boldsymbol{H})\bar{\sigma}_{sk}(\boldsymbol{H})\frac{\partial^2 V_1}{\partial H_r \partial H_s}\Bigg\}. \end{aligned} \tag{12}$$

For system (6) with performance index (9), the associated HJI equation is

$$\begin{aligned}\gamma = \inf_{\bar{\boldsymbol{u}}\in U^0}\sup_{\tilde{s}_l,\tilde{c}_{ij},\tilde{w}_i} \Bigg\{ & f_4(\boldsymbol{H}, \langle\bar{\boldsymbol{u}}\rangle) + \Bigg[\bar{m}_r(\boldsymbol{H}) \\ &+ \left\langle \frac{\partial H_r}{\partial P_i}\left(\bar{u}_i - \tilde{s}_l(t)g_{il}(\boldsymbol{Q}) - \tilde{c}_{ij}(t)\frac{\partial H}{\partial P_j} + \tilde{w}_i(t)\right)\right\rangle\Bigg]\frac{\partial V_2}{\partial H_r} \\ &+ \frac{1}{2}\bar{\sigma}_{rk}(\boldsymbol{H})\bar{\sigma}_{sk}(\boldsymbol{H})\frac{\partial^2 V_2}{\partial H_r \partial H_s}\Bigg\} \end{aligned} \tag{13}$$

where $V_1 = V(\boldsymbol{H}, t)$ and $V_2 = V_2(\boldsymbol{H}, t)$ are the value functions, γ is a constant and U^0 denotes the following domain of bounded control force $\bar{\boldsymbol{u}}$

$$U^0 : |\bar{u}_i| \le U_i^0, \quad U_i^0 > 0, i = 1, 2, \ldots, n. \tag{14}$$

Let function f_4 (or f_3) be of the form

$$f_4(\boldsymbol{H}, \langle\bar{\boldsymbol{u}}\rangle) = f_c(\boldsymbol{H}) + \langle \bar{\boldsymbol{u}}^T \boldsymbol{R}\bar{\boldsymbol{u}}\rangle \tag{15}$$

where $\boldsymbol{R}$ is a positive-definite diagonal matrix, $\boldsymbol{R} = \mathrm{diag}(R_i)$ and $f_c(\boldsymbol{H}) > 0$ is a convex function. Then, the worst-case disturbances can be determined by maximizing the right-hand side of Eq. (12) (or Eq. (13)) with respect to $\tilde{c}_{ij}(t), \tilde{s}_l(t)$ and $\tilde{w}_i(t)$. Due to the boundness of disturbances, the worst-case disturbances are determined by the following expressions

$$\begin{aligned} \tilde{s}_l^*(t) &= -s_l^0 \mathrm{sgn}\left[g_{il}(\boldsymbol{Q})\frac{\partial H_r}{\partial p_i}\frac{\partial V_2}{\partial H_r}\right] \\ \tilde{c}_{ij}^*(t) &= -c_{ij}^0 \mathrm{sgn}\left[\frac{\partial H}{\partial p_j}\frac{\partial H_r}{\partial p_i}\frac{\partial V_2}{\partial H_r}\right] \\ \tilde{w}_i^*(t) &= w_i^0 \mathrm{sgn}\left[\frac{\partial H_r}{\partial p_i}\frac{\partial V_2}{\partial H_r}\right] \end{aligned} \tag{16}$$

where sgn[·] is the sign function. Minimizing the right-hand side of Eq. (13) (or Eq. (12)) with respect to $\bar{u}_i$, the optimal control forces can be obtained. Due to the control constraints in Eq. (14), the optimal control forces for system (5) are of the form

$$\bar{u}_i^* = \begin{cases} -\dfrac{1}{2R_i}\dfrac{\partial H_r}{\partial P_i}\dfrac{\partial V_2}{\partial H_r}, & \left|\dfrac{1}{2R_i}\dfrac{\partial H_r}{\partial P_i}\dfrac{\partial V_2}{\partial H_r}\right| < U_i^0 \\ -U_i^0 \mathrm{sgn}\left(\dfrac{\partial V_2}{\partial H_r}\dfrac{\partial H_r}{\partial P_i}\right), & \left|\dfrac{1}{2R_i}\dfrac{\partial H_r}{\partial P_i}\dfrac{\partial V_2}{\partial H_r}\right| \ge U_i^0, \end{cases} \quad i = 1, 2, \ldots, n. \tag{17}$$

Substituting the worst-case disturbances $\tilde{s}_l^*, \tilde{c}_{ij}^*, \tilde{w}_i^*$ in Eq. (16) and the optimal bounded control forces $\bar{u}_i^*$ in Eq. (17) into HJI Eq. (12) or (13) and completing the averaging yield the final HJI equations. $\partial V_2/\partial H_r$ in Eqs. (16) and (17) can be obtained from solving these final HJI equations. Then, the worst-case disturbances $\tilde{s}_l^*, \tilde{c}_{ij}^*, \tilde{w}_i^*$ and the optimal controls $\bar{u}_i^*$ for system (5) are determined by substituting $\partial V_2/\partial H_r$ into Eqs. (16) and (17).

By using Eq. (4), the system states without time delay in optimal control forces $\bar{u}_i^*$ can be expressed approximately in terms of system state variables with time delay as following

$$\begin{aligned} Q_i(t) &\approx Q_{\tau i}\cos\omega_i\tau_i + \frac{P_{\tau i}}{\omega_i}\sin\omega_i\tau_i \\ P_i(t) &\approx -Q_{\tau i}\omega_i\sin\omega_i\tau_i + P_{\tau i}\cos\omega_i\tau_i. \end{aligned} \tag{18}$$

By substituting Eq. (18) into Eq. (17), the following optimal time-delayed controls u_i^* for system (1) can be obtained.

$$\begin{aligned} u_i^*(\boldsymbol{Q}_\tau, \boldsymbol{P}_\tau) &= \begin{cases} -F_i(\boldsymbol{Q}_\tau, \boldsymbol{P}_\tau), & |F_i(\boldsymbol{Q}_\tau, \boldsymbol{P}_\tau)| < U_i^0 \\ -U_i^0 \mathrm{sgn}(F_i(\boldsymbol{Q}_\tau, \boldsymbol{P}_\tau)), & |F_i(\boldsymbol{Q}_\tau, \boldsymbol{P}_\tau)| \ge U_i^0 \end{cases} \\ F_i(\boldsymbol{Q}_\tau, \boldsymbol{P}_\tau) &= \frac{1}{2R_i}\frac{\partial V_{2\tau}}{\partial H_{\tau r}}\frac{\partial H_{\tau r}}{\partial P_{\tau i}} \\ &= \frac{1}{2R_i}\left(\frac{\partial V_2}{\partial H_r}\frac{\partial H_r}{\partial P_i}\right)\Bigg|_{\substack{Q_i = Q_{\tau i}\cos\omega_i\tau_i + \frac{P_{\tau i}}{\omega_i}\sin\omega_i\tau_i \\ P_i = -Q_{\tau i}\omega_i\sin\omega_i\tau_i + P_{\tau i}\cos\omega_i\tau_i}} \end{aligned} \tag{19}$$

$\partial V_2/\partial H_r$ in Eq. (17) are functions of Hamiltonian $\boldsymbol{H}$. For quasi-integrable Hamiltonian system, $\boldsymbol{H}$ generally represents slowly varying quantitative. For small delay time, $\partial V_{2\tau}/\partial H_{\tau r}$ can be approximately replaced by $\partial V_2/\partial H_r$. Then, the second equation of Eq. (19) can be rewritten as

$$F_i(\boldsymbol{Q}_\tau, \boldsymbol{P}_\tau) = \frac{1}{2R_i}\frac{\partial V_2}{\partial H_r}\left(\frac{\partial H_r}{\partial P_i}\right)\Bigg|_{\substack{Q_i = Q_{\tau i}\cos\omega_i\tau_i + \frac{P_{\tau i}}{\omega_i}\sin\omega_i\tau_i \\ P_i = -Q_{\tau i}\omega_i\sin\omega_i\tau_i + P_{\tau i}\cos\omega_i\tau_i}}. \tag{20}$$

For comparison, the optimal control strategy without considering time delay is also considered. The control forces of this control strategy are of the form

$$u_{wodi}^*(\boldsymbol{Q}_\tau, \boldsymbol{P}_\tau) = \begin{cases} -\dfrac{1}{2R_i}P_\tau\dfrac{\partial V_2}{\partial H_r}, & \left|\dfrac{1}{2R_i}P_\tau\dfrac{\partial V_2}{\partial H_r}\right| < U_i^0 \\ -U_i^0 \mathrm{sgn}\left(P_\tau\dfrac{\partial V_2}{\partial H_r}\right), & \left|\dfrac{1}{2R_i}P_\tau\dfrac{\partial V_2}{\partial H_r}\right| \ge U_i^0. \end{cases} \tag{21}$$

Substituting the worst-case disturbances $\tilde{s}_l^*, \tilde{c}_{ij}^*, \tilde{w}_i^*$ and the optimal control forces u_i^* in Eqs. (19) and (20) (or u_{wodi}^* in Eq. (21)) into the partially averaged Itô differential equation (6) and completing the averaging, the fully averaged Itô equations can be obtained. By solving the associated Fokker–Planck-Kolmogorov equation (FPK), the stationary probability density of controlled and uncontrolled $(p_c(\boldsymbol{H}), p_u(\boldsymbol{H}))$ can be predicted. Then, the mean-square values of the controlled and uncontrolled displacements can be calculated as follows

$$E[Q_{ci}^2] = \int_0^\infty \left\langle Q_{ci}^2\right\rangle p_c(\boldsymbol{H})\mathrm{d}\boldsymbol{H}, \qquad E[Q_{ui}^2] = \int_0^\infty \left\langle Q_{ui}^2\right\rangle p_u(\boldsymbol{H})\mathrm{d}\boldsymbol{H}. \tag{22}$$

To measure the performance of the proposed control strategy, the control effectiveness k_i and control efficiency μ_i are evaluated. They are defined as follows

$$k_i = \frac{E[Q_{ui}^2] - E[Q_{ci}^2]}{E[Q_{ui}^2]}, \qquad \mu_i = \frac{k_i}{E[u_i^{*2}]}, \quad i = 1, 2, \ldots, n \tag{23}$$

where $E[u_i^{*2}]$ is mean-square optimal controls for the worst case. k_i represents the percentage reduction in the mean-square displacement of the controlled systems while μ_i denotes the relative reduction per unit of the mean-square control. $E[Q_{ui}^2]$, $E[Q_{ci}^2]$ and $E[u_i^{*2}]$ can be calculated by using Eq. (22). Obviously, higher k and μ indicate a better control strategy.

5. Examples

To illustrate the application and efficacy of the proposed optimal control strategy, consider the following two stochastically excited nonlinear systems with parametric and external disturbances and time-delayed bounded feedback control forces.

5.1. Example 1

Consider the following single degree-of-freedom nonlinear system

$$\begin{aligned} \dot{Q} &= P, \\ \dot{P} &= -[\bar{a} + \tilde{a}(t)]Q + [\bar{c} + \tilde{c}(t)]P - [\bar{d} + \tilde{d}(t)]Q^2P \\ &\quad + u(Q_\tau, P_\tau) + e\xi(t) + \tilde{w}(t) \\ |u| &\le U^0 \end{aligned} \tag{24}$$

where $\bar{a}$, $\bar{c}$ and $\bar{d}$ are the nominal values of linear stiffness, linear and nonlinear damping coefficients; e is excitation amplitude; $\xi(t)$ is Gaussian white noise with intensity $2D$; $u(Q_\tau, P_\tau)$ is time-delayed feedback control, which is bounded due to actuator saturation. $\tilde{a}(t)$, $\tilde{c}(t)$ and $\tilde{d}(t)$ are parameter disturbances and $\tilde{w}(t)$ is external disturbance. They are bounded, i.e., $\tilde{a}(t) \in [-a^0, a^0]$, $\tilde{c}(t) \in [-c^0, c^0]$, $\tilde{d}(t) \in [-d^0, d^0]$ and $\tilde{w}(t) \in [-w^0, w^0]$.

Following Eq. (4), the time-delayed feedback control force in system (24) can be expressed in terms of system state variables without time delay as follows

$$\begin{aligned} u(Q_\tau, P_\tau) &\doteq u\left(Q\cos\omega\tau - \frac{P}{\omega}\sin\omega\tau, P\cos\omega\tau + Q\omega\sin\omega\tau\right) \\ &= \bar{u}(Q, P, \tau). \end{aligned} \tag{25}$$

Substituting Eq. (25) into system (24), the optimal control problem of system (24) with time-delayed bounded feedback control is converted into the one without time delay, in the form of Eq. (5). Then, by using the stochastic averaging method for quasi-integrable Hamiltonian systems, the following partially averaged Itô stochastic differential equation can be obtained

$$\begin{aligned} \mathrm{d}H &= \left[\bar{m}(H) + \left\langle \frac{\partial H}{\partial P}(\bar{u}(Q, P, \tau) - \tilde{a}Q + \tilde{c}P - \tilde{d}Q^2P + \tilde{w})\right\rangle\right] \\ &\quad \times \mathrm{d}t + \bar{\sigma}(H)\mathrm{d}B(t) \\ |\bar{u}| &\le U^0 \end{aligned} \tag{26}$$

where

$$\bar{m}(H) = e^2D + \bar{c}H - \frac{\bar{d}}{\bar{a}}H^2, \qquad \bar{\sigma}^2(H) = 2e^2DH. \tag{27}$$

For the semi-infinite time-interval ergodic control, let the performance index be (9) with

$$\begin{aligned} f_4(H, \bar{u}) &= f_c(H) + \langle R\bar{u}^2\rangle, \\ f_c(H) &= s_0 + s_1H + s_2H^2 + s_3H^3. \end{aligned} \tag{28}$$

Following Eq. (16), the worst-case disturbances can be obtained

$$\begin{aligned} \tilde{a}^* &= -a^0\mathrm{sgn}\left[\frac{\mathrm{d}V}{\mathrm{d}H}QP\right], \qquad \tilde{c}^* = c^0\mathrm{sgn}\left[\frac{\mathrm{d}V}{\mathrm{d}H}P^2\right], \\ \tilde{d}^* &= -d^0\mathrm{sgn}\left[\frac{\mathrm{d}V}{\mathrm{d}H}Q^2P^2\right], \qquad \tilde{w}^* = w^0\mathrm{sgn}\left[\frac{\mathrm{d}V}{\mathrm{d}H}P\right] \end{aligned} \tag{29}$$

and the expression of the optimal control forces without time delay are of the form

$$\bar{u}^* = \begin{cases} -\dfrac{1}{2R}\dfrac{\partial V}{\partial H}P, & \left|\dfrac{1}{2R}\dfrac{\partial V}{\partial H}P\right| < U^0 \\ -U^0\mathrm{sgn}(P), & \left|\dfrac{1}{2R}\dfrac{\partial V}{\partial H}P\right| \ge U^0. \end{cases} \tag{30}$$

Substituting the worst-case disturbances $\tilde{a}^*(t)$, $\tilde{c}^*(t)$, $\tilde{d}^*(t)$, $\tilde{w}^*(t)$ and the optimal controls $\bar{u}^*$ into HJI equation (13) and completing the averaging yield the following final HJI equation

$$\begin{aligned} &\frac{1}{2}\sigma^2(H)\frac{\mathrm{d}^2V}{\mathrm{d}H^2} + \left[\bar{m}(H) + \frac{2a^0H}{\pi\sqrt{\bar{a}}} + c^0H + d^0\frac{H^2}{2\bar{a}} + \frac{2w^0}{\pi}\sqrt{2H}\right] \\ &\quad \times \frac{\mathrm{d}V}{\mathrm{d}H} + m_{oV}(H) + f_c(H) = \lambda \end{aligned} \tag{31}$$

where

$$\begin{aligned} m_{oV}(H) &= RU^{02}\frac{T_1(H)}{T(H)} - 4U^0\frac{x_{cr}(H)}{T(H)}\frac{\mathrm{d}V}{\mathrm{d}H} \\ &\quad - \frac{1}{4R}[H - G_1(H)]\left(\frac{\mathrm{d}V}{\mathrm{d}H}\right)^2 \\ T_1(H) &= 2\int_{-x_{cr}}^{x_{cr}}\frac{\mathrm{d}q}{\sqrt{2H - \bar{a}q^2}}, \qquad T(H) = \frac{2\pi}{\sqrt{\bar{a}}}, \\ G_1(H) &= \frac{2}{T(H)}\int_{-x_{cr}}^{x_{cr}}\sqrt{2H - \bar{a}q^2}\mathrm{d}q \\ x_{cr} &= \frac{\sqrt{2H - (2RU^0/(\partial V/\partial H))^2}}{\sqrt{\bar{a}}}, \qquad \gamma = e^2D\mathrm{d}V/\mathrm{d}H|_{H=0} + s_0 \end{aligned} \tag{32}$$

$\partial V/\partial H$ can be obtained from solving this equation. Then the worst-case disturbances $\tilde{a}^*(t)$, $\tilde{c}^*(t)$, $\tilde{d}^*(t)$, $\tilde{w}^*(t)$ and the optimal controls $\bar{u}_i^*$ without time delay are determined by substituting $\partial V/\partial H$ into Eqs. (29) and (30).

By using the transformation in Eq. (18), the optimal time-delayed controls u_i^* for system (24) are determined as follows:

$$\begin{aligned} u^* &= \begin{cases} -F, & |F| < U^0 \\ -U^0\mathrm{sgn}(F), & |F| \ge U^0 \end{cases} \\ F &= \frac{1}{2R}\frac{\partial V}{\partial H}(-Q_\tau\omega\sin\omega\tau + P_\tau\cos\omega\tau). \end{aligned} \tag{33}$$

By using the proposed procedure, some numerical results of control effectiveness and control efficiency for system (24) are obtained for parameter values: $\bar{a} = 1.0$, $\bar{c} = 0.2$, $\bar{d} = 0.5$, $e = 1.0$, $D = 0.2$, $a^0 = 0.04$, $d^0 = 0.005$, $c^0 = 0.01$, $w^0 = 0.01$, $R = 0.25$, $s_1 = s_3 = 0.0$, $s_2 = 1.5$, $\mathrm{d}V(0)/\mathrm{d}H = 4.0$, $U^0 = 1.0$, $\tau = 0.1$, except otherwise specified in the figures. The control effectiveness and control efficiency of the proposed control strategy for various control bounds, delay time and intensities of excitation are shown in Figs. 1–4. It can be seen from these figures that the proposed control strategy performs very well in the entire range of parameter values used. Figs. 2 and 3 indicate that the proposed control strategy has slightly less control effectiveness and higher control efficiency than that of the control without considering time delay. Fig. 2 also shows that delay time τ has a significant negative effect on the control effectiveness of the proposed control strategy. The effects of the bounds of uncertain disturbances $\tilde{c}(t)$ and $\tilde{w}(t)$

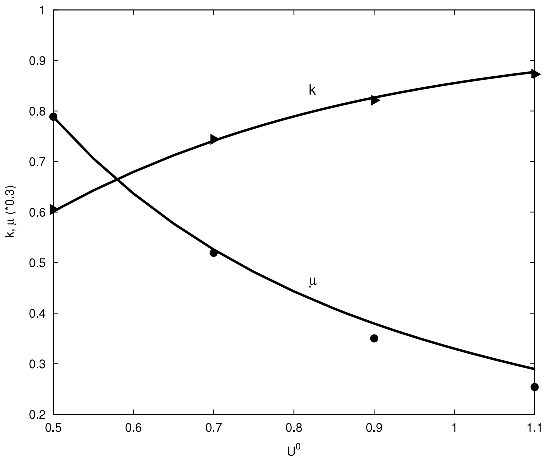

Fig. 1. Control effectiveness K and control efficiency μ for displacement of system (24) as functions of control bounds U^0. — Analytical result. ● ▲ Numerical simulation.

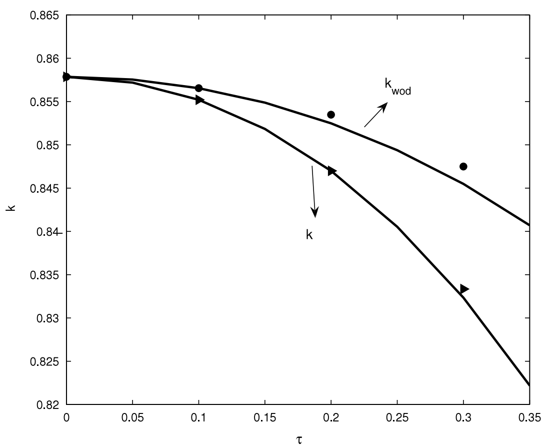

Fig. 2. Control effectiveness for displacement of system (24) as functions of delay time τ. K for the proposed optimal control and K_{wod} for the control without considering time delay. — Analytical result. ● ▲ Numerical simulation.

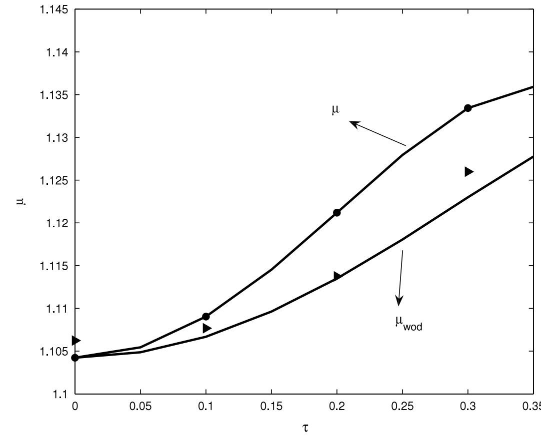

Fig. 3. Control efficiency for displacement of system (24) as functions of delay time τ. μ for the proposed optimal control and μ_{wod} for the control without considering time delay. — Analytical result. ● ▲ Numerical simulation.

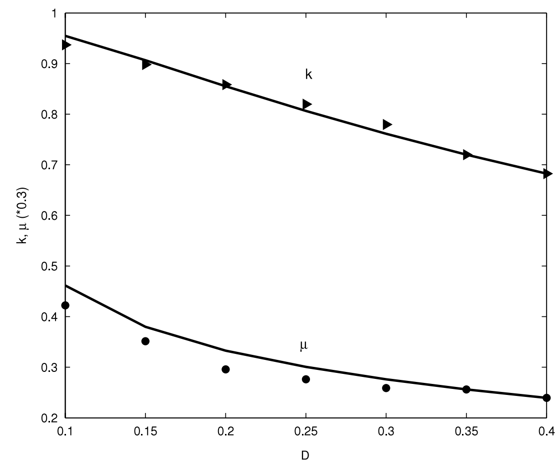

Fig. 4. Control effectiveness K and control efficiency μ for displacement of system (24) as functions of intensities of excitation D. — Analytical result. ● ▲ Numerical simulation.

on the control effectiveness and efficiency are also shown in Figs. 5 and 6, respectively. From these figures we can see that the proposed control strategy is robust with respect to the bound of $\tilde{c}(t)$ but sensitive to the bound of $\tilde{w}(t)$. Samples of the displacement, velocity and acceleration of uncontrolled and optimal controlled systems are shown in Fig. 7, from which the effects of optimal time-delayed control on the displacement, velocity and acceleration can be visualized intuitively.

5.2. Example 2

As the second example, consider the controlled system of two nonlinearly coupled oscillators

$$\begin{aligned}
&\dot{Q}_i = P_i, \\
&\dot{P}_i = -[a_i + \tilde{a}_i]Q_i - [c_{ij} + \tilde{c}_{ij}]P_j - [d_i + \tilde{d}_i]Q_jQ_jP_i \\
&\qquad + u_i(\boldsymbol{Q}_\tau, \boldsymbol{P}_\tau) + e_i\xi_i(t) + \tilde{w}_i(t) \\
&|u_i| \le b_i, \quad i, j = 1, 2
\end{aligned} \tag{34}$$

where a_i, c_{ij}, and d_i are the nominal values of linear stiffness, linear and nonlinear damping coefficients; e_i are amplitude of excitations; ξ_i are independent Gaussian white noises with intensities $2D_i$; $u_i(\boldsymbol{Q}_\tau, \boldsymbol{P}_\tau)$ are time-delayed bounded feedback controls due to actuator saturation. $\tilde{a}_i$, $\tilde{c}_{ij}$, $\tilde{d}_i$ and $\tilde{w}_i(t)$ are parameter and external disturbances, which are bounded, i.e., $\tilde{a}_i(t) \in [-a_i^0, a_i^0]$, $\tilde{c}_{ij}(t) \in [-c_{ij}^0, c_{ij}^0]$, $\tilde{d}_i(t) \in [-d_i^0, d_i^0]$ and $\tilde{w}_i(t) \in [-w_i^0, w_i^0]$.

Following Eq. (4), the time-delayed feedback control forces in system (34) can be expressed in terms of system state variables without time delay as follows

$$\begin{aligned}
u_i(\boldsymbol{Q}_\tau, \boldsymbol{P}_\tau) \approx \bar{u}_i\bigg(&Q_i\cos\omega_i\tau_i - \frac{P_i}{\omega_i}\sin\omega_i\tau_i, P_i\cos\omega_i\tau_i \\
&+ Q_i\omega_i\sin\omega_i\tau_i\bigg) = \bar{u}_i(\boldsymbol{Q}, \boldsymbol{P}, \boldsymbol{\tau}).
\end{aligned} \tag{35}$$

Substituting Eq. (35) into system (34), the optimal control problem of system (34) with time-delayed bounded feedback control is converted into the one without time delay, as expressed in Eq. (5). Then, by using the stochastic averaging method for quasi-integrable Hamiltonian systems, the partially averaged Itô stochastic differential equations can be obtained as the form of Eq. (6) with

$$\begin{aligned}
&\bar{m}_1(\boldsymbol{H}) = -c_{11}H_1 - \frac{d_1}{2a_1}H_1^2 - \frac{d_1}{a_2}H_1H_2 + D_1 \\
&\bar{m}_2(\boldsymbol{H}) = -c_{22}H_2 - \frac{d_2}{2a_2}H_2^2 - \frac{d_2}{a_1}H_1H_2 + D_2 \\
&\bar{\sigma}_1^2 = 2D_1H_1; \qquad \bar{\sigma}_2^2 = 2D_2H_2 \\
&H = \sum_{r=1}^{2} H_r, H_r = (p_r^2 + a_rq_r^2)/2, \quad r = 1, 2.
\end{aligned} \tag{36}$$

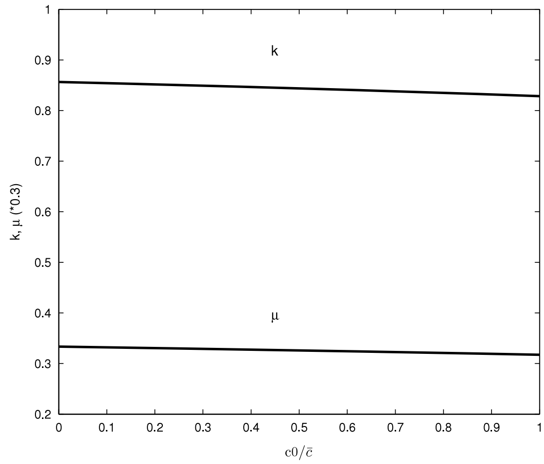

Fig. 5. Control effectiveness K and control efficiency μ for displacement of system (24) as functions of the ratio of parameter disturbance c^0 to $\bar{c}$.

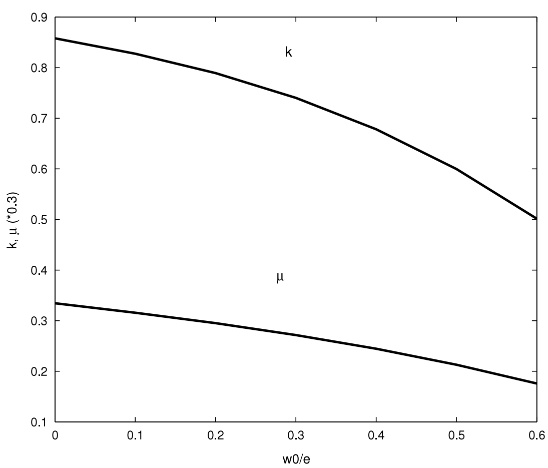

Fig. 6. Control effectiveness K and control efficiency μ for displacement of system (24) as functions of the ratio of parameter disturbance w^0 to e.

For the proposed control strategy, the partially averaged cost function f_4 is of the form of Eq. (15) with $\bar{\boldsymbol{u}} = [\bar{u}_1, \bar{u}_2]^T$, $\boldsymbol{R} = \mathrm{diag}(R_1, R_2)$ and

$$
\begin{aligned}
f_c(\boldsymbol{H}) = {} & s_0 + s_{11}H_1 + s_{12}H_2 + s_{21}H_1^2 + s_{22}H_1H_2 \\
& + s_{23}H_2^2 + s_{31}H_1^3 + s_{32}H_1^2H_2 + s_{33}H_2^2H_1 + s_{34}H_2^3.
\end{aligned} \tag{37}
$$

Following Eqs. (16) and (17), the worst-case disturbances for system (34) can be obtained

$$
\begin{aligned}
\tilde{a}_i^* &= -a_i^0 \mathrm{sgn}\left[\frac{\mathrm{d}V}{\mathrm{d}H_i} Q_i P_i\right], \qquad \tilde{c}_{ij}^* = -c_{ij}^0 \mathrm{sgn}\left[\frac{\mathrm{d}V}{\mathrm{d}H_i} P_i P_j\right], \\
\tilde{d}_i^* &= -d_i^0 \mathrm{sgn}\left[\frac{\mathrm{d}V}{\mathrm{d}H_i} Q_j^2 P_i^2\right], \qquad \tilde{w}_i^* = w_i^0 \mathrm{sgn}\left[\frac{\mathrm{d}V}{\mathrm{d}H_i} P_i\right]
\end{aligned} \tag{38}
$$

and the optimal control forces without time delay are of the form

$$
\bar{u}_i^* = \begin{cases} -\dfrac{1}{2R_i}\dfrac{\partial V}{\partial H_i} P_i, & \left|\dfrac{1}{2R_i}\dfrac{\partial V}{\partial H_i} P_i\right| < b_i \\ -b_i \mathrm{sgn}(P_i), & \left|\dfrac{1}{2R_i}\dfrac{\partial V}{\partial H_i} P_i\right| \geq b_i \end{cases} \tag{39}
$$

where b_i are the control bounds due to the saturation. Substituting the worst-case disturbances $\tilde{a}_i^*$, $\tilde{c}_{ij}^*$, $\tilde{d}_i^*$, $\tilde{w}_i^*$ and the optimal controls $\bar{u}^*$ into HJI equation (13) and completing the averaging yield the final HJI equation

$$
\begin{aligned}
& \frac{1}{2}\sigma_{11}^2(\boldsymbol{H})\frac{\mathrm{d}^2V}{\mathrm{d}H_1^2} + \frac{1}{2}\sigma_{22}^2(\boldsymbol{H})\frac{\mathrm{d}^2V}{\mathrm{d}H_2^2} + \bar{\bar{m}}_1(\boldsymbol{H})\frac{\mathrm{d}V}{\mathrm{d}H_1} + \bar{\bar{m}}_2(\boldsymbol{H})\frac{\mathrm{d}V}{\mathrm{d}H_2} \\
& \quad + m'(\boldsymbol{H}) = \lambda - f_c(\boldsymbol{H})
\end{aligned} \tag{40}
$$

where

$$
\begin{aligned}
\bar{\bar{m}}_1(\boldsymbol{H}) = {} & \bar{m}_1(\boldsymbol{H}) + \frac{2a_1^0 H_1}{\pi\sqrt{\bar{a}_1}} + c_{11}^0 H_1 + \frac{8c_{12}^0}{\pi^2}\sqrt{H_1H_2} \\
& + d_1^0\left(\frac{H_1^2}{2\bar{a}_1} + \frac{H_1H_2}{\bar{a}_2}\right) + \frac{2w_1^0}{\pi}\sqrt{2H_1} \\
\bar{\bar{m}}_2(\boldsymbol{H}) = {} & \bar{m}_2(\boldsymbol{H}) + \frac{2a_2^0 H_2}{\pi\sqrt{\bar{a}_2}} + c_{22}^0 H_2 + \frac{8c_{21}^0}{\pi^2}\sqrt{H_1H_2} \\
& + d_2^0\left(\frac{H_2^2}{2\bar{a}_2} + \frac{H_1H_2}{\bar{a}_1}\right) + \frac{2w_2^0}{\pi}\sqrt{2H_2}
\end{aligned}
$$

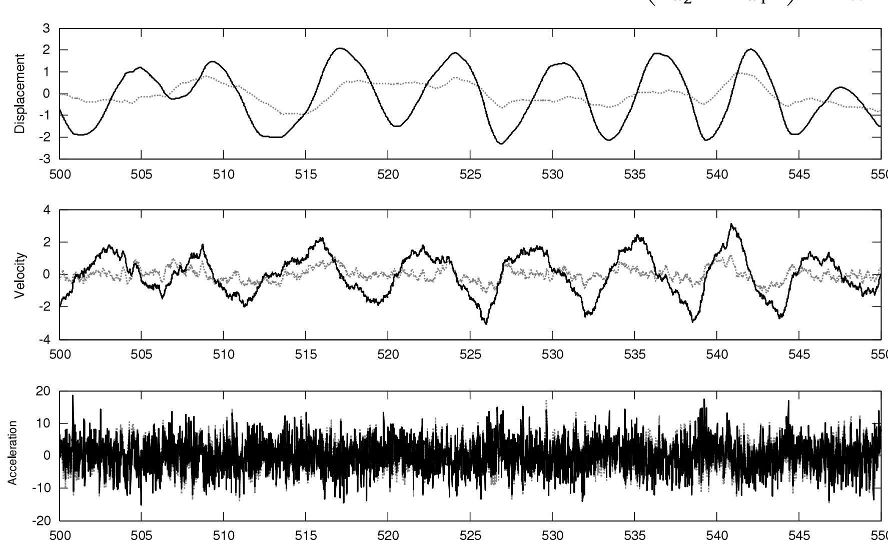

Fig. 7. Time histories of the displacement, velocity and acceleration response of system (24). – – – Optimally controlled; — Uncontrolled.

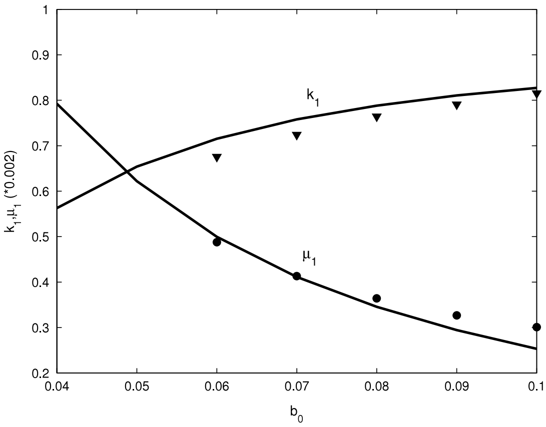

Fig. 8. Control effectiveness K_1 and control efficiency μ_1 for displacement of the first DOF of system (34) as functions of control bounds ($b_1 = b_2 = b_0$). — Analytical result. ● ▼ Numerical simulation.

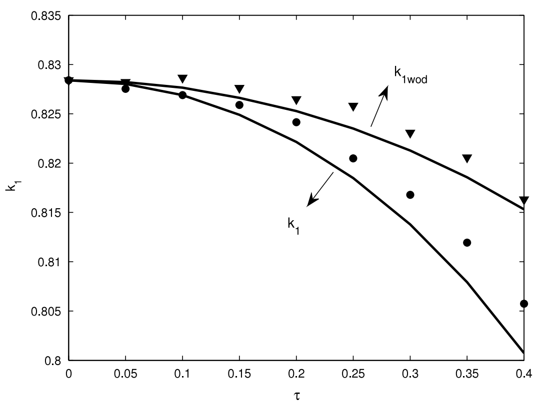

Fig. 9. Control effectiveness for displacement of the first DOF of system (34) as functions of delay time τ. K_1 for the proposed optimal control and K_{1wod} for the control without considering time delay. — Analytical result. ● ▼ Numerical simulation.

$$
\begin{aligned}
m'(\boldsymbol{H}) \;=\;& \frac{2b_1^2R_1}{T_1}\int_{-Rd_1}^{Rd_1}\frac{1}{p_1}\mathrm{d}q_1-\frac{4b_1*Rd_1}{T_1}\frac{\partial V}{\partial H_1}\\
&-\frac{1}{4R_1}\left(\frac{\partial V}{\partial H_1}\right)^2H_1+\frac{(\partial V/\partial H_1)^2}{2R_1T_1}\int_{-Rd_1}^{Rd_1}p_1\mathrm{d}q_1\\
&+\frac{2b_2^2R_2}{T_2}\int_{-Rd_2}^{Rd_2}\frac{1}{p_2}\mathrm{d}q_2-\frac{4b_2*Rd_2}{T_2}\frac{\partial V}{\partial H_2}\\
&-\frac{1}{4R_2}\left(\frac{\partial V}{\partial H_2}\right)^2H_2+\frac{(\partial V/\partial H_2)^2}{2R_2T_2}\int_{-Rd_2}^{Rd_2}p_2\mathrm{d}q_2 \qquad (41)
\end{aligned}
$$

$$
Rd_i=\frac{\sqrt{2H_i-(2R_ib_i/(\partial V/\partial H_i))^2}}{\sqrt{a_i}},\qquad T_1=\frac{2\pi}{\sqrt{a_1}},
$$

$$
T_2=\frac{2\pi}{\sqrt{a_2}}
$$

$\partial V/\partial H_i$ can be obtained from solving this equation. Then the worst-case disturbances $\tilde{a}_i^*, \tilde{c}_{ij}^*, \tilde{d}_i^*, \tilde{w}_i^*$ and the optimal controls $\bar{u}_i^*$ without time delay are determined by substituting $\partial V/\partial H_i$ into Eqs. (38) and (39).

By using the transformation expressed in Eq. (18), the optimal time-delayed controls u_i^* for system (34) can be obtained similarly

$$
u_i^*=\begin{cases}-F_i(Q_{i\tau},P_{i\tau}), & |F_i(Q_{i\tau},P_{i\tau})|<b_i\\ -b_i\mathrm{sgn}(F_i(Q_{i\tau},P_{i\tau})), & |F_i(Q_{i\tau},P_{i\tau})|\ge b_i\end{cases} \qquad (42)
$$

$$
F_i(Q_{i\tau},P_{i\tau})=\frac{1}{2R_i}\frac{\partial V}{\partial H_i}(-Q_{i\tau}\omega_i\sin\omega_i\tau_i+P_{i\tau}\cos\omega_i\tau_i).
$$

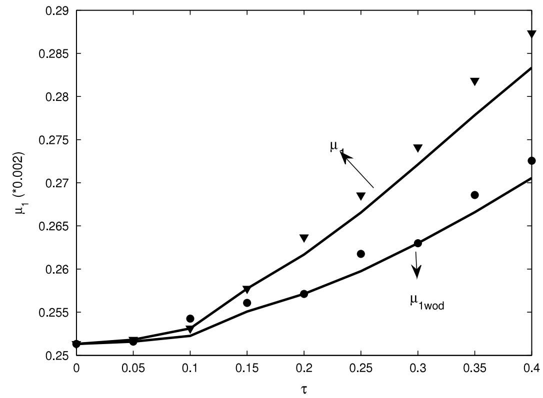

Fig. 10. Control efficiency for displacement of the first DOF of system (34) as functions of delay time τ. μ_1 for the proposed optimal control and μ_{1wod} for the control without considering time delay. — Analytical result. ● ▼ Numerical simulation.

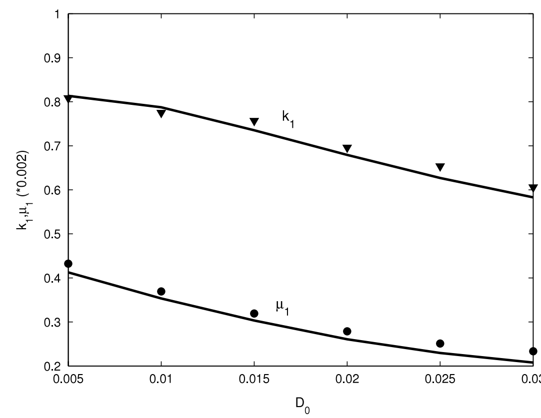

Fig. 11. Control effectiveness K_1 and control efficiency μ_1 for displacement of the first DOF of system (34) as functions of intensities of excitation ($D_1 = D_2 = D_0$). — Analytical result. ● ▼ Numerical simulation.

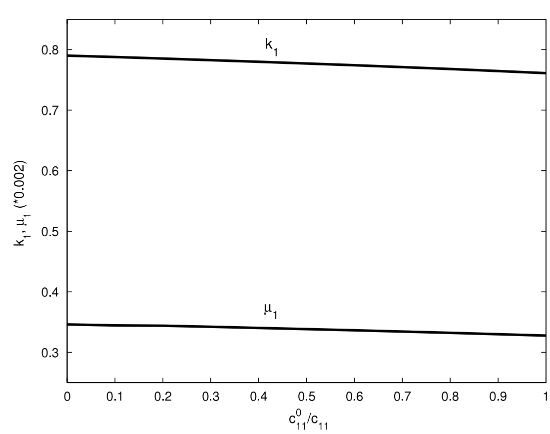

Fig. 12. Control effectiveness K_1 and control efficiency μ_1 for displacement of first DOF of system (34) as functions of the ratio of parameter disturbance c_{11}^0 to c_{11}.

Numerical results are shown in Figs. 8–13 for system (34) with the following parameter values: $a_1 = 1.0, a_2 = 2.0, c_{11} = 0.02, c_{12} = 0.02, c_{21} = 0.02, c_{22} = 0.02, d_1 = 0.05, d_2 = 0.02, D_1 = 1.0, D_2 = 1.0, e_1 = e_2 = 0.1, R_1 = R_2 = 0.01, s_{31} = s_{34} = 2.0, b_i = 0.08, a_1^0 = 0.01, a_2^0 = 0.02, c_{ij}^0 = 0.001, d_i^0 = 0.001, w_i^0 = 0.001, \tau_i = 0.1$, except otherwise specified in the figures. It is seen from these figures that the proposed control strategy has high con-

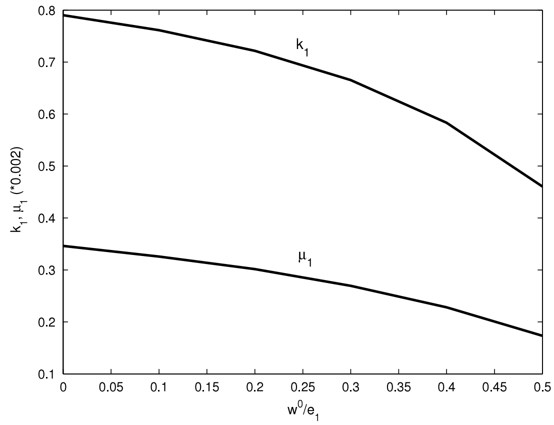

Fig. 13. Control effectiveness K_1 and control efficiency μ_1 for displacement of first DOF of system (34) as functions of the ratio of parameter disturbance w_1^0 to e_1.

trol effectiveness and control efficiency even for long delay time or strong intensities of excitations. The effects of the bounds of uncertain disturbances $\tilde{c}_{11}(t)$ and $\tilde{w}(t)$ on the control effectiveness and efficiency are shown in Figs. 12 and 13, respectively. From these figures we can see that the proposed control strategy is robust with respect to the bound of $\tilde{c}_{11}(t)$ but sensitive to the bound of $\tilde{w}(t)$.

6. Concluding remarks

To apply the theory of stochastic optimal control to real engineering structures, many practical ingredients such as boundness and time delay of control forces, the uncertainty of system parameters and excitation, should be considered. In this paper, a minimax optimal control strategy for quasi-integrable Hamiltonian systems with all these ingredients has been proposed based on the stochastic averaging method for quasi-integrable Hamiltonian systems and the stochastic differential game. First, the optimal control problem with time-delayed feedback control was converted into the one without time delay. Then, the optimal time-delayed controls were obtained by using the method proposed by the present corresponding author [20]. Two examples are worked out in detail. Numerical results showed that delay time τ causes deterioration of the control performance; and the control effectiveness and control efficiency reduce as the bounds of disturbance increase. However, the proposed control strategy generally performs very well in the entire range of parameter values used. So, the proposed control strategy is very promising.

Acknowledgements

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