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A new finite-dimensional pair coherent state studied by virtue of the entangled state representation and its statistical behavior

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ABSTRACT

In this paper we construct a new type of finite-dimensional pair coherent states $|\xi, q\rangle$ as realizations of SU(2) Lie algebra. Using the technique of integration within an ordered product of operator, the nonorthogonality and completeness properties of the state $|\xi, q\rangle$ are investigated. Based on the Wigner operator in the entangled state $|\tau\rangle$ representation, the Wigner function of $|\xi, q\rangle$ is obtained. The properties of $|\xi, q\rangle$ are discussed in terms of the negativity of its Wigner function. The tomogram of $|\xi, q\rangle$ is calculated with the aid of the Radon transform between the Wigner operator and the projection operator of the entangled state $|\eta, \kappa_1, \kappa_2\rangle$. In addition, using the entangled state $|\tau\rangle$ representation of $|\xi, q\rangle$ to show that the states $|\xi, q\rangle$ are just a set of energy eigenstates of time-independent two coupled oscillators.

1. Introduction

In recent years there has been considerable attention in constructing some new multimode quantum states based on the fundamental principles in quantum mechanics. The motivation to construct the kind of states is considering some important applications, for instance, as necessary resources in multiuser quantum communication network, of the multimode entangled properties in quantum communications. Among multimode quantum states, pair coherent state (PCS) [1,2] is seen as an important two-mode correlated state because it shows strong entanglement and remarkable nonclassical properties. Moreover, some schemes for generation of the PCS have been introduced [2–4]. For example, it can be generated in competition between nondegenerate two-photon absorption and nondegenerate parametric amplification [2]. Also it can be generated in two-mode photon matching process using weak cross-Kerr media [3]. Wu et al. [5] have investigated the entanglement of the PCS in the phase damping channel and proposed a protocol of teleportation via the PCS. Arvind [6] has analyzed the squeezing properties of the PCS using the U(2) invariant methodology. Agarwal [1,2] has discussed the violations of the Cauchy–Schwarz, Bell inequalities and many-photon anti-bunching.

As an important generalization of PCS, Refs. [7–9] have introduced the superposition of the PCS and present its some nonclassical properties, such as phase distribution and single-mode Wigner and Weyl functions. Ref. [10] has constructed a finite-dimensional pair coherent state (FDPCS) and investigated some of its nonclassical properties such as sub-Poissonian distribution and phase properties. Moreover, they have also proposed a concrete scheme generating the FDPCS in the vibrational motion of a trapped ion in two dimensional harmonic potential. In this paper a new type of finite-dimensional pair coherent states $|\xi, q\rangle$ is constructed, where the states are the two-mode bosonic realizations of the SU(2) Lie groups. The physical meaning of the two states is: creating one quantum of the mode a and meanwhile annihilating one quantum in mode b will not change the finite quantum sum q between the modes a and b in the whole system.

Representation theory in quantum mechanics was invented by Dirac [11]. Dirac emphasized that "when one has a particular problem to work out in quantum mechanics, one can minimize the labour by using a representation in which the representatives of the more important abstract quantities occurring in that problem are as simple as possible". We believe that it will be very convenient for us to treat many problems in quantum optics based on some newly constructing entangled state representations. In history it is Einstein–Podolsky–Rosen (EPR) who first used the commutative property of two particles' relative position and total momentum to initiate the concept of quantum entanglement. Following EPR's idea of quantum entanglement [12] Fan et al. [13–16] have constructed the bipartite entangled state representations and discussed many applications in quantum optics and quantum information. In this work we shall sufficiently use these newly introduced entangled state representations to simplify the study of the state $|\xi, q\rangle$. For details, we want to use the Wigner operator in the entangled state representation $|\tau\rangle$ and the Radon transform between the Wigner operator and the projection

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operator of the entangled state $|\eta, \kappa_1, \kappa_2\rangle$ to derive Wigner function and tomogram of the states $|\xi, q\rangle$, and then use the entangled state $|\tau\rangle$ representation of $|\xi, q\rangle$ to show that the Hamiltonian of two coupled oscillators possesses the states $|\xi, q\rangle$ as its energy eigenstates.

Our main work is arranged as follows: in Section 2 the nonorthogonality and completeness relation of the FDPCS is investigated using the technique of integration within an ordered product (IWOP) of operator [13,14]. In Section 3 Wigner function of the state $|\xi, q\rangle$ is obtained based on the Wigner operator in the entangled state $|\tau\rangle$ representation, then nonclasscial properties of the state $|\xi, q\rangle$ are discussed in terms of the negativity of its Wigner function. In Section 4 we derive marginal distribution of Wigner function of $|\xi, q\rangle$. In Section 5 the tomogram of $|\xi, q\rangle$ is calculated with the aid of the Radon transform between the Wigner operator and the projection operator of the entangled state $|\eta, \kappa_1, \kappa_2\rangle$ [16]. In final section using the entangled state $|\tau\rangle$ representation of $|\xi, q\rangle$ to show that the Hamiltonian of two coupled oscillators possesses the states $|\xi, q\rangle$ as its energy eigenstates.

2. The nonorthogonality and completeness relation of the FDPCS

To obtain the FDPCS, we define the generators of SU(2) in this case as follows:

$$K_+ = a^\dagger b, \quad K_- = ab^\dagger, \quad K_0 = \frac{1}{2}(a^\dagger a - b^\dagger b), \tag{1}$$

which yields the following SU(2) Lie algebra

$$[K_+, K_-] = 2K_0, \quad [K_0, K_\pm] = \pm K_\pm, \tag{2}$$

where a, b are the annihilation operators of the optical field, then the vacuum state $|0, 0\rangle$ is annihilated by either a or b. So we define a unitary evolution operator $D(\xi) = \exp[\xi K_+ - \xi^* K_-]$, its standard factorization is

$$D(\xi) = \exp(\varsigma K_+)\exp\left[K_0 ln(1 + |\varsigma|^2)\right]\exp(-\varsigma^* K_-), \tag{3}$$

where $\varsigma = e^{-i\phi} tg\frac{\theta}{2}$, $\xi = \frac{\theta}{2}e^{-i\phi}$. Using Eq. (3) we derive

$$\exp(\xi K_+ - \xi^* K_-) a^\dagger \exp(\xi^* K_- - \xi K_+) = a^\dagger \cos\frac{\theta}{2} - b^\dagger e^{i\phi}\sin\frac{\theta}{2}, \tag{4}$$

$$\exp(\xi K_+ - \xi^* K_-) b^\dagger \exp(\xi^* K_- - \xi K_+) = b^\dagger \cos\frac{\theta}{2} + a^\dagger e^{-i\phi}\sin\frac{\theta}{2}, \tag{5}$$

if operating this unitary operator $D(\xi)$ on two-mode Fock state $|q, 0\rangle$ we find that the explicit form of the FDPCS in two-mode Fock space is

$$\begin{aligned} |\xi, q\rangle &= \left(1 + |\varsigma|^2\right)^{-q/2} e^{\varsigma K_+}|q, 0\rangle \\ &= \frac{1}{\sqrt{q!}}\left(a^\dagger \cos\frac{\theta}{2} - b^\dagger e^{i\phi}\sin\frac{\theta}{2}\right)^q |0, 0\rangle \\ &= \sum_{n=0}^{q}\binom{q}{n}^{1/2}\left(\cos\frac{\theta}{2}\right)^{q-n}\left(-e^{i\phi}\sin\frac{\theta}{2}\right)^n |q-n, n\rangle \\ &= \left(1 + |\varsigma|^2\right)^{-q/2}\sum_{n=0}^{q}\binom{q}{n}^{1/2}\varsigma^n |q-n, n\rangle, \end{aligned} \tag{6}$$

which has the same form of the binomial state in the light field. In addition, we also find that the state $|\xi, q\rangle$ is completely different from the FDPCS constructed in Ref. [10], so it can be named as a new type of FDPCS. Using the relation (6) and the normal ordering form of the two-mode vacuum state projector [13,14]

$$|00\rangle\langle 00| = :e^{-a^\dagger a - b^\dagger b}:, \tag{7}$$

where the symbol : : denotes normal ordering, we obtain the completeness relation of the FDPCS in the whole two-mode Fock space,

$$\begin{aligned} &\sum_{q=0}^{\infty}(q+1)\int\frac{d\Omega}{4\pi}|\xi, q\rangle\langle\xi, q| \\ &= \sum_{q=0}^{\infty}\frac{q+1}{q!}\int_0^\pi d\theta\sin\theta\int_0^{2\pi}d\phi : \left(a^\dagger\cos\frac{\theta}{2} - b^\dagger e^{i\phi}\sin\frac{\theta}{2}\right)^q \\ &\quad\times\left(a\cos\frac{\theta}{2} - be^{-i\phi}\sin\frac{\theta}{2}\right)^q \exp(-a^\dagger a - b^\dagger b) : \\ &= \sum_{q=0}^{\infty} : \frac{(a^\dagger a + b^\dagger b)^q}{q!}\exp(-a^\dagger a - b^\dagger b) : = 1. \end{aligned} \tag{8}$$

Thus $|\xi, q\rangle$ is capable of making up a quantum mechanical representation. The inner product $\langle\xi', q.|\xi, q\rangle$ is

$$\langle\xi', q|\xi, q\rangle = \left(1 + \varsigma\varsigma'\right)^q / \left(1 + |\varsigma|^2\right)^{q/2}\left(1 + |\varsigma'|^2\right)^{q/2}. \tag{9}$$

From Eq. (9) one can see that $\langle\xi', q.|\xi, q\rangle$ is nonorthogonal, only when $\varsigma = \varsigma'$, Eq. (9) reduces to $\langle q, \xi.|\xi, q\rangle = 1$.

3. Wigner function of the state $|\xi, q\rangle$

Wigner function is a very important quasiprobability distribution function in studying quantum optics and quantum statistics. It gives the most analogous description of quantum mechanics in the phase space to classical statistical mechanics of Hamilton systems and is also a useful measure for studying the nonclassical features of quantum states. In order to obtain the Wigner function of the state $|\xi, q\rangle$ we first recall the features of the entangled state $|\tau\rangle$. In Ref. [13] we have shown that the entangled state $|\tau\rangle$ simultaneously obeys the eigenvector equations

$$\left(a - b^\dagger\right)|\tau\rangle = \tau|\tau\rangle, \quad \left(a^\dagger - b\right)|\tau\rangle = \tau^*|\tau\rangle, \tag{10}$$

where $|\tau\rangle$ is the entangled state defined as in two-mode Fock space

$$|\tau\rangle = \exp\left[-\frac{1}{2}|\tau|^2 + \tau a^\dagger - \tau^* b^\dagger + a^\dagger b^\dagger\right]|00\rangle, \tau = \tau_1 + i\tau_2, \tag{11}$$

The state $|\tau\rangle$ is the common eigenstate of the operators $(Q_a - Q_b)$ and $(P_a + P_b)$, i.e.,

$$(Q_a - Q_b)|\tau\rangle = \sqrt{2}\tau_1|\tau\rangle, \quad (P_a + P_b)|\tau\rangle = \sqrt{2}\tau_2|\tau\rangle, \tag{12}$$

where

$$Q_a = \frac{a + a^\dagger}{\sqrt{2}}, \quad P_a = \frac{a - a^\dagger}{\sqrt{2}i}, \quad Q_b = \frac{b + b^\dagger}{\sqrt{2}}, \quad P_b = \frac{b - b^\dagger}{\sqrt{2}i}. \tag{13}$$

Using Eq. (7) and the IWOP technique we can immediately prove the completeness relation of the state $|\tau\rangle$

$$\int\frac{d^2\tau}{\pi}|\tau\rangle\langle\tau| = \int\frac{d^2\tau}{\pi} : e^{-|\tau|^2 + (a^\dagger - b)\tau + (a - b^\dagger)\tau^* - (a^\dagger - b)(a - b^\dagger)} : = 1. \tag{14}$$

From Eq. (10) we can see the orthonormalized property of the state $|\tau\rangle$

$$\langle\tau|\tau'\rangle = \pi\delta(\tau - \tau')\delta(\tau^* - \tau'^*) \equiv \pi\delta^{(2)}(\tau - \tau'). \tag{15}$$

Recalling the generating function formula of the two-variable Hermite polynomials $H_{m,n}(\epsilon, \epsilon^*)$

$$\sum_{m,n=0}^{\infty} \frac{t^m t'^n}{m!n!} H_{m,n}(\epsilon, \epsilon^*) = \exp\left(-tt' + t\epsilon + t'\epsilon^*\right), \tag{16}$$

where

$$\begin{aligned} H_{m,n}(\epsilon, \epsilon^*) &= \sum_{l=0}^{\min(n,m)} \frac{(-)^l n!m!}{l!(m-l)!(n-l)!} \epsilon^{m-l} \epsilon^{*n-l} \\ &= \frac{\partial^{m+n}}{\partial t^m \partial t'^n} \exp\left(-tt' + t\epsilon + t'\epsilon^*\right)\Big|_{t=t'=0}, \end{aligned} \tag{17}$$

which yields

$$\overset{*}{H}_{m,n}(\epsilon, \epsilon^*) = H_{n,m}(\epsilon, \epsilon^*). \tag{18}$$

The entangled state $|\tau\rangle$ can be expanded as

$$|\tau\rangle = e^{-|\tau|^2/2} \sum_{m,n=0}^{\infty} \frac{(-)^n H_{m,n}(\tau, \tau^*)}{\sqrt{m!n!}} |m, n\rangle, \tag{19}$$

where $|m, n\rangle = \frac{a^{\dagger m} b^{\dagger n}}{\sqrt{m!n!}} |0, 0\rangle$ is the two-mode number state. Using Eqs. (6), (18) and (19), we derive the inner product,

$$\langle \tau | \xi, q\rangle = \left(1 + |\varsigma|^2\right)^{-q/2} e^{-|\tau|^2/2} \sum_{n=0}^{q} H_{n,q-n}(\tau, \tau^*) \frac{(-\varsigma)^n \sqrt{q!}}{n!(q-n)!}. \tag{20}$$

In the entangled state $|\tau\rangle$ representation, the two-mode Wigner operator is neatly expressed as [17]

$$\Delta\left(\sigma, \gamma\right) = \int \frac{d^2\tau}{\pi^3} |\sigma - \tau\rangle\langle \sigma + \tau| \exp\left(\tau\gamma^* - \tau^*\gamma\right), \tag{21}$$

where σ, γ are complex parameters. Using Eq. (21) and the IWOP technique, we derive

$$\begin{aligned} \Delta\left(\sigma, \gamma\right) &= \pi^{-2} : \exp\left[-2\left(a^\dagger - \alpha^*\right)(a - \alpha) - 2\left(b^\dagger - \beta^*\right)(b - \beta)\right] : \\ &= \Delta(\alpha, \alpha^*) \otimes \Delta(\beta, \beta^*), \end{aligned} \tag{22}$$

where

$$\alpha = (\sigma + \gamma)/2, \quad \beta^* = (\gamma - \sigma)/2, \tag{23}$$

Using Eqs. (6), (20) and (21), we have the Wigner function of the state $|\xi, q\rangle$

$$\begin{aligned} W\left(\sigma, \gamma\right) &= \langle \xi, q | \Delta\left(\sigma, \gamma\right) | \xi, q\rangle \\ &= (1 + |\varsigma|^2)^{-q} \int \frac{d^2\tau}{\pi^3} \langle \xi, q | \sigma - \tau\rangle\langle \sigma + \tau | \xi, q\rangle \exp\left(\tau\gamma^* - \tau^*\gamma\right) \\ &= (1 + |\varsigma|^2)^{-q} \sum_{m,n=0}^{q} \frac{(-1)^{m+n} \varsigma^{*m} \varsigma^n q!}{m!n!(q-n)!(q-m)!} \int \frac{d^2\tau}{\pi^3} H_{q-m,m}(\sigma - \tau, \sigma^* - \tau^*) \\ &\quad \times H_{n,q-n}(\sigma + \tau, \sigma^* + \tau^*) \exp(-|\sigma|^2 - |\tau|^2 + \tau\gamma^* - \tau^*\gamma) \\ &= (1 + |\varsigma|^2)^{-q} \sum_{m,n=0}^{q} \frac{(-1)^{m+n} \varsigma^{*m} \varsigma^n q!}{m!n!(q-n)!(q-m)!} \frac{\partial^q}{\partial t^{q-m} \partial t'^m} \frac{\partial^q}{\partial r^n \partial r'^{q-n}} \\ &\quad \times \int \frac{d^2\tau}{\pi^3} \exp[-|\sigma|^2 - |\tau|^2 + \tau\gamma^* - \tau^*\gamma - tt' + t(\sigma - \tau) \\ &\quad + t'(\sigma - \tau)^* - rr' + r(\sigma + \tau) + r'(\sigma + \tau)^*]|_{t=t'=r=r'=0}. \end{aligned} \tag{24}$$

Further, using the following integration formula

$$\begin{aligned} &\int \frac{d^2 z}{\pi} \exp\left(\zeta|z|^2 + \xi z + \eta z^* + f z^2 + g z^{*2}\right) \\ &= \frac{1}{\sqrt{\zeta^2 - 4fg}} \exp\left[\frac{-\zeta\xi\eta + \xi^2 g + \eta^2 f}{\zeta^2 - 4fg}\right], \end{aligned} \tag{25}$$

with the convergence condition

$$\mathrm{Re}(\zeta \pm f \pm g) < 0, \quad \mathrm{Re}\left(\frac{\zeta^2 - 4fg}{\zeta \pm f \pm g}\right) < 0, \tag{26}$$

we have

$$\begin{aligned} W\left(\sigma, \gamma\right) &= \frac{\exp(-|\gamma|^2 - |\sigma|^2)}{\pi^2(1 + |\varsigma|^2)^q} \sum_{m,n=0}^{q} \frac{(-1)^{m+n} \varsigma^{*m} \varsigma^n q!}{m!n!(q-n)!(q-m)!} \frac{\partial^q}{\partial t^{q-m} \partial t'^m} \frac{\partial^q}{\partial r^n \partial r'^{q-n}} \\ &\quad \times \exp[-tr' + (\sigma + \gamma)t + (\sigma + \gamma)^* r' - rt' + (\sigma - \gamma) r + (\sigma - \gamma)^* t']|_{t=t'=r=r'=0}. \\ &= \frac{\exp(-|\gamma|^2 - |\sigma|^2)}{\pi^2(1 + |\varsigma|^2)^q} \sum_{m,n=0}^{q} \frac{(-1)^{m+n} \varsigma^{*m} \varsigma^n q!}{m!n!(q-n)!(q-m)!} \\ &\quad \times H_{q-m,q-n}(\gamma + \sigma, \gamma^* + \sigma^*) H_{n,m}(\sigma - \gamma, \sigma^* - \gamma^*). \end{aligned} \tag{27}$$

which show the function $W(\sigma,\gamma)$ is related to the two-mode Hermite polynomials. By means of simple calculations, we find that the Wigner function $W(\sigma,\gamma)$ can also be written as a sum of the mixture part $W^M(\sigma,\gamma)$ and the quantum interference part $W^I(\sigma,\gamma)$, where $W^M(\sigma,\gamma)$ and $W^I(\sigma,\gamma)$ are given as, respectively

$$\begin{aligned} W^M\left(\sigma, \gamma\right) &= \frac{\exp(-|\gamma|^2 - |\sigma|^2)}{\pi^2(1 + |\varsigma|^2)^q} \\ &\quad \times \sum_{m=0}^{q} \sum_{k=0}^{m} \sum_{l=0}^{q-m} \frac{(-1)^{l+k} q! |\varsigma|^{2m} |\sigma - \gamma|^{2(m-k)} |\sigma + \gamma|^{2(q-m-l)}}{l!k![(m-k)!(q-m-l)!]^2}, \end{aligned} \tag{28}$$

and

$$\begin{aligned} W^I\left(\sigma, \gamma\right) &= \frac{exp\left(-|\gamma|^2 - |\sigma|^2\right)}{\pi^2(1 + |\varsigma|^2)^q} \left(\sum_{n>m} \sum_{k=0}^{m} \sum_{l=0}^{q-m} + \sum_{m>n} \sum_{k=0}^{n} \sum_{l=0}^{q-n}\right) \\ &\quad \times \frac{(-1)^{l+k} q! \varsigma^{*m} \varsigma^n}{k!l!(m-k)!(n-k)!(q-n-l)!(q-m-l)!} \\ &\quad \times (\sigma - \gamma)^{n-k} (\sigma - \gamma)^{*m-k} (\sigma + \gamma)^{q-m-l} (\sigma + \gamma)^{*q-n-l}. \end{aligned} \tag{29}$$

Now we would like to discuss changes in the Wigner function of $|\xi, q\rangle$ as we vary the parameters q and ς. When $q = 0$ and ς is taken any value, the shape of the Wigner function $W(\sigma,\gamma)$ is a round hill centered at the origin of phase space, which is the same as that of the vacuum state. In fact, this result is also obtained from Eq. (6). Since the function $W(\sigma,\gamma)$ is positive and has a Gaussian form, nonclassical effects cannot be established. When $\varsigma = 0$ and q is taken any value, the variation of the function $W(\sigma,\gamma)$ is very inerratic. This result causes by the superposition of two-mode number states in Eq. (6). With increasing ς the negativity of $W(\sigma,\gamma)$ begin to reduce. This feature indicates that nonclassical effects of the state $|\xi, q\rangle$ begin to be weakened.

In Fig. 1(a)–(c) we plot the Wigner function $W(\sigma,\gamma)$ of the state $|\xi, q\rangle$ while for taking $\varsigma = 0.1$ and different values of $q(q = 2, 3$ and $4)$ for the three three-dimensional figures. From Fig. 1(a)–(c) it is clearly that, at the center position, there exists a downward main peak when q is an odd number, however there exists an upward main peak when q is an even number. With varying q, the number of the downward minor peaks is equivalent to $q - 1$, however the number of the upward minor peaks is q. So the quantum interference property is connected with the two-mode number sum q. The larger the number sum q of two modes is, the more remarkable the interference property is.

Moreover, from the three figures we can also see that the negativity of $W(\sigma,\gamma)$ relies on two-mode number sum q. This means the state $|\xi, q\rangle$ exhibits different nonclassical statistical properties when q is taken different values. The nonclassicality is more pronounced when q is an odd number for $|\xi, q\rangle$. Furthermore, from Fig. 1(a)–(c) it is obviously that the function $W(\sigma,\gamma)$ has downward peaks when $q=3$ and the function $W(\sigma,\gamma)$ has upward peaks when $q=2$ or $q=4$ at the same position. The results are in general valid for all values of q for the state $|\xi, q\rangle$. The feature indicates that the states $|\xi, q\rangle$ for q being odd and the states $|\xi, q\rangle$ for q being even are orthogonal. For fixed value of q, the peak values of the function $W(\sigma,\gamma)$ are decreased as ς is increased, but the shape of $W(\sigma,\gamma)$ becomes very irregular. This result causes by the quantum interference part $W^I(\sigma, \gamma)$ for $|\xi, q\rangle$. When ς is large enough, the multi-peak structure of $W(\sigma,\gamma)$ is the same as the shape of $W(\sigma,\gamma)$ for $\varsigma=0.1$, but the peak values become small. In conclusion, the behavior of the function $W(\sigma,\gamma)$ is in a good agreement with the quantum features of the states $|\xi, q\rangle$.

4. Marginal distributions of Wigner function of $|\xi, q\rangle$

Now we want to obtain the marginal distributions of Wigner function $W(\sigma, \gamma)$ in Eq. (27). However, we find that it will be very difficult to evaluate the integration directly using the Wigner function $W(\sigma, \gamma)$ owing to the existence of two-variable Hermite polynomial $H_{2j,2j}$ in Eq. (18). Fortunately, we can use the relationship between the Wigner operator $\Delta(\sigma,\gamma)$ and two entangled states. From Eq. (21) carrying out the integral over $d^2\gamma$ for $\Delta(\sigma,\gamma)$ yields [17]

$$\int d^2\gamma\Delta(\sigma,\gamma) = \frac{1}{\pi}|\tau\rangle\langle\tau|_{\tau=\sigma}, \tag{30}$$

using Eq. (20) we have a marginal distribution of the Wigner function of the state $|\xi, q\rangle$ in the σ variable,

$$\begin{aligned}\int d^2\gamma W\left(\sigma,\gamma\right) &= \frac{1}{\pi}|\langle\tau|\xi,q\rangle|^2_{\tau=\sigma} \\ &= \frac{q!e^{-|\sigma|^2}}{\pi\left(1+|\varsigma|^2\right)^q}\left|\sum_{n=0}^{q}H_{n,q-n}\left(\sigma,\sigma^*\right)\frac{(-\varsigma)^n}{n!(q-n)!}\right|^2.\end{aligned} \tag{31}$$

Performing the integration of $\Delta(\sigma,\gamma)$ over $d^2\sigma$ leads to another projection operator, i.e.,

$$\int d^2\sigma\Delta(\sigma,\gamma) = \frac{1}{\pi}|\chi\rangle\langle\chi|_{\chi=\gamma}. \tag{32}$$

where $|\chi\rangle$ also has complete orthonormalized relation and is the canonical conjugate state to $|\tau\rangle$. The concrete expression of $|\chi\rangle$ is [17]

$$|\chi\rangle = \exp\left[-\frac{1}{2}|\chi|^2 + \chi a^\dagger + \chi^* b^\dagger - a^\dagger b^\dagger\right]|00\rangle, \quad \chi = \chi_1 + i\chi_2, \tag{33}$$

which is the common eigenstate of (Q_a+Q_b) and P_a-P_b in two-mode Fock space, i.e.,

$$(Q_a+Q_b)|\chi\rangle = \sqrt{2}\chi_1|\chi\rangle, \quad (P_a-P_b)|\chi\rangle = \sqrt{2}\chi_2|\chi\rangle. \tag{34}$$

Thus we obtain

$$\begin{aligned}\int d^2\sigma W(\sigma,\gamma) &= \frac{1}{\pi}|\langle\chi|\xi,q\rangle|^2_{\chi=\gamma} \\ &= \frac{q!e^{-|\gamma|^2}}{\pi(1+|\varsigma|^2)^q}\left|\sum_{n=0}^{q}H_{n,q-n}(\gamma,\gamma^*)\frac{\varsigma^n}{n!(q-n)!}\right|^2,\end{aligned} \tag{35}$$

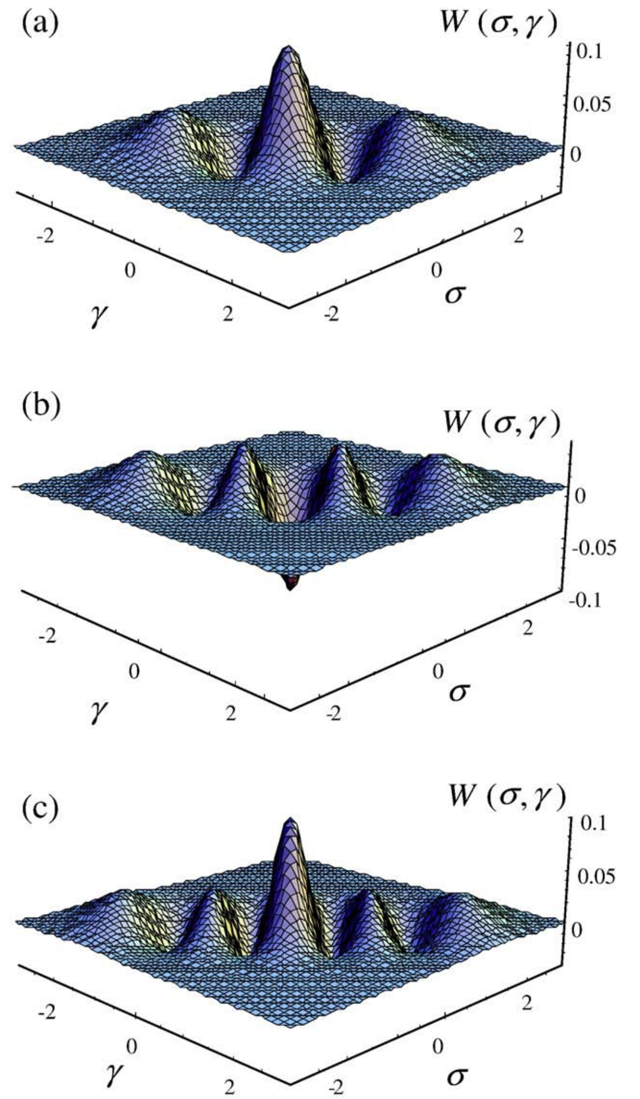

Fig. 1. The Wigner function of the state $|\xi, q\rangle$ for $\varsigma=0.1$ and (a) $q=2$, (b) $q=3$, and (c) $q=4$.

which is another marginal distribution of the Wigner function of $|\xi, q\rangle$ in the γ variable. Here we have used the expansion of $|\chi\rangle$ in two-mode Fock space

$$|\chi\rangle = e^{-|\chi|^2/2}\sum_{m,n=0}^{\infty}\frac{H_{m,n}(\chi,\chi^*)}{\sqrt{m!n!}}|m,n\rangle. \tag{36}$$

Eq. (31) (or Eq. (35)) is proportional to the probability for finding the two particles, which have total momentum $\sqrt{2}\sigma_2$(or relative momentum $\sqrt{2}\gamma_2$) and simultaneously relative position $\sqrt{2}\sigma_1$(or center-of-mass position $\sqrt{2}\gamma_1$), in the state $|\xi, q\rangle$. Therefore, for an entangled particle system, the physical meaning of the Wigner function $W(\sigma,\gamma)$ should lie in that its marginal distributions give the probability of finding the particles in an entangled way in the $\sigma-\gamma$ phase space.

5. Tomogram of $|\xi, q\rangle$

The use of tomogram in quantum mechanics and quantum optics provides the possibility of describing a quantum state with a positive probability distribution. A direct description of quantum states by means of quantum tomogram for the system observable is interesting from both the theoretical and experimental points of view. Therefore, in recent years the tomogram approach has brought much interest to physicists. In this section we continue to derive the tomogram of the state $|\xi, q\rangle$, which is defined as [16]

$$T(\eta,\kappa_1,\kappa_2) = \pi \int_{-\infty}^{\infty}\int d^2\sigma d^2\gamma \delta(\eta_1-\mu_1\gamma_1-\nu_1\sigma_2)\delta(\eta_2-\nu_2\gamma_2-\mu_2\sigma_1)W(\sigma,\gamma), \quad (37)$$

where κ_1, κ_2, σ and γ are complex numbers, $\kappa_j=|\kappa_j|e^{i\theta}=\mu_j+i\nu_j$, $(j=1,2)$, $\sigma=\sigma_1+i\sigma_2$ and $\gamma=\gamma_1+i\gamma_2$. However, it will be very difficult to evaluate the integration if we directly substitute Eq. (27) into Eq. (37). But we can use the following relation between the Wigner operator and the projection operator of the state $|\eta, \kappa_1, \kappa_2\rangle$

$$|\eta,\kappa_1,\kappa_2\rangle\langle\eta,\kappa_1,\kappa_2| = \pi \int_{-\infty}^{\infty}\int d^2\sigma d^2\gamma \delta(\eta_1-\mu_1\gamma_1-\nu_1\sigma_2)\delta(\eta_2-\nu_2\gamma_2-\mu_2\sigma_1)\Delta(\sigma,\gamma). \quad (38)$$

Thus the tomogram of the state $|\xi, q>$ is

$$\begin{aligned} T(\eta,\kappa_1,\kappa_2) &= \pi \int_{-\infty}^{\infty}\int d^2\sigma d^2\gamma \delta(\eta_1-\mu_1\gamma_1-\nu_1\sigma_2)\delta(\eta_2-\nu_2\gamma_2-\mu_2\sigma_1)\langle\xi,q|\Delta\left(\sigma,\gamma\right)|\xi,q\rangle \\ &= |\langle\eta,\kappa_1,\kappa_2|\xi,q\rangle|^2, \end{aligned} \quad (39)$$

which shows that for tomographic approach there exists the entangled state $|\eta, \kappa_1, \kappa_2\rangle$, and the Radon transforms of the Wigner operator are just the entangled state density matrices $|\eta, \kappa_1, \kappa_2\rangle\langle\eta, \kappa_1, \kappa_2|$. As a result, the tomogram of quantum states can be considered as the module-square of the states' wave function in this entangled state representation. This is a new way to derive tomogram of quantum states. Here the entangled state $|\eta, \kappa_1, \kappa_2\rangle$ is expressed as

$$|\eta,\kappa_1,\kappa_2\rangle = A\exp[B + Ca_1^+ + Da_2^+ + Ea_1^+a_2^+ - Fa_1^{+2} - Fa_2^{+2}|00\rangle, \quad (40)$$

where

$$A = \frac{1}{\sqrt{|\kappa_1\kappa_2|}}, \quad B = -\frac{\eta_1^2}{2|\kappa_1|^2} - \frac{\eta_2^2}{2|\kappa_2|^2}, \quad C = \frac{\eta_1}{\kappa_1^*} + \frac{\eta_2}{\kappa_2^*}, \quad (41)$$

$$D = -\frac{\eta_1}{\kappa_1^*} + \frac{\eta_2}{\kappa_2^*}, \quad E = \frac{1}{2}\left(e^{i2\theta_1} - e^{i2\theta_2}\right), \quad F = \frac{1}{4}\left(e^{i2\theta_1} + e^{i2\theta_2}\right), \quad (42)$$

then using Eq. (6) the tomogram amplitude of the state $|\xi, q\rangle$ is

$$\langle\xi,q|\eta,\kappa_1,\kappa_2\rangle = \left(1 + |\varsigma|^2\right)^{-q/2}\sum_{n=0}^{q}\frac{\sqrt{q!}\varsigma^{*n}}{n!(q-n)!}\langle 00|a^{q-n}b^n|\eta,\kappa_1,\kappa_2\rangle. \quad (43)$$

In order to obtain the tomogram amplitude, we first obtain

$$\begin{aligned} \langle 00|\langle a^{q-n}b^n|\eta,\kappa_1,\kappa_2\rangle &= A\langle 00|a^{q-n}b^n\int\frac{d^2z_1d^2z_2}{\pi^2}|z_1z_2\rangle\langle z_1z_2| \\ &\quad\times\exp[B + Ca_1^+ + Da_2^+ + Ea_1^+a_2^+ - Fa_1^{+2} - Fa_2^{+2}]|00\rangle \\ &= A\int\frac{d^2z_1d^2z_2}{\pi^2}z_1^{q-n}z_2^n\exp[-|z_1|^2 - |z_2|^2 + B + Cz_1^* + Dz_2^* \\ &\quad + Ez_1^*z_2^* - Fz_1^{*2} - Fz_2^{*2}] \\ &= A\frac{\partial^q}{\partial\lambda^{q-n}\partial\mu^n}\int\frac{d^2z_1d^2z_2}{\pi^2}\exp(-|z_1|^2 - |z_2|^2 + B + \lambda z_1 + \mu z_2 + Cz_1^* \\ &\quad + Dz_2^* + Ez_1^*z_2^* - Fz_1^{*2} - Fz_2^{*2})|_{\lambda=\mu=0}, \end{aligned} \quad (44)$$

using Eq. (25) the right-hand side (r.h.s) of Eq. (44) is converted into

$$\begin{aligned} \text{r.h.s of Eq. (44)} &= A\frac{\partial^q}{\partial\lambda^{q-n}\partial\mu^n}\exp\left[-F\lambda^2 + (C+\mu E)\lambda - F\mu^2 + D\mu + B\right]|_{\lambda=\mu=0} \\ &= AF^{(q-n)/2}e^BF^{(q-n)/2}\frac{\partial^n}{\partial\mu^n}H_{q-n}\left(\frac{C+\mu E}{2\sqrt{F}}\right)\exp(-F\mu^2 + D\mu)|_{\mu=0} \\ &= Ae^BF^{(q-n)/2}\sum_{l=0}^{n}\binom{n}{l}\left[\frac{\partial^l}{\partial\mu^l}H_{q-n}\left(\frac{C+\mu E}{2\sqrt{F}}\right)\right]\left[\frac{\partial^{n-l}}{\partial\mu^{n-l}}\exp(-F\mu^2 + D\mu)\right]|_{\mu=0} \\ &= Ae^B\sum_{l=0}^{n}\binom{n}{l}\binom{q-n}{l}l!E^lF^{(q-2l)/2}H_{q-n-l}\left(\frac{C}{2\sqrt{F}}\right)H_{n-l}\left(\frac{D}{2\sqrt{F}}\right), \end{aligned} \quad (45)$$

where we have used the generating function formula of $H_n(x)$

$$H_n(x) = \frac{\partial^n}{\partial t^n}e^{2xt-t^2}\bigg|_{t=0,} \quad (46)$$

and the recurrence relations of $H_n(x)$

$$\frac{d^l}{dx^l}H_n(x) = \frac{2^l n!}{(n-l)!}H_{n-l}(x). \quad (47)$$

So the tomogram amplitude of the state $|\xi, q\rangle$ is

$$\begin{aligned} \langle\xi,q|\eta,\kappa_1,\kappa_2\rangle = \frac{Ae^B\sqrt{q!}}{\sqrt{(1+|\varsigma|^2)^q}}\sum_{n=0}^{q}\sum_{l=0}^{n}\frac{\varsigma^{*n}l!E^lF^{(q-2l)/2}}{l!(n-l)!(q-n-l)!} \\ H_{q-n-l}\left(\frac{C}{2\sqrt{F}}\right)H_{n-l}\left(\frac{D}{2\sqrt{F}}\right). \end{aligned} \quad (48)$$

Using Eq. (39) we can obtain the tomogram of the state $|\xi, q\rangle$

$$\begin{aligned} T(\eta,\kappa_1,\kappa_2) = \frac{A^2e^{2B}q!}{(1+|\varsigma|^2)^q}\Bigg|\sum_{n=0}^{q}\sum_{l=0}^{n}\frac{\varsigma^{*n}l!E^lF^{(q-2l)/2}}{l!(n-l)!(q-n-l)!} \\ H_{q-n-l}\left(\frac{C}{2\sqrt{F}}\right)H_{n-l}\left(\frac{D}{2\sqrt{F}}\right)\Bigg|^2. \end{aligned} \quad (49)$$

Therefore, experimentally one can measure the modole-square of the wave function $|\xi, q\rangle$ in the entangled state $|\eta, \kappa_1, \kappa_2\rangle$ representation, then the tomogram of the state $|\xi, q\rangle$ is obtained. Since quantum state tomogram provides a means of fully reconstructing the density matrix for the state, the state $|\xi, q\rangle$ can be measured based on the results in Eq. (49).

6. Eigenstate of time-independent two coupled oscillators

In order to simplify the process of proving that the states $|\xi, q\rangle$ are just the eigenstates of the time-independent Hamiltonian of two coupled oscillators, we first give the wave function of the state $|\xi, q\rangle$ in the un-normalized entangled state $\langle\tau||$ related to $\langle\tau|$ by a normalization factor $\exp(-|\tau|^2/2)$. The inner product of the states $\langle\tau||$ and $|\xi, q\rangle$ is,

$$\langle\tau||\xi,q\rangle = \frac{\sqrt{q!}}{(1+|\varsigma|^2)^{q/2}}\sum_{n=0}^{q}H_{n,q-n}(\tau,\tau^*)\frac{(-\varsigma)^n}{n!(q-n)!}, \quad (50)$$

using the integral expression of $H_{m,n}(\zeta, \xi)$ [18],

$$H_{m,n}(\zeta,\xi) = (-1)^n e^{\zeta\xi}\int\frac{d^2z}{\pi}z^nz^{*m}\exp\left\{-|z|^2 + \zeta z - \xi z^*\right\}, \quad (51)$$

Eq. (50) is rewritten as

$$\begin{aligned} \langle\tau||\xi,q\rangle &= \frac{(-1)^q\sqrt{q!}e^{|\tau|^2}}{(1+|\varsigma|^2)^{q/2}}\sum_{n=0}^{q}\int\frac{d^2z}{\pi}\frac{z^{q-n}(\varsigma z^*)^n}{n!(q-n)!}\exp\left\{-z^2 + \tau z - \tau^*z^*\right\} \\ &= \frac{(-1)^qe^{|\tau|^2}}{\sqrt{q!}(1+|\varsigma|^2)^{q/2}}\int\frac{d^2z}{\pi}(z+\varsigma z^*)^q\exp\left\{-z^2 + \tau z - \tau^*z^*\right\}. \end{aligned} \quad (52)$$

For convenience, making the integration variable transform

$$z = z' - \varsigma z'^*, \quad z + \varsigma z^* = \left(1-|\varsigma|^2\right)z', \quad d^2z = \left(1-|\varsigma|^2\right)d^2z' \quad (53)$$

and setting

$$\kappa = \tau + \varsigma^*\tau^*, \quad (54)$$

then Eq. (52) becomes

$$\langle\tau||\xi,q\rangle=\frac{e^{|\tau|^2}\left(1-\varsigma^2\right)^{q+1}}{\sqrt{q!}(1+|\varsigma|^2)^{q/2}}\left(-\frac{\partial}{\partial\kappa}\right)^q\int\frac{d^2z'}{\pi}\times\exp\left\{-\left(1+|\varsigma|^2\right)|z'|\cdot+\varsigma^*z'^2+\varsigma z'^{*2}+\kappa z'-\kappa^*z'^*\right\}. \tag{55}$$

Further, using the mathematical integral formula (25) and the single-variable Hermite polynomial $H_m(x)$

$$H_m(x)=e^{x^2}\left(-\frac{d}{dx}\right)^m e^{-x^2}, \tag{56}$$

we derive wave function of the state $|\xi,q\rangle$ in $\langle\tau||$

$$\langle\tau||\xi,q\rangle=\frac{(-1)^{q/2}\varsigma^{q/2}}{\sqrt{q!}(1+|\varsigma|^2)^{q/2}}H_q\left(i\frac{\tau^*-\varsigma\tau}{2\sqrt{\varsigma}}\right), \tag{57}$$

which is just proportional to a single-variable ordinary Hermite polynomial of order q.

Let us consider the Hamiltonian of two coupled oscillators

$$H=\omega_1a^\dagger a+\omega_2b^\dagger b+\epsilon\left(a^\dagger b+ab^\dagger\right), \tag{58}$$

where ϵ is the coupling constant. By virtue of Eq. (10) and the differential relations

$$\frac{\partial}{\partial\tau^*}\langle\tau||=\langle\tau||a,\quad -\frac{\partial}{\partial\tau}\langle\tau||=\langle\tau||b, \tag{59}$$

in the $\langle\tau||$ basis we have the corresponding relations

$$a\rightarrow\frac{\partial}{\partial\tau^*},\quad b\rightarrow-\frac{\partial}{\partial\tau},\quad a^\dagger\rightarrow-\frac{\partial}{\partial\tau}+\tau^*,\quad b^\dagger\rightarrow\frac{\partial}{\partial\tau^*}-\tau. \tag{60}$$

Supposing the state $|\xi,\ q\rangle$ is an eigenstate of H with energy eigenvalue E,

$$\langle\tau||H|\xi,q\rangle=E\langle\tau||\xi,q\rangle, \tag{61}$$

then according to Eq. (60) we derive

$$\langle\tau||H|\xi,q\rangle=\left[\omega_1\left(-\frac{\partial}{\partial\tau}+\tau^*\right)\frac{\partial}{\partial\tau^*}-\omega_2\left(\frac{\partial}{\partial\tau^*}-\tau\right)\frac{\partial}{\partial\tau}+\epsilon\left(\frac{\partial^2}{\partial\tau^2}+\frac{\partial^2}{\partial\tau^{*2}}-\tau^*\frac{\partial}{\partial\tau}-\tau\frac{\partial}{\partial\tau^*}\right)\right]\langle\tau||\xi,q\rangle. \tag{62}$$

Substituting Eq. (57) into r.h.s of the Eq. (62) leads to

$$\langle\tau||H|\xi,q\rangle=D(\varsigma,q)\left[\left(i\omega_1\frac{\tau^*}{2\sqrt{\varsigma}}-i\omega_2\frac{\tau\sqrt{\varsigma}}{2}+i\epsilon G\right)H_q'(\lambda)-\frac{K}{4}H_q''(\lambda)\right], \tag{63}$$

where

$$G=\frac{\tau^*\sqrt{\varsigma}}{2}-\frac{\tau}{2\sqrt{\varsigma}},\quad K=(\omega_1+\omega_2)+\epsilon\left(\frac{1}{\varsigma}+\varsigma\right), \tag{64}$$

$$D(\varsigma,q)=\frac{(-1)^{q/2}\varsigma^{q/2}}{\sqrt{q!}(1+|\varsigma|^2)^{q/2}},\quad \lambda=i\frac{\tau^*-\varsigma\tau}{2\sqrt{\varsigma}}. \tag{65}$$

Further, using Eq. (61) and the property of Hermite polynomials,

$$2\lambda H_m'(\lambda)-2mH_m(\lambda)=H_m''(\lambda), \tag{66}$$

we obtain

$$EH_q(\lambda)=\left(i\omega_1\frac{\tau^*}{2\sqrt{\varsigma}}-i\omega_2\frac{\tau\sqrt{\varsigma}}{2}+i\epsilon G-\frac{K\lambda}{2}\right)H_q'(\lambda)+\frac{Kq}{2}H_q(\lambda). \tag{67}$$

Since

$$H_q'(\lambda)=2qH_{q-1}(\lambda) \tag{68}$$

and the Hermite polynomials of different orders are mutual orthogonal, so the coefficient of $H_q'(\lambda)$ is taken as zero, i.e.,

$$i\omega_1\frac{\tau^*}{2\sqrt{\varsigma}}-i\omega_2\frac{\tau\sqrt{\varsigma}}{2}+i\epsilon G-\frac{K\lambda}{2}=0. \tag{69}$$

Eq. (69) is satisfied for any value of τ and τ^*, so we have

$$\epsilon\varsigma^2+(\omega_1-\omega_2)\varsigma-\epsilon=0, \tag{70}$$

its solutions are

$$\varsigma_\pm=\frac{(\omega_2-\omega_1)\pm\sqrt{(\omega_1-\omega_2)^2+4\epsilon^2}}{2\epsilon}. \tag{71}$$

then the states $|\varsigma_\pm\rangle$, expressed by Eq. (6), are the energy eigenstates of H with the eigenvalues

$$E_\pm=\frac{q}{2}\left[(\omega_1+\omega_2)+\epsilon\left(\frac{1}{\varsigma_\pm}+\varsigma_\pm\right)\right]. \tag{72}$$

Therefore, we find that a set of energy eigenstates of time-independent two coupled oscillators are classified as the states $|\xi,q\rangle$ in terms of the values of q in the SU(2) Lie algebra realization, where is determined by the dynamics parameter in the Hamiltonian. In particular, when $\omega_1=\omega_2=\omega$, thus the Hamiltonian H describes two coupled isotropic harmonic oscillators, from Eqs. (71) and (72) we know $\varsigma_\pm=\pm1$, $E_\pm=(\omega\pm\epsilon)q$. Further, taking $q=1$, from Eq. (6) we know the eigenstate of H is

$$|\xi,q\rangle_\pm=\frac{1}{\sqrt{2}}(|1,0\rangle\pm|0,1\rangle). \tag{73}$$

From Eq. (8) we see that all values of are allowed because the states $|\xi,q\rangle$ shall form an over complete set. However, from Eqs. (71) and (72) we find that, for the definite Hamiltonian, only two very specific eigenstates corresponding to two specific values of are selected.

In summary, we have introduced a new type of FDPCS as the two-mode bosonic realizations of the SU(2) Lie groups [19,20]. Using the IWOP technique and the entangled state $|\tau\rangle$ representation of Wigner operator, we have obtained the Wigner function $W(\sigma,\gamma)$ of the state $|\xi,q\rangle$ and discussed its nonclasscial properties based on the negativity of $W(\sigma,\gamma)$. By virtue of the Radon transform between the Wigner operator and the projection operator of the entangled state $|\eta,\kappa_1,\kappa_2\rangle$ the tomogram of $|\xi,q\rangle$ is computed. Through our discussions we have noticed that the entangled state representations provide us with a convenient and direct approach for the calculation of Wigner function and tomogram of the two-mode correlated quantum state. Finally, using wave function of $|\xi,q\rangle$ in $\langle\tau|$ representation, it also proved that the states $|\xi,\ q\rangle$ are just a set of energy eigenstates of two coupled oscillators. In doing so, we wish that these results in this work may enrich PCS theory in quantum optics and be further used in the future works.

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