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SOME EXACT BLOWUP SOLUTIONS TO MULTIDIMENSIONAL SCHRÖDINGER MAP EQUATION ON HYPERBOLIC SPACE AND CONE

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Exact solutions for the multidimensional Schrödinger map equation (SM for short) on hyperbolic 2-space $\mathcal{H}^2$ cone are obtained. Consequently, we show the non-traveling wave solution on $\mathcal{H}^2$ is a finite energy solution on the finite spacial domain. The question of whether a solution of SM can develop a finite time singularity on $\mathcal{H}^2$ with smooth initial data is not clear. Our result show that blowup can really happen on this initial data. In addition, some exact global smooth solutions are constructed.

Keywords: Schrödinger map; Landau–Lifshitz; solution; blow up.

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1. Introduction

The Landau–Lifshitz (LL) equation is a well-known equation, which is one of the most important nonlinear equations in physics. LL bears a fundamental role in the understanding of non-equilibrium magnetism, just as the Navier–Stokes equation does in that of fluid dynamics. The fundamental form of LL is described with the help of the spin vector $s = (s_1, s_2, s_3)$ restricted to live on the unit sphere S^2:

$$s_1^2 + s_2^2 + s_3^2 = 1$$

while obeying the motion equation

$$s_t = s \times \Delta_{R^n} s\,, \tag{1}$$

where $\times$ denotes the cross-product in the Euclidean 3-space R^3. From the differential geometry point of view, this equation can be regarded as a special case of

Schrödinger map equation (SM) from spacetime into a Kähler manifold $\mathcal{M}$ with metric h and complex structure J satisfying:

$$u_t = J \sum_l D_l \partial^l u \,, \tag{2}$$

where D denotes the covariant derivative on $u^{-1}\mathcal{TM}$. Equations (1) and (2) are equivalent when $\mathcal{M} = S^2$.

If the target manifold is chosen to be the hyperbolic space, $\mathcal{H}^2 = \{(s_1, s_2, s_3) \in R^{2+1} : |s|^2 = s_1^2 + s_2^2 - s_3^2 = -1, s_3 > 0\}$. Equation (2) turns into the following equation which can be regarded as the dual Landau–Lifshitz (DLL) equation[1,10,11] because it is similar with (1) in the form:

$$s_t = s \dot{\times} \Delta_{R^n} s \,, \tag{3}$$

where $\dot{\times}$ denotes the pseudo-cross product in R^{2+1}, $\mathbf{a} \dot{\times} \mathbf{b} = (\mathbf{a} \times \mathbf{b})\,\mathrm{diag}\{1, 1, -1\}$.

Usually, the formal equivalence of Eq. (3), to a nonlinear Schrödinger equation can be seen by applying the stereographic projection from $\mathcal{H}^2$ to C_∞, the extended complex plane:

$$u = \frac{s_1 + is_2}{1 - s_3} \,, \qquad iu_t = -\Delta u - \frac{2\bar{u}}{1 - |u|^2} \sum_{j=1}^{n} (\partial_j u)^2 \,. \tag{4}$$

In this paper, we mainly discuss constructing the exact solution of (3) (or (4)) which we briefly note it as SM. SMs are the natural Schrödinger equation when the target space is a complex manifold (such as the sphere or hyperbolic space). So, we can regard SM as a general version of Schrödinger equations. We mention that a geometric derivative nonlinear Schrödinger equation that has been intensively studied is SM. On the target S^2, it is also known that (1) is equivalent to integrable cubic Schrödinger equation in space dimension $n = 1$ (see, e.g., Ref. 2). However, Eq. (1) is not equivalent to a standard Schrödinger equation without first-order derivative in higher dimensions ($n \geq 2$). As the SM is nonintegrable[3] in multidimensional case, we can only find out some particular exact solutions[7,9] by various direct methods such as Hirota bilinear method and auxiliary function method.[4–6] It is natural to expect that these similar situations will arise for the $\mathcal{H}^2$ case. In fact, the bad derivatives and the fraction part of (4) are hard to deal with. These two reasons make it hard to search for the exact solution of (3) or (4).

An open question, one can ask regarding (1) (or (3)) is the following. Given smooth initial data as in (1) (or (3)), does a smooth solution exist for all time? This question has been extensively studied in the past years. Local existence for smooth initial data that can be found, goes back to Refs. 16 and 17. The global existence and scattering was proved for general H^1, small initial data case was obtained by Bejenaru, Kenig and Tataru.[18] The global existence for H^2 data was established by Gustafson and Koo[19] in the radial case. While it is known that weak solutions are non-unique (global or not) in general[20] for LL, exactly speaking, a solution for PDE may blow up in finite time with respect to one norm, yet be continuable as a solution in an appropriately weakened sense. An inquiry into the property of the solution is a natural start to the problem of non-uniqueness as well as the broader issues regarding the validity of the model and selection criteria for correct solutions. As we know, a standard existence and continuation theorem asserts that finite time blowup is equivalent to global nonexistence. So, we can study the singularity topic about the solution for a better understanding about the non-uniqueness of the weak solution. Furthermore, the interpretation of blowup theorems in physical problems often poses difficulties; blowup may indicate either a real phenomenon or a failure of the physical model. In fact, in magnetically ordered materials, non-equilibrium magnetism shows a large variety of localized nonlinear excitations. The study of blowup phenomena of LL will be useful for better understanding about the self-focusing of magnetization evolution.

Recently, Merle, Raphael and Rodnianski[12] proved the existence of 1-equivariant blowup solutions and presented the blowup rate. For higher-dimensional cases, i.e. $n \geq 2$, SM is a critical ($n = 2$) (or supper-critical ($n > 2$)) equation. So, we expect that the solution of SM may develop finite-time singularities for some special setting of the initial data about the solution. For the S^2 case, we recommend Refs. 12 and 13 for some results about the blowup topic about SM or LL. Although the blowup properties are confirmed under some special norm of the solutions, we have to mention that whether smooth solutions always exist is still unknown. However, the blowup property of the solution on $\mathcal{H}^2$ is also not clear. In fact, we have never seen any blowup results about this case as far as we know. In this paper, we provide some results about the blowup properties of the SM (or DLL).

Since, now the motion takes place on pseudo-sphere $\mathcal{H}^2$, it is convenient to introduce the pseudo-spherical coordinates by analogy with spherical, e.g.

$$s_1 = \sinh\theta(t,x)\cos\varphi(t,x)\,, \qquad s_2 = \sinh\theta(t,x)\sin\varphi(t,x)\,, \qquad s_3 = \cosh\theta(t,x)\,.$$

Furthermore, if we endowed with the metric $\frac{4}{1-|u|^2}|du|^2$ about the complex plane $\mathbb{C}$ in which $u \in \mathbb{C}$, we can build the connection between (3) and (4). Also, by analogy with spherical case we can use the stereographic projection: from pseudo-sphere to hyperbolic plane. Exactly, we have

$$(s_1, s_2, s_3) = \left(\frac{\pm 2\,\mathrm{Re}(u)}{1-|u|^2}, \frac{\pm 2\,\mathrm{Im}(u)}{1-|u|^2}, \frac{1+|u|^2}{1-|u|^2}\right). \tag{5}$$

So according to (4) and (5), one can do the exact equivalent transform between the exact solutions of (3) and (4). In fact, this fact can be seen in Sec. 3.

The paper is organized as follows. In Sec. 2, the exact blowup solution on the cone about DLL is obtained under the radially symmetrical coordinates. In Sec. 3, some exact solutions which cover traveling wave and non-traveling wave solutions about the SM on $\mathcal{H}^2$ manifold will be presented. If the coefficients of the solutions are suitably chosen, these exact solutions can be a finite time blowup solution.

2. Blowup Solution on the Cone

In this section, we provide an exact blowup solution of (3) under the condition that $|V|$ is not a constant. We construct the exact solution of the following LL

$$s_t = s \dot{\times} \left(s_{rr} + \frac{n-1}{r} s_r \right). \tag{6}$$

Inspired by Refs. 8 and 14, we will find the explicit solution of (6) in the form of

$$\begin{cases} s_1(t,r) = A\cos(M(t,r))F(r)\,, \\ s_2(t,r) = A\sin(M(t,r))F(r)\,, \\ s_3(r) = BF(r)\,, \end{cases} \tag{7}$$

where $M(t,r)$ and $F(r)$ are functions to be determined, A and B are constants.

Substituting (7) into (6), we get

$$\begin{aligned} &AB(F(r))^2 \sin(M(t,r))\left(\frac{\partial}{\partial r}M(t,r)\right)^2 - AB(F(r))^2\cos(M(t,r))\frac{\partial^2}{\partial r^2}M(t,r) \\ &\quad - 2ABF(r)\cos(M(t,r))\frac{\partial}{\partial r}M(t,r)\frac{d}{dr}F(r) - AB\frac{n}{r}(F(r))^2 \\ &\quad \times \cos(M(t,r))\frac{\partial}{\partial r}M(t,r) + AB\frac{1}{r}(F(r))^2\cos(M(t,r))\frac{\partial}{\partial r}M(t,r) \\ &\quad + A\sin(M(t,r))\left(\frac{\partial}{\partial t}M(t,r)\right)F(r) = 0\,, \end{aligned} \tag{8}$$

$$\begin{aligned} &-AB(F(r))^2 \cos(M(t,r))\left(\frac{\partial}{\partial r}M(t,r)\right)^2 - AB(F(r))^2\sin(M(t,r))\frac{\partial^2}{\partial r^2}M(t,r) \\ &\quad - 2ABF(r)\sin(M(t,r))\frac{\partial}{\partial r}M(t,r)\frac{d}{dr}F(r) - AB\frac{n}{r}(F(r))^2 \\ &\quad \times \sin(M(t,r))\frac{\partial}{\partial r}M(t,r) + AB\frac{1}{r}(F(r))^2\sin(M(t,r))\frac{\partial}{\partial r}M(t,r) \\ &\quad - A\cos(M(t,r))\left(\frac{\partial}{\partial t}M(t,r)\right)F(r) = 0\,, \end{aligned} \tag{9}$$

$$\begin{aligned} &-A^2(F(r))^2\frac{\partial^2}{\partial r^2}M(t,r) - 2A^2F(r)\frac{\partial}{\partial r}M(t,r)\frac{d}{dr}F(r) \\ &\quad - \frac{n}{r}A^2(F(r))^2\frac{\partial}{\partial r}M(t,r) + \frac{1}{r}A^2(F(r))^2\frac{\partial}{\partial r}M(t,r) = 0\,. \end{aligned} \tag{10}$$

Furthermore, if we insert (10) into (8) and (9), we have

$$\begin{cases} r\left(\dfrac{\partial^2}{\partial r^2}M(t,r)\right)F(r) + 2r\left(\dfrac{\partial}{\partial r}M(t,r)\right)\dfrac{d}{dr}F(r) + F(r)\left(\dfrac{\partial}{\partial r}M(t,r)\right)n \\ \qquad - F(r)\dfrac{\partial}{\partial r}M(t,r) = 0\,, \\ B\left(\dfrac{\partial}{\partial r}M(t,r)\right)^2 F(r) + \dfrac{\partial}{\partial t}M(t,r) = 0\,. \end{cases} \tag{11}$$

If we settle down on the following ansatzs

$$M(t,r) = \frac{G(r)}{t-T} + C\,, \tag{12}$$

where $G(r)$ is the function in terms of r which is to be determined T and C are constants.

Substituting (12) into (11), we have

$$\begin{cases} r\left(\dfrac{d^2}{dr^2}G(r)\right)F(r) + 2r\left(\dfrac{d}{dr}G(r)\right)\dfrac{d}{dr}F(r) + F(r)\left(\dfrac{d}{dr}G(r)\right)n \\ \qquad - F(r)\dfrac{d}{dr}G(r) = 0\,, \\ B\left(\dfrac{d}{dr}G(r)\right)^2 F(r) - G(r) = 0\,. \end{cases} \tag{13}$$

Solving the above ordinary differential equations, we find out the exact solution of (13) as follows

$$\begin{cases} G(r) = \dfrac{(2C_2 + 3C_1 r^{2/3+n/3} + C_2 n)^3}{27(n+2)^3}\,, \\ F(r) = \dfrac{3(n+2)r^{2/3-2n/3}}{(2C_2 + 3C_1 r^{2/3+n/3} + C_2 n)BC_1^2}\,, \end{cases} \tag{14}$$

where γ, C_1, C_2 are any constants.

According to the above deduction, we have

Theorem 1. *Let $T > 0$, C_1, C_2, C are any constants, then*

$$\begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} \begin{pmatrix} A\cos\left(\frac{(2C_2+3C_1 r^{2/3+n/3}+C_2 n)^3}{27(n+2)^3(t-T)} + C\right) \frac{3(n+2)r^{2/3-2n/3}}{(2C_2+3C_1 r^{2/3+n/3}+C_2 n)BC_1^2} \\ A\sin\left(\frac{(2C_2+3C_1 r^{2/3+n/3}+C_2 n)^3}{27(n+2)^3(t-T)} + C\right) \frac{3(n+2)r^{2/3-2n/3}}{(2C_2+3C_1 r^{2/3+n/3}+C_2 n)BC_1^2} \\ \frac{3(n+2)r^{2/3-2n/3}}{(2C_2+3C_1 r^{2/3+n/3}+C_2 n)C_1^2} \end{pmatrix} \tag{15}$$

is a blowup solution of (6).

Proof. It is not difficult to verify the blowup property of the solution. When $r_0 \le r \le r_1$, $T > 0$, we can verify directly

$$|s_r|^2 = \left(\frac{d}{dr}F(r)\right)^2 (A^2 + B^2) + \frac{A^2(\frac{d}{dr}G(r))^2(F(r))^2}{(-t+T)^2}\,,$$

which is not bounded as $t \to T$. □

Remark 1.

(i) For the solution (15), a calculation shows

$$|s(t,r)|^2 = \frac{9(n+2)^2 r^{4/3-4n/3}(A^2+B^2)}{(2C_2 + 3C_1 r^{2/3+n/3} + C_2 n)^2 B^2 C_1^4}\,,$$

which is independent of t, so $\int_{r_0}^{r_1} r|s(t,r)|^2\, dr$ $(0 < r_0 \le r_1 < \infty)$ is a conservation quantity.

(ii) Since the solution has no bound as $r \to 0$ at the initial time $t = 0$, it does not belong to finite energy solution.

3. Global Solution and Blowup Solution about SM

In Sec. 2, the blowup solution on the cone is proposed. In this section, we investigate the other ansatz which live on the $\mathcal{H}^2$. In the following parts, some new traveling wave solutions and some new non-traveling wave solutions will be presented. We will see that non-traveling wave solutions can develop a finite time blowup behavior if the coefficients are suitably selected.

3.1. *Traveling wave solutions about the SM*

Let

$$u = e^{i\left(\sum_{j=1}^n p_j x_j + wt + \xi_1\right)} c_0\,, \tag{16}$$

where p_j, w, ξ_1, c_0 are constants.

According to (16), we get the exact smooth solution about (4)

$$u = e^{i\left(\sum_{j=1}^n p_j x_j + wt + \xi_1\right)} c_0\,, \tag{17}$$

where $w = \sum_{j=1}^n p_i^2 \frac{c_0^2+1}{c_0^2-1}$.

Employing (5), the exact solution of (3) is

$$\begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} \pm\sqrt{-1+s_0^2}\cos\left(\sum_{j=1}^n p_j x_j + wt + \xi_1\right) \\ \pm\sqrt{-1+s_0^2}\sin\left(\sum_{j=1}^n p_j x_j + wt + \xi_1\right) \\ s_0 \end{pmatrix}, \tag{18}$$

where $s_0 = \frac{1+c_0^2}{1-c_0^2}$, $w = -\sum_{j=1}^n p_i^2 s_0$ (p_j, w, ξ_1, c_0 are constants).

We can find out some other traveling wave solutions in the Euclid coordinates. Here, we consider the two-dimensional case on (x, y) coordinates. Let

$$\begin{cases} s_1(t,x,y) = \cos(\omega t + \zeta)\sin(\theta(x,y))\,, \\ s_2(t,x,y) = \sin(\omega t + \zeta)\sin(\theta(x,y))\,, \\ s_3(x,y) = \cos(\theta(x,y))\,, \end{cases} \tag{19}$$

where w, ζ are constants.

According to (19), (3) can be transformed into

$$\frac{\partial^2\theta(x,y)}{\partial x^2} + \frac{\partial^2\theta(x,y)}{\partial y^2} = \omega \sinh\theta(x,y)\,. \tag{20}$$

Equation (20) can be regarded as a Sinh-Gordon equation. Conveniently, we propose two different kinds of exact solutions of (20) in the following content.

The traveling wave solutions is:

$$\theta(x,y) = \pm 2\ln\left[\tan\frac{\omega(C_1x + C_2y + C_3)}{2(\omega(C_1^2 + C_2^2))^{1/2}}\right], \tag{21}$$

$$\theta(x,y) = \pm 4\arctan\left[\exp\frac{\omega(C_1x + C_2y + C_3)}{(\omega(C_1^2 + C_2^2))^{1/2}}\right], \tag{22}$$

where C_1, C_2 and C_3 are arbitrary constants.

We can also find out some functional separable solutions. Similar to the deduction process of Ref. 15, we can obtain the functional separable solutions of (20). Assume

$$\theta(x,y) = 4\,\mathrm{arctanh}[f(\xi_1)g(\xi_2)]\,, \tag{23}$$

where $\xi_1 = ax + b$, $\xi_2 = cy + d$, functions $f = f(\xi_1)$ and $g = g(\xi_2)$ are determined by the first-order autonomous Ordinary Differential Equations as follows

$$(f'_{\xi_1})^2 = Af^4 + Bf^2 + C\,, \qquad (g'_{\xi_2})^2 = -Cg^4 + (1-B)g^2 - A\,, \tag{24}$$

where A, B and C are arbitrary constants.

We omit some details about the deduction and present some exact solutions of (20). If $A \neq 0$, we have the following Jacobi elliptic function solution of (24).

$$f = \frac{\sqrt{2}C\,\mathrm{sn}\left(\frac{1}{2}\sqrt{-2B - 2\sqrt{B^2 - 4CA}}\xi_1 + C_1, \frac{\sqrt{2}\sqrt{-(2CA - B^2 - B\sqrt{B^2-4CA})CA}}{2CA - B^2 - B\sqrt{B^2-4CA}}\right)}{\sqrt{-C(B + \sqrt{B^2 - 4CA})}}, \tag{25}$$

$$g = \frac{\sqrt{2}A\,\mathrm{sn}\left(\frac{1}{2}\sqrt{-2 + 2B + 2\sqrt{B^2 - 2B + 1 + 4CA}}\xi_2 + C_2, \frac{\sqrt{-2C_3CA}}{C_3}\right)}{\sqrt{A(-1 + B + \sqrt{B^2 - 2B + 1 + 4CA})}}, \tag{26}$$

where C_1 and C_2 are constants,

$$C_3 = 2CA - 2B + B^2 + B\sqrt{B^2 - 2B + 1 + 4CA} + 1 - \sqrt{B^2 - 2B + 1 + 4CA}\,.$$

If $A = 0$, then we obtain

$$f = -\frac{\left(Ce^{2C_1\sqrt{B}} - e^{2\xi_1\sqrt{B}}\right)}{2\sqrt{B}e^{\sqrt{B}(C_1+\xi_1)}}\,, \tag{27}$$

$$g = -\frac{2e^{G(C_2+\xi_2)}(1-B)(1-C_3)}{-4Ce^{2\xi_2 G} + 4BCe^{2\xi_2 G} - e^{2C_2 G}}\,, \tag{28}$$

where $C_3 = \sqrt{1-B}$, C_1 and C_2 are constants.

In fact, according to the deduction process above, we have proven the following theorem.

Theorem 2.

(i) *Equations* (18) *and* (19) (*combined with* (21) *and* (22)) *are exact traveling wave solutions about* (3).

(ii) *Equation* (19) (*combined with* (23), (25)–(26) *and* (27)–(28)) *are exact solutions about* (3).

3.2. *Non-traveling wave solution about the SM*

Here, we choose another ansatz in our calculation for finding out the solution of SME. In fact, the ansatz which in a radial coordinates situation imply that our solution will be a non-traveling wave solution. Our ansatz for (4) is in the following form:

$$u(r,t) = e^{i(f(t)r^a)}g(t)\,, \tag{29}$$

where $r = \left(\sum_{j=1}^{n} x_j^2\right)^{1/2}$, a is a constant, $f(t)$ and $g(t)$ are the functions to be determined.

In the radial coordinates, (4) can be changed into

$$iu_t = -u_{rr} - \frac{n-1}{r}u_r - \frac{2\bar{u}}{1-|u|^2}u_r^2\,. \tag{30}$$

Substituting (29) into (30), then separating the real part and imaginary part of it, we have

$$\begin{aligned} r^{a+2}\frac{d}{dt}f(t) - r^{a+2}\left(\frac{d}{dt}f(t)\right)(g(t))^2 + (f(t))^2 r^{2a}a^2 \\ + (g(t))^2(f(t))^2 r^{2a}a^2 = 0\,, \end{aligned} \tag{31}$$

$$\left(\frac{d}{dt}g(t)\right)r^2 + f(t)r^a a^2 g(t) - 2f(t)r^a a g(t) + f(t)r^a a g(t)n = 0\,. \tag{32}$$

If $n = a = 2$, we can deduce the exact solutions of (31) and (32) as follows

$$f(t) = \pm\frac{C_1}{4\sqrt{C_2^2 + 2C_2C_1t + C_1^2t^2 - 4}}\,,$$

$$g(t) = \frac{1}{2}C_2 + \frac{1}{2}C_1t \mp \frac{1}{2}\sqrt{C_2^2 + 2C_2C_1t + C_1^2t^2 - 4}\,,$$

where C_1 and C_2 are arbitrary constants.

Accordingly, the exact solution of (4) is

$$u(t,r) = -\frac{1}{2}e^{\pm\frac{iC_1r^2}{4\sqrt{C_2^2+2C_2C_1t+C_1^2t^2-4}}}\Big(\pm\sqrt{C_2^2 + 2C_2C_1t + C_1^2t^2 - 4} - C_2 - C_1t\Big). \tag{33}$$

Employing (5), the exact solution of (3) is

$$\begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} \cos\left(\frac{C_1r^2\,\mathrm{sign}(C_1t+C_2)}{4\sqrt{-1+C_1^2t^2+2C_1C_2t+C_2^2}} - C_3\right)\frac{1}{\sqrt{-1+C_1^2t^2+2C_1C_2t+C_2^2}} \\ -\sin\left(\frac{C_1r^2\,\mathrm{sign}(C_1t+C_2)}{4\sqrt{-1+C_1^2t^2+2C_1C_2t+C_2^2}} - C_3\right)\frac{1}{\sqrt{-1+C_1^2t^2+2C_1C_2t+C_2^2}} \\ -\frac{(C_1t+C_2)\mathrm{sign}(C_1t+C_2)}{\sqrt{-1+C_1^2t^2+2C_1C_2t+C_2^2}} \end{pmatrix}, \tag{34}$$

where C_1, C_2 and C_3 are arbitrary constants.

One easily sees that (34) satisfies (3). Indeed, noticing that $\Delta s_3 = 0$, it suffices to show

$$\begin{pmatrix} \partial_t s_1 \\ \partial_t s_2 \\ \partial_t s_3 \end{pmatrix} = \begin{pmatrix} -s_3\Delta s_2 \\ s_3\Delta s_1 \\ -s_1\Delta s_2 + s_2\Delta s_1 \end{pmatrix}. \tag{35}$$

One can easily check (35) using MATHEMATICA or MAPLE. So (34) is a solution of (3).

In fact, we can construct solution (34) in another way. Let $n = 2$, we choose the following form of the solution

$$\begin{cases} s_1(t,r) = \cos(m(t,r))a(t)\,, \\ s_2(t,r) = \sin(m(t,r))a(t)\,, \\ s_3(t) = -\sqrt{1+(a(t))^2}\,, \end{cases} \tag{36}$$

where $m(t,r)$ and $a(t)$ are functions to be determined.

Substituting (36) into (3), similar to the deduction process of Sec. 2, we find out the equations for $m(t,r)$ and $a(t)$ as follows

$$\begin{cases} a(t)r\sqrt{1+(a(t))^2}\dfrac{\partial^2}{\partial r^2}m(t,r)+a(t)\left(\dfrac{\partial}{\partial r}m(t,r)\right)\sqrt{1+(a(t))^2} \\ \qquad -\left(\dfrac{d}{dt}a(t)\right)r=0\,, \\ \sqrt{1+(a(t))^2}\left(\dfrac{\partial}{\partial r}m(t,r)\right)^2-\dfrac{\partial}{\partial t}m(t,r)=0\,. \end{cases} \tag{37}$$

Accordingly, we find out the solution about (37) as follows

$$\begin{cases} m(t,r)=-\dfrac{C_1r^2\,\mathrm{sign}(C_1t+C_2)}{4\sqrt{-1+C_1^2t^2+2C_1C_2t+C_2^2}}+C_3\,, \\ a(t)=\dfrac{1}{\sqrt{-1+C_1^2t^2+2C_1C_2t+C_2^2}}\,, \end{cases} \tag{38}$$

where C_1, C_2, C_3 are any constants. According to the above deduction, we find out the exact solution of (3):

Theorem 3. *Equation* (33) *is a solution of* (4); (34) *is a solution of* (3).

Remark 2.

(i) If the coefficients of (34) are suitably selected, (34) can be a blowup solution. We can calculate directly that

$$|s_r|^2=\frac{C_1^2r^2}{4(C_2+1+C_1t)^2(C_2-1+C_1t)^2}\,. \tag{39}$$

In fact, if $C_2^2>1$ and $\pm\frac{1}{C_1}-\frac{C_2}{C_1}>0$, (39) is not bounded as $t\to\pm\frac{1}{C_1}-\frac{C_2}{C_1}$. According to (39), obviously, if $C_2^2=1$, (34) blows up at $t=0$.

It will be easy to find out the suitable coefficients which lead to a blowup solution of (4). For example, if $C_1=-1$, $C_2=2$ and $C_3=0$, exactly we have the following blowup (blow up at $t=1$) solution

$$\left(\frac{\cos\left(\frac{r^2}{4\sqrt{3+t^2-4t}}\right)}{\sqrt{3+t^2-4t}}\cdot\frac{\sin\left(\frac{r^2}{4\sqrt{3+t^2-4t}}\right)}{\sqrt{3+t^2-4t}}\cdot\frac{t-2}{\sqrt{3+t^2-4t}}\right).$$

(ii) Equation (34) can be a global solution. If $C_2^2>1$ and $\pm\frac{1}{C_1}-\frac{C_2}{C_1}<0$, (39) is bounded as for any $t>0$.

(iii) We mention that $(-s_1,s_2,-s_3)$ is also a solution of (4). Since the solution has no decay as $r\to\infty$, it does not belong to any Sobolev spaces.

Similar to the deduction of Theorem 3, we can extent the Theorem 3 to the three-dimensional case. We seek the solution to (4) with the form

$$\begin{cases} s_1(t,x_1,x_2,x_3) = \cos(m(t,x_1,x_2,x_3))h(t)\,, \\ s_2(t,x_1,x_2,x_3) = \sin(m(t,x_1,x_2,x_3))h(t)\,, \\ s_3(t) = -\sqrt{1+(h(t))^2}\,, \end{cases} \tag{40}$$

where $m(t,x_1,x_2,x_3)$ and $h(t)$ are functions to be determined.

Substituting (40) into (4), we get

$$\begin{cases} \sqrt{1+(h(t))^2}h(t)\dfrac{\partial^2}{\partial x_1^2}m(t,x_1,x_2,x_3) + \sqrt{1+(h(t))^2}h(t)\dfrac{\partial^2}{\partial x_2^2}m(t,x_1,x_2,x_3) \\ \qquad + \sqrt{1+(h(t))^2}h(t)\dfrac{\partial^2}{\partial x_3^2}m(t,x_1,x_2,x_3) - \dfrac{d}{dt}h(t) = 0\,, \\ \sqrt{1+(h(t))^2}\left(\dfrac{\partial}{\partial x_1}m(t,x_1,x_2,x_3)\right)^2 + \sqrt{1+(h(t))^2}\left(\dfrac{\partial}{\partial x_2}m(t,x_1,x_2,x_3)\right)^2 \\ \qquad + \sqrt{1+(h(t))^2}\left(\dfrac{\partial}{\partial x_3}m(t,x_1,x_2,x_3)\right)^2 - \dfrac{\partial}{\partial t}m(t,x,y,z) = 0\,. \end{cases} \tag{41}$$

Furthermore, we look for the functional separable solution of (41). Let $m(t,x_1,x_2,x_3) = p(t)q(x_1,x_2,x_3)$, we have

$$\begin{cases} \sqrt{1+(h(t))^2}(p(t))^2 + \dfrac{d}{dt}p(t) = 0\,, \\ \sqrt{1+(h(t))^2}h(t)p(t) + \dfrac{d}{dt}p(t) = 0\,, \\ \dfrac{\partial^2}{\partial x_1^2}q(x_1,x_2,x_3) + \dfrac{\partial^2}{\partial x_2^2}q(x_1,x_2,x_3) + \dfrac{\partial^2}{\partial x_3^2}q(x_1,x_2,x_3) + 1 = 0\,, \\ \left(\dfrac{\partial}{\partial x_1}q(x_1,x_2,x_3)\right)^2 + \left(\dfrac{\partial}{\partial x_2}q(x_1,x_2,x_3)\right)^2 + \left(\dfrac{\partial}{\partial x_3}q(x_1,x_2,x_3)\right)^2 \\ \qquad + q(x_1,x_2,x_3) = 0\,. \end{cases} \tag{42}$$

Solving (42), exactly we have

$$\begin{cases} p(t) = \pm\dfrac{1}{\sqrt{-1+t^2-2tC_1+C_1^2}}\,, \qquad h(t) = p(t)\,, \\ q(x,y,z) = -\dfrac{1}{12}((x_1-x_2)^2+(x_1-x_3)^2+(x_2-x_3)^2)\,, \end{cases} \tag{43}$$

where C_1 is a constant.

According to (40) and (43), we find out the exact solution of (4) as follows:

Theorem 4. *Let $C_1^2 \geq 1$, then*

$$\begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} \cos\left(\pm\dfrac{x_1^2 - x_1x_2 + x_2^2 - x_1x_3 + x_3^2 - x_2x_3}{6\sqrt{-1+t^2-2tC_1+C_1^2}}\right)\dfrac{1}{\sqrt{-1+t^2-2tC_1+C_1^2}} \\ \sin\left(\pm\dfrac{x_1^2 - x_1x_2 + x_2^2 - x_1x_3 + x_3^2 - x_2x_3}{6\sqrt{-1+t^2-2tC_1+C_1^2}}\right)\dfrac{1}{\sqrt{-1+t^2-2tC_1+C_1^2}} \\ \dfrac{(C_1-t)\mathrm{sign}(t-C_1)}{\sqrt{-1+t^2-2tC_1+C_1^2}}\,, \end{pmatrix}$$

is a solution of (4).

Remark 3. Following the above deduction, we can construct the exact solution of (4) for any spacial dimension n. In fact, let $x = (x_1, x_2, \ldots, x_n)$, $K = (K_1, K_2, \ldots, K_n)$ and $|K| = 1$, then $s(x_1, x_2, \ldots, x_n, t) = v(x_1, x_2, y, t)$ (v is a solution of (4) and $y = K \cdot x$) is the n-dimensional solution of (4).

4. Concluding Remarks

In the case when the curvature of the target manifold is S^2, under the radially symmetrical coordinates, the evolution of SM is given by the equation

$$iu_t = -u_{rr} - \frac{1}{r}u_r + \frac{2\bar{u}}{1+|u|^2}u_r^2\,. \tag{44}$$

In fact, a global solution of (3) was first presented in Ref. 7 as follows

$$u(t,r) = e^{i\frac{br^2}{4\sqrt{1+b^2(t-T)^2}}}\left(\sqrt{1+b^2(t-T)^2} - b(t-T)\right), \tag{45}$$

where $r = \left(\sum_{j=1}^2 x_j^2\right)^{1/2}$, $T > 0$ and b is a constant.

We remove the north pole from this sphere and apply the adverse stereographic projection

$$(s_1, s_2, s_3) = \left(\frac{\pm 2\,\mathrm{Re}(u)}{1+|u|^2}, \frac{\pm 2\,\mathrm{Im}(u)}{1+|u|^2}, \frac{1-|u|^2}{1+|u|^2}\right) \in S^2\backslash N\,, \tag{46}$$

then (44) transforms to an equivalent LL

$$s_t = s \times \left(s_{rr} + \frac{1}{r}s_r\right). \tag{47}$$

Employing (46), the exact solution of (47) is

$$(s_1, s_2, s_3) = \left(\pm\frac{\cos\frac{br^2}{4\sqrt{1+b^2(t-T)^2}}}{\sqrt{1+b^2(t-T)^2}}, \pm\frac{\sin\frac{br^2}{4\sqrt{1+b^2(t-T)^2}}}{\sqrt{1+b^2(t-T)^2}}, \frac{b(t-T)}{\sqrt{1+b^2(t-T)^2}}\right). \tag{48}$$

We can see that (45) and (48) are the global smooth solutions of (44) and (47), respectively. However, from the analysis of Sec. 3, we see that the solutions of SM and DLL can develop a finite time blowup behavior when the target manifold are replaced by $\mathcal{H}^2$. It reveals an important factor that the geometry of the target manifold should play an important role in the behavior of solutions since it affects the nature of the nonlinear term in the SM.

From Sec. 3, it will be clear that given smooth initial data as in SM, a smooth solution can develop a finite time singular behavior. Furthermore, on the finite spacial domain, this blowup solution is a finite energy solution in its initial time. However, the energy of the cone solution proposed in Sec. 2 will not be a convergent one in the initial time.

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