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Multiple solutions of some fourth-order boundary value problems

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Abstract

For some fourth-order boundary value problems, several new existence theorems on multiple positive, negative and sign-changing solutions are obtained. The critical point theory and the supersolution and subsolution method are employed to discuss this problem.

Keywords: Boundary value problems; Critical points; Cone; Supersolution and subsolution

1. Introduction

In this paper, we consider the existence and multiplicity of positive, negative and sign-changing solutions to the following fourth-order boundary value problem (BVP):

$$\begin{cases} u^{(4)} = f(t, u(t)), \quad t \in [0, 1] \\ u(0) = u(1) = 0, \\ u''(0) = u''(1) = 0, \end{cases} \tag{1}$$

where $f \in C([0, 1] \times \mathbb{R}^1, \mathbb{R}^1)$.

Owing to the importance of high order differential equations in physics, the existence and multiplicity of the solutions to such problems have been studied by many authors, see [2,4–6, 10–12,14,15]. They all obtained the existence of positive solutions provided f is superlinear or sublinear in u by employing the cone expansion or compression fixed point theorem. In [9],

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by using the strongly monotone operator principle and the critical point theory to discuss BVP (1), the authors established some sufficient conditions for f to guarantee that the problem has a unique solution, at least one nonzero solution, or infinitely many solutions.

In the present paper, by combining the critical point theory and the method of subsolutions and supersolutions, some new existence theorems on multiple positive, negative and sign-changing solutions are obtained. Our theorem can deal with the nonlinearity composed by a sublinear function and a superlinear function; see Example 21 in Section 3.

This paper is organized as follows. In Section 2 we give some preliminaries, including a critical point theorem which will be used in our main result, the definition of supersolution and subsolution, and some concepts concerning the partially ordered Banach space. The main results are established in Section 3.

2. Preliminaries

Let H be a real Hilbert space, E a real Banach space such that E is embedded in H. Let J be a C^{2-0} functional defined on H, that is, the differential J' of J is locally Lipschitz continuous from H to H. Assume that $J'(u) = u - Au$ and J' is also locally Lipschitz continuous as an operator from E to E.

We need the following concept of E-regular.

Definition 1. An operator A is said to be E-regular if there exists a finite sequence $\{E_i\}$, $i = 0, 1, \dots, n + 1$, of real Banach spaces such that

(i) $E = E_0 \hookrightarrow E_1 \hookrightarrow E_2 \hookrightarrow \cdots \hookrightarrow E_n \hookrightarrow E_{n+1} = H$;
(ii) $A(E_i) \subset E_{i-1}$ and $A\,|_{E_i} \in C(E_i, E_{i-1})$ for $i = 1, 2, \dots, n + 1$.

Remark 2. This definition was introduced by Hofer [8] to overcome the difficulty that in applications the natural cone in H has empty interior but has nonempty interior in E. Similar definition also presented in Chang [3].

Theorem 3 ([13]). *Assume that A is E-regular, J satisfies the PS condition on H and there are two open convex subsets D_1 and D_2 of E with the properties that $D_1 \cap D_2 \neq \emptyset$, $A(\partial_E D_1) \subset D_1$ and $A(\partial_E D_2) \subset D_2$. If there exists a path $h : [0, 1] \to E$ such that*

$$h(0) \in D_1 \setminus D_2, \quad h(1) \in D_2 \setminus D_1,$$

and

$$\inf_{u \in \overline{D}_1^E \cap \overline{D}_2^E} J(u) > \sup_{t \in [0,1]} J(h(t)),$$

then J has at least four critical points, one in $D_1 \cap D_2$, one in $D_1 \setminus \overline{D}_2^E$, one in $D_2 \setminus \overline{D}_1^E$ and one in $E \setminus (\overline{D}_1^E \cup \overline{D}_2^E)$. Here $\partial_E D$ and $\overline{D}^E$ mean respectively the boundary and the closure of D relative to E.

Definition 4. $\alpha \in C^4[0, 1]$ is called a subsolution for BVP (1.1) if

$$\begin{cases} \alpha^{(4)} \le f(t, u(t)), & t \in [0, 1] \\ \alpha(0) \le 0, & \alpha(1) \le 0, \\ \alpha''(0) \ge 0, & \alpha''(1) \ge 0. \end{cases}$$

A subsolution which is not a solution is called a strict subsolution. Supersolutions and strict supersolutions are defined by reversing the signs of the above inequalities.

We also need some basic concepts of ordered Banach spaces.

Definition 5. An ordered real Banach space is a pair (X, P), where X is a real Banach space and P a closed convex subset of X such that $(-P) \cap P = \{0\}$ and $\mathbb{R}^+ \cdot P \subset P$. The partial order on X is given by the cone P. For $u, v \in X$ we write

$$u \le v \Leftrightarrow v - u \in P;$$
$$u < v \Leftrightarrow u \le v \quad \text{but } u \ne v;$$
$$u \ll v \Leftrightarrow v - u \in \text{ int } (P).$$

If P has nonempty interior, then it is call a solid cone. If every ordered interval is bounded, then P is called a normal cone. An operator $A : \mathcal{D}(A) \to X$ is called order preserving (in the literature sometimes increasing) if

$$u \le v \Rightarrow Au \le Av,$$

strictly order preserving if

$$u < v \Rightarrow Au < Av,$$

and strongly order preserving if

$$u < v \Rightarrow Au \ll Av.$$

3. Main results

In this section, we will employ the abstract result in Section 2 to establish some existence theorems on positive, negative and sign-changing solutions of BVP (1). Firstly, we give some lemmas to change BVP (1) to a variational problem. Let $C[0, 1]$ denote the usual real Banach space with the norm $\|u\|_C = \max_{t\in[0,1]} |u(t)|$ for all $u \in C[0, 1]$. We can easily verify that

$$C_0[0, 1] = \{u \in C[0, 1] : u(0) = u(1) = 0\}$$

is also a Banach space with respect to $\|\cdot\|_C$. Let

$$P = \{u \in C_0[0, 1] : u(t) \ge 0 \text{ for all } t \in [0, 1]\},$$

then P is a normal solid cone in $C_0[0, 1]$ and

$$\text{int}(P) = \{u \in C_0[0, 1] : u(t) > 0 \text{ for all } t \in (0, 1)\}.$$

By $L^2[0, 1]$ we denote the usual real reflexive Banach space with the norm $\|u\| = (\int_0^1 |u(t)|^2 \mathrm{d}t)^{1/2}$ for all $u \in L^2[0, 1]$ and the real Hilbert space with the inner product $(u, v) = \int_0^1 u(t)v(t)\mathrm{d}t$ for all $u, v \in L^2[0, 1]$.

It is well known that the solution of BVP (1) in $C^4[0, 1]$ is equivalent to the solution of the following integral equation in $C[0, 1]$:

$$u(t) = \int_0^1 G(t, s) \left[\int_0^1 G(s, \tau) f(\tau, u(\tau))\mathrm{d}\tau\right] \mathrm{d}s, \quad t \in [0, 1], \tag{2}$$

where $G : [0,1] \times [0,1] \to [0,1]$ is Green's function for $-u''(t) = 0$ for all $t \in [0,1]$ subject to $u(0) = u(1) = 0$, i.e.

$$G(t,s) = \begin{cases} s(1-t), & 0 \le s \le t \le 1, \\ t(1-s), & 0 \le t \le s \le 1. \end{cases}$$

Define operators $K, \mathbf{f} : C[0,1] \to C[0,1]$, respectively, by

$$\begin{aligned} &Ku(t) = \int_0^1 G(t,s)u(s)\mathrm{d}s, \quad t \in [0,1], \forall u \in C[0,1], K^2 = K \circ K, \\ &\mathbf{f}u(t) = f(t, u(t)), \quad t \in [0,1], \forall u \in C[0,1]. \end{aligned} \tag{3}$$

Since $K : C[0,1] \to C_0[0,1]$, integral equation (2) is equivalent to the following operator equation in $C_0[0,1]$:

$$u = K^2 \mathbf{f} u. \tag{4}$$

Remark 6. It is easy to see that

(i) $G : [0,1] \times [0,1] \to [0,1]$ is nonnegative continuous;
(ii) $\max_{(t,s) \in [0,1] \times [0,1]} G(t,s) = 1/4$;
(iii) $\mathbf{f} : C[0,1] \to C[0,1]$ is bounded and continuous.

The operator K defined in (3) can also be defined on $L^2[0,1]$. In fact, we have the following lemma.

Lemma 7. *$K : L^2[0,1] \to C_0[0,1]$ is a linear completely continuous operator and $K : L^2[0,1] \to L^2[0,1]$ is also a linear completely continuous operator. In addition, $K : C_0[0,1] \to C_0[0,1]$ is strongly order preserving.*

Proof. For given $u \in L^2[0,1]$, it follows from (ii) of Remark 6 that

$$\begin{aligned} |Ku(t)| &= \left| \int_0^1 G(t,s)u(s)\mathrm{d}s \right| \le \frac{1}{4} \int_0^1 |u(s)|\mathrm{d}s \le \frac{1}{4} \left(\int_0^1 |u(s)|^2 \mathrm{d}s \right)^{1/2} \\ &= \frac{1}{4} \|u\|, \quad t \in [0,1]. \end{aligned} \tag{5}$$

So Ku is a function well defined on $[0,1]$. For any given $\varepsilon > 0$, since G is continuous on $[0,1] \times [0,1]$, there exists $\delta > 0$ such that $|G(t_1,s) - G(t_2,s)| < \varepsilon$ for all t_1, t_2 and s in $[0,1]$ with $|t_1 - t_2| < \delta$. And then for all t_1, t_2 with $|t_1 - t_2| < \delta$, it follows that:

$$\begin{aligned} |Ku(t_1) - Ku(t_2)| &= \left| \int_0^1 (G(t_1,s) - G(t_2,s))u(s)\mathrm{d}s \right| \\ &\le \int_0^1 |G(t_1,s) - G(t_2,s)||u(s)|\mathrm{d}s \\ &\le \varepsilon \int_0^1 |u(s)|\mathrm{d}s \\ &\le \varepsilon \|u\|. \end{aligned} \tag{6}$$

This implies that $Ku \in C[0, 1]$. And from (5) we have

$$\|Ku\|_C \leq \frac{1}{4}\|u\|, \quad u \in L^2[0, 1]. \tag{7}$$

It is obvious that K is linear and $Ku(0) = Ku(1) = 0$ for all $u \in L^2[0, 1]$. So, from (7), we obtain that $K : L^2[0, 1] \to C_0[0, 1]$ is continuous.

Let $S \subset L^2[0, 1]$ be a bounded subset. Then there exists $M > 0$ such that $\|u\| \leq M$ for all $u \in S$. It follows from (7) and (6) that $K(S)$ is bounded and equicontinuous. According to the Arzela–Ascoli theorem, $K(S)$ is a precompact subset of $C[0, 1]$. Observing that $K(S) \subset C_0[0, 1]$ and $C_0[0, 1]$ is a Banach space with respect to the norm of $C[0, 1]$, we have that $K(s)$ is a precompact subset of $C_0[0, 1]$. Therefore, $K : L^2[0, 1] \to C_0[0, 1]$ is completely continuous. Moreover, $K : L^2[0, 1] \to L^2[0, 1]$ is completely continuous because $C[0, 1]$ can be continuously embedded into $L^2[0, 1]$.

As for the last part of this lemma, it is enough to prove that $Ku \gg 0$ for any $u \in P \setminus \{0\}$ because $K : C_0[0, 1] \to C_0[0, 1]$ is a linear operator. For any $u \in P \setminus \{0\}$, i.e. $u(t) \geq 0$ for all $t \in [0, 1]$ and $u(t) \not\equiv 0$ in $t \in [0, 1]$, according to the property of $G(t, s)$, it is easy to see that $Ku(t) > 0$ for all $t \in (0, 1)$, thus $Ku \in \text{int}(P)$, i.e. $Ku \gg 0$. The proof is completed. □

Remark 8. From the definition of K, we can obtain that $Ku \neq 0$ for all $u \in L^2[0, 1]$ with $u \neq 0$. Therefore, $Ku_1 \neq Ku_2$ for all $u_1, u_2 \in L^2[0, 1]$ with $u_1 \neq u_2$. It is easy to see that the eigenvalues of K^2 are $1/(k^4\pi^4)$ and the corresponding eigenvectors are $\sin k\pi t$, $k = 1, 2, \ldots$. K^2 is linear compact, symmetric and the norm $\|K^2\| = 1/\pi^4$, where K^2 maps $L^2[0, 1]$ into itself.

Since the critical point theory will be employed to deal with BVP (1), we also need the following lemmas. Please find their proof in [9].

Lemma 9. (i) *The operator equation* (4) *has a solution in* $C_0[0, 1]$ *if and only if the operator equation*

$$v = K\mathbf{f}Kv \tag{8}$$

has a solution in $L^2[0, 1]$.

(ii) *The uniqueness of the solution for these two above equations is also equivalent.*

(iii) *If* (8) *has a nonzero solution in* $L^2[0, 1]$, *then* (4) *has a nonzero solution in* $C_0[0, 1]$.

Lemma 10. *Let* $\Phi(u) = \int_0^1 \int_0^{u(t)} f(t, v)\mathrm{d}v\, \mathrm{d}t$, $u \in C[0, 1]$. *Then*

(i) Φ *is Fréchet differentiable on* $C[0, 1]$ *and* $(\Phi'(u))(w) = (\mathbf{f}, w)$ *for all* $u, v \in C[0, 1]$;

(ii) $\Phi \circ K$ *is Fréchet differentiable on* $L^2[0, 1]$ *and* $(\Phi \circ K)'(v) = K\mathbf{f}Kv$ *for all* $v \in L^2[0, 1]$.

Choose $H = L^2[0, 1]$ and $E = C_0[0, 1]$ to be our Hilbert space and Banach space, respectively. Define a functional $J : H \to \mathbb{R}^1$

$$J(v) = \frac{1}{2}\|v\|^2 - \Phi(Kv), \quad v \in H. \tag{9}$$

Then, according to Lemma 10, we have

$$J'(v) = v - K\mathbf{f}Kv \quad \text{for all } v \in H. \tag{10}$$

Hence, Lemma 9 implies that the operator equation $u = K^2\mathbf{f}u$ has a solution in E if and only if the functional J has a critical point in H. Thus BVP (1) has been transformed into a variational problem. Before we state our main results, we list some conditions as follows which will play roles.

(H3.1) There exist a strict subsolution α and a strict supersolution β of BVP (1) with $\alpha < \beta$, $\alpha(0) = \alpha(1) = \alpha''(0) = \alpha''(1) = 0$ and $\beta(0) = \beta(1) = \beta''(0) = \beta''(1) = 0$;

(H3.2) $f(t, u)$ is strictly increasing in u;

(H3.3) $f(t, u)$ is locally Lipschitz continuous in u;

(H3.4) there exist $\mu \in (0, 1/2)$ and $M > 0$ such that $0 < F(t, u) \triangleq \int_0^u f(t, v)\mathrm{d}v \leq \mu u f(t, u)$ for all $|u| \geq M$ and $t \in [0, 1]$;

(H3.5) $\alpha_1 < \beta_1, \alpha_2 < \beta_2$ are two pairs of strict subsolutions and strict supersolutions of BVP (1) with $\alpha_i(0) = \alpha_i(1) = \alpha_i''(0) = \alpha_i''(1) = 0$, $\beta_i(0) = \beta_i(1) = \beta_i''(0) = \beta_i''(1) = 0$ for $i = 1, 2$.

Remark 11. There are several cases in which (H3.1) is satisfied. For example:

(a) $f(t, 0) = 0$, $\lim_{u\to 0+} \frac{f(t,u)}{u} < \pi^4$ and $\lim_{u\to 0-} \frac{f(t,u)}{u} < \pi^4$ uniformly for $t \in [0, 1]$. In this case, $\alpha = -\delta \sin \pi t$ is a strict subsolution and $\beta = \delta \sin \pi t$ is a strict supersolution for suitable positive number δ.

(b) There exists a number $k > 0$ such that $|f(t, u)| \leq k$ in $u \in [-c_k, c_k]$, where $c_k = \max_{t\in[0,1]} e_k(t)$ and e_k satisfies

$$\begin{cases} e_k^{(4)}(t) = k, & t \in [0, 1] \\ e_k(0) = e_k(1) = 0, \\ e_k''(0) = e_k''(1) = 0. \end{cases}$$

In this case, $\alpha = -e_k$ and $\beta = e_k$ are a strict subsolution and a strict supersolution, respectively.

Remark 12. (H3.3) is used only to deduce that J' is locally Lipschitz continuous. If $f \in C^1([0, 1] \times \mathbb{R}^1, \mathbb{R}^1)$, this assumption can be removed. In fact, if $f \in C^1([0, 1] \times \mathbb{R}^1, \mathbb{R}^1)$, it is easy to see that J is a C^2 functional and thus J' is locally Lipschitz continuous. See [7, P. 456, Remark 4].

The following lemmas will lead to the main results of this section.

Lemma 13. *Define $A = K\mathbf{f}K : H \to H$. Then A is E-regular.* (i) *If* (H3.2) *holds, then $A : C_0[0, 1] \to C_0[0, 1]$ is strongly order preserving.* (ii) *If* (H3.3) *holds, then J' is locally Lipschitz continuous both as an operator from H to H and as one from E to E.*

Proof. By Lemma 7 and (iii) of Remark 6, it is clear that A is E-regular. (i) can be directly obtained by the fact that $K : C_0[0, 1] \to C_0[0, 1]$ is strongly order preserving and $f(t, u)$ is strictly increasing in u. Next we prove (ii). Let $U \subset H$ be a bounded set. For any u and $v \in U$,

$$\begin{aligned} \|Au - Av\| &= \|K\mathbf{f}Ku - K\mathbf{f}Kv\| \\ &\leq \|K\|_{\mathcal{L}(H,H)} \cdot \|\mathbf{f}Ku - \mathbf{f}Kv\| \\ &= \|K\|_{\mathcal{L}(H,H)} \cdot \left(\int_0^1 |f(t, Ku(t)) - f(t, Kv(t))|^2 \mathrm{d}t \right)^{1/2}. \end{aligned} \tag{11}$$

Since $K(U)$ is bounded and noticing (H3.3), there exists a positive number L such that

$$|f(t, Ku(t)) - f(t, Kv(t))| \leq L|Ku(t) - Kv(t)| \quad \text{for all } t \in [0, 1]. \tag{12}$$

Combining (11) and (12), we get

$$\begin{aligned}\|Au - Av\| &\le L\|K\|_{\mathcal{L}(H,H)} \cdot \left(\int_0^1 |Ku(t) - Kv(t)|^2 \mathrm{d}t\right)^{1/2} \\ &= L\|K\|_{\mathcal{L}(H,H)} \cdot \|Ku - Kv\| \\ &\le L\|K\|^2_{\mathcal{L}(H,H)} \cdot \|u - v\|.\end{aligned}$$

Hence A is locally Lipschitz continuous, and so is J'. In a similar way, it is easy to prove $J' : E \to E$ is locally Lipschitz continuous. So we sketch it. The proof is completed. □

Lemma 14. *If* (H3.1) *and* (H3.2) *hold, then there exist* φ *and* $\psi \in E$ *with*

$$\varphi \ll \psi, \quad \varphi \ll A\varphi, \quad A\psi \ll \psi.$$

Proof. By (H3.1) and Definition 4, a direct computation shows that

$$\alpha < K^2\mathbf{f}\alpha, \quad K^2\mathbf{f}\beta < \beta.$$

Operating on both sides of the above two inequalities with $K\mathbf{f}$ and noticing that K is strongly order preserving and (H3.2), we obtain that

$$K\mathbf{f}\alpha \ll K\mathbf{f}K(K\mathbf{f}\alpha), \quad K\mathbf{f}K(K\mathbf{f}\beta) \ll K\mathbf{f}\beta.$$

Let $\varphi = K\mathbf{f}\alpha$ and $\psi = K\mathbf{f}\beta$, then $\varphi, \psi \in E$,

$$\varphi \ll \psi, \quad \varphi \ll A\varphi, \quad A\psi \ll \psi.$$

The proof is done. □

Theorem 15. *Assume that* (H3.1), (H3.2), (H3.3)*and* (H3.4) *hold. Then BVP (1.1) has at least four solutions.*

Proof. According to Lemma 14, there exist φ and $\psi \in E$ with

$$\varphi \ll \psi, \quad \varphi \ll A\varphi, \quad A\psi \ll \psi.$$

Define

$$D_1 = \{u \in E : u \ll \psi\} \quad \text{and} \quad D_2 = \{u \in E : u \gg \varphi\}.$$

Clearly, D_1 and D_2 are nonempty open convex subsets of E and $D_1 \cap D_2 = \{u \in E : \varphi \ll u \ll \varphi\} \neq \emptyset$. Moreover, if $u \in \partial_E D_1$, then $u \le \psi$. Thus, by Lemma 13, $Au \ll \psi$ and $A(\partial_E D_1) \subset D_1$. Similarly, $A(\partial_E D_2) \subset D_2$.

Step 1. J satisfies the PS condition on H. Since $F(t,u) - \mu u f(t,u)$ is continuous on $[0,1] \times [-M, M]$, there exists $C > 0$ such that

$$F(t,u) \le \mu u f(t,u) + C \quad \text{for all } t \in [0,1] \text{ and } u \in [-M, M].$$

By assumption (H3.4), we obtain

$$F(t,u) \le \mu u f(t,u) + C \quad \text{for all } t \in [0,1] \text{ and } u \in \mathbb{R}^1. \tag{13}$$

Suppose $\{v_n\} \subset H$ and there exists $M_1 > 0$ such that $|J(v_n)| \le M_1$ and $J'(v_n) = v_n - K\mathbf{f}Kv_n \to 0$ with respect to the H-norm. Notice that

$$(J'(v_n), v_n) = (v_n - K\mathbf{f}Kv_n, v_n) = \|v_n\|^2 - \int_0^1 f(t, Kv_n(t))Kv_n(t)\mathrm{d}t.$$

It follows from (13) and the definition of J that

$$\begin{aligned} M_1 \geq J(v_n) &= \frac{1}{2}\|v_n\|^2 - \int_0^1 F(t, Kv_n(t))\mathrm{d}t \\ &\geq \frac{1}{2}\|v_n\|^2 - \mu \int_0^1 f(t, Kv_n(t))Kv_n(t)\mathrm{d}t - C \\ &= \left(\frac{1}{2} - \mu\right)\|v_n\|^2 + \mu(J'(v_n), v_n) - C \\ &\geq \left(\frac{1}{2} - \mu\right)\|v_n\|^2 - \mu\|J'(v_n)\|\|v_n\| - C, \quad n = 1, 2, \ldots. \end{aligned}$$

Since $J'(v_n) \to 0$ as $n \to \infty$, there exists $N_0 \in \mathbb{N}$ such that

$$M_1 \geq \left(\frac{1}{2} - \mu\right)\|v_n\|^2 - \|v_n\| - C, \quad n > N_0.$$

This implies that $\{v_n\} \subset H$ is bounded. And then, the properties of K and $\mathbf{f}$ guarantee that $\{v_n\}$ has a convergent subsequence in H. Thus, J satisfies the PS condition on H.

Step 2. J is bounded from below on $\overline{D}_1^E \cap \overline{D}_2^E = [\varphi, \psi]$. For any $v \in [\varphi, \psi]$, it follows from Lemma 7 and (H3.2) that

$$K\varphi \leq Kv \leq K\psi, \quad \mathbf{f}(K\varphi) \leq \mathbf{f}(Kv) \leq \mathbf{f}(K\psi).$$

Since P is normal, there exists $M_0 > 0$ such that

$$\|Kv\|_C < M_0 \quad \text{and} \quad \|\mathbf{f}(Kv)\|_C < M_0 \quad \text{for all } v \in [\varphi, \psi].$$

Notice that $E \hookrightarrow H$, there is a number $N_0 > 0$ such that $\|u\| \leq N_0\|u\|_C$. So,

$$\|\mathbf{f}(Kv)\| \cdot \|Kv\| \leq N_0^2\|\mathbf{f}(Kv)\|_C \cdot \|Kv\|_C \leq N_0^2 M_0^2. \tag{14}$$

Therefore, according to (13), Hölder's inequality and (14), we have

$$\begin{aligned} J(v) &= \frac{1}{2}\|v\|^2 - \int_0^1 F(t, Kv(t))\mathrm{d}t \\ &\geq \frac{1}{2} \cdot \min\{\|\varphi\|^2, \|\psi\|^2\} - \mu \int_0^1 f(t, Kv(t))Kv(t)\mathrm{d}t - C \\ &\geq \frac{1}{2} \cdot \min\{\|\varphi\|^2, \|\psi\|^2\} - \mu \int_0^1 |f(t, Kv(t))||Kv(t)|\mathrm{d}t - C \\ &\geq \frac{1}{2} \cdot \min\{\|\varphi\|^2, \|\psi\|^2\} - \mu\|\mathbf{f}(Kv)\| \cdot \|Kv\| - C \\ &\geq \frac{1}{2} \cdot \min\{\|\varphi\|^2, \|\psi\|^2\} - \mu N_0^2 M_0^2 - C \\ &= \text{constant} \quad \text{for all } v \in [\varphi, \psi]. \end{aligned}$$

As a result, J is bounded from below on $[\varphi, \psi]$.

Step 3. Define a path $h_R : [0, 1] \to E$ as

$$h_R(s) = R\cos((1-s)\pi)\sin\pi t + R\sin((1-s)\pi)\sin 2\pi t, \quad R > 0.$$

Then $h_R(0) = -R\sin\pi t$ and $h_R(1) = R\sin\pi t$. Let $g_R(t) = R\sin\pi t + \varphi(t)$, then $g_R(0) = g_R(1) = 0$, i.e. $g_R \in E$. It is easy to see that g_R is strictly increasing at 0 and strictly decreasing at 1 as $R > \max\{-\varphi'(0)/\pi, \varphi'(1)/\pi\}$. So there exist two sufficiently small numbers $\delta_1, \delta_2 > 0$ such that $g_R(t) > 0$ for $t \in (0, \delta_1)$ and $(\delta_2, 1)$. For $t \in [\delta_1 - \varepsilon, \delta_2 - \varepsilon]$, $g_R(t) > 0$ as $R > M_2 = \max_{t\in[\delta_1-\varepsilon,\delta_2-\varepsilon]} |\frac{\varphi(t)}{\sin \pi t}|$, where ε is a sufficiently small positive number. Therefore,

$$g_R(t) > 0 \quad \text{for } t \in (0, 1) \quad \text{as } R > M_3 = \max\left\{M_2, -\frac{\varphi'(0)}{\pi}, \frac{\varphi'(1)}{\pi}\right\}.$$

Thus, if $R > M_3$, then $h_R(0) \ll \varphi \ll \psi$, i.e. $h_R(0) \in D_1 \backslash D_2$. In a similar way, $h_R(1) \in D_2 \backslash D_1$ as R is sufficiently large. Defining $\nu = 1/\mu > 2$, from (H3.4), we have

$$\frac{\nu}{u} \le \frac{f(t,u)}{F(t,u)} \quad \text{for all } t \in [0,1] \text{ and } u \ge M,$$
$$\frac{\nu}{u} \ge \frac{f(t,u)}{F(t,u)} \quad \text{for all } t \in [0,1] \text{ and } u \le -M.$$

Integrating the above two inequalities on $[M, u]$ and $[u, -M]$ respectively, we have

$$\nu \ln \frac{u}{M} \le \ln \frac{F(t,u)}{F(t,M)} \quad \text{for all } t \in [0,1] \text{ and } u \ge M,$$
$$\nu \ln \frac{-M}{u} \ge \ln \frac{F(t,-M)}{F(t,u)} \quad \text{for all } t \in [0,1] \text{ and } u \le -M,$$

that is,

$$F(t,u) \ge F(t,M)\left(\frac{u}{M}\right)^{\nu} \quad \text{for all } t \in [0,1] \text{ and } u \ge M,$$
$$F(t,u) \ge F(t,-M)\left(\frac{-u}{M}\right)^{\nu} \quad \text{for all } t \in [0,1] \text{ and } u \le -M.$$

Combining the above two inequalities, we get

$$F(t,u) \ge C_1|u|^{\nu} \quad \text{for all } t \in [0,1] \text{ and } |u| \ge M, \tag{15}$$

where $C_1 = M^{-\nu} \cdot \min\left\{\min_{t\in[0,1]} F(t,M), \min_{t\in[0,1]} F(t,-M)\right\} > 0$. Since $F(t,u)$ is bounded on $[0,1] \times [-M, M]$, there exists $C_2 > 0$ such that

$$F(t,u) \ge -C_2 \ge -C_2 + C_1|u|^{\nu} - C_1 M^{\nu} \quad \text{for all } (t,u) \in [0,1] \times [-M, M]. \tag{16}$$

It follows from (15) and (16) that

$$F(t,u) \ge C_1|u|^{\nu} - C_3 \quad \text{for all } (t,u) \in [0,1] \times \mathbb{R}^1, \tag{17}$$

where $C_3 = C_1 M^{\nu} + C_2 > 0$. Since $\nu > 2$, $L^{\nu}[0,1] \hookrightarrow H$, i.e., there exists $C_4 > 0$ such that $\|\cdot\| \le C_4 \|\cdot\|_{L^{\nu}[0,1]}$. Thus, from (17), a direct computation shows that

$$\begin{aligned} J(h_R(s)) &= \frac{1}{2}\|h_R(s)\|^2 - \int_0^1 F(t, Kh_R(s))\mathrm{d}t \\ &\le \frac{1}{4}R^2 - \int_0^1 \left(C_1|Kh_R(s)|^{\nu} - C_3\right)\mathrm{d}t \\ &= \frac{1}{4}R^2 - C_1\|Kh_R(s)\|^{\nu}_{L^{\nu}[0,1]} + C_3 \\ &\le \frac{1}{4}R^2 - C_1 C_4^{-\nu}\|Kh_R(s)\|^{\nu} + C_3 \end{aligned}$$

$$= \frac{1}{4}R^2 - C_1 C_4^{-\nu}\pi^{-2\nu}\left(\frac{1}{32} + \frac{15}{32}\cos^2((1-s)\pi)\right)^{\nu/2} R^\nu + C_3$$
$$\le \frac{1}{4}R^2 - C_5 R^\nu + C_3,$$

where $C_5 = C_1 C_4^{-\nu}\pi^{-2\nu}\min_{s\in[0,1]}\left(\frac{1}{32} + \frac{15}{32}\cos^2((1-s)\pi)\right)^{\nu/2} > 0$. As a result,

$$\lim_{R\to+\infty}\sup_{s\in[0,1]} J(h_R(s)) = -\infty.$$

Therefore,

$$\inf_{u\in\overline{D}_1^E\cap\overline{D}_2^E} J(u) > \sup_{s\in[0,1]} J(h_R(s))$$

as R is sufficiently large.

Up to now, all the conditions of Theorem 3 are satisfied and we get the result by Theorem 3. The proof is completed. □

It seems that the hypothesis (H3.2) is rather restrictive. However, by means of a simple technique it is possible to replace this hypothesis by the following weaker one:

(H3.2′) there exists a constant $m > 0$ such that

$$f(t,\xi) - f(t,\eta) > -m(\xi - \eta)$$

for all $t \in [0,1]$ and $\xi, \eta \in \mathbb{R}^1$ satisfying $\xi > \eta$.

Clearly, BVP (1.1) is equivalent to

$$\begin{cases} u^{(4)} + mu = f_1(t, u(t)), & t \in [0,1] \\ u(0) = u(1) = 0, \\ u''(0) = u''(1) = 0, \end{cases} \tag{1′}$$

where $f_1(t,u) = f(t,u) + mu$ for all $(t,u) \in [0,1]\times\mathbb{R}^1$. Let $G_1(t,s)$ be the Green's function for the linear boundary value problem

$$-u'' + mu = f_1(t,u(t)), \quad u(0) = u(1) = 0,$$

which is explicitly given by

$$G_1(t,s) = \begin{cases} (\omega\sinh\omega)^{-1}\cdot\sinh\omega s\cdot\sinh\omega(1-t), & 0\le s\le t\le 1, \\ (\omega\sinh\omega)^{-1}\cdot\sinh\omega t\cdot\sinh\omega(1-s), & 0\le t\le s\le 1, \end{cases}$$

where $\omega = \sqrt{m}$, $\sinh x = (\mathrm{e}^x - \mathrm{e}^{-x})/2$ is the hyperbolic sine function. It is easy to verify that $G_1(t,s) > 0$ for all $t, s \in [0,1]$. Define operators $K_1, \mathbf{f}_1 : C[0,1] \to C[0,1]$, respectively, by

$$\begin{aligned} &K_1 u(t) = \int_0^1 G_1(t,s)u(s)\mathrm{d}s, \quad t\in[0,1], \forall u \in C[0,1], K_1^2 = K_1\circ K_1, \\ &\mathbf{f}_1 u(t) = f_1(t,u(t)), \quad t\in[0,1], \forall u\in C[0,1]. \end{aligned} \tag{3′}$$

Then BVP (1′) is equivalent to the following operator equation in $C_0[0,1]$:

$$u = K_1^2\mathbf{f}_1 u. \tag{4′}$$

Obviously, $K_1, \mathbf{f}_1$ have the same properties as K and $\mathbf{f}$ in Lemmas 7, 9 and 10. Replacing (H3.2) by (H3.2′), Lemmas 13 and 14 still hold for the operator $A_1 = K_1\mathbf{f}_1K_1$. Therefore, we have the following theorem, which is an improvement of Theorem 15.

Theorem 16. *Assume that* (H3.1), (H3.2′), (H3.3) *and* (H3.4) *hold. Then BVP* (1) *has at least four solutions.*

Remark 17. The conditions of Theorems 15 and 16 permit the possibility that $f(t, 0) \neq 0$, therefore it is possible that the four solutions guaranteed by these two theorems are all nontrivial solutions.

Corollary 18. *In addition to the conditions in Theorem* 15 *(or* 16*), if* $f(t, 0) = 0$ *for all* $t \in [0, 1]$, $\alpha < 0$ *and* $\beta > 0$, *then BVP* (1) *possesses at least a positive solution, a negative solution, and a sign-changing solution.*

Proof. According to Lemma 14, there exist φ and $\psi \in E$ with

$$\varphi \ll \psi, \quad \varphi \ll A\varphi, \quad A\psi \ll \psi.$$

Besides this, from the conditions, we have $\varphi \ll 0$ and $\psi \gg 0$.

Let

$$D_1 = \{u \in E : u \ll \psi\} \quad \text{and} \quad D_2 = \{u \in E : u \gg 0\}.$$

Then it is easy to verify that D_1 and D_2 satisfy all the conditions of Theorem 3. According to Theorem 3, J has a critical point $v_1 \in D_2 \setminus \overline{D_1}^E$. Clearly, $v_1 \gg 0$. By [9, Lemma 3.1], $u_1 = Kv \gg 0$ is a positive solution of BVP (1). Similarly, if we define

$$D_1 = \{u \in E : u \ll 0\} \quad \text{and} \quad D_2 = \{u \in E : u \gg \varphi\},$$

then BVP (1) has a negative solution u_2 by Theorem 3.

Define D_1 and D_2 as in the proof of Theorem 15, then J has a critical point $v_3 \in E \setminus (\overline{D_1}^E \cup \overline{D_2}^E)$. By the structures of D_1 and D_2, we know that v_3 is sign-changing. So $u_3 = Kv_3$ is a sign-changing solution of BVP (1). The proof is done. □

By combining Theorem 2.1 and Amann's three-solution theorem (see [1, Theorem 14.2]), we can get more solutions of BVP (1).

Theorem 19. *Assume that* (H3.2) *(or* (H3.2′)*),* (H3.3), (H3.4) *and* (H3.5) *hold. Then BVP* (1) *has at least six solutions.*

Proof. Suppose that (H3.2) holds. Let $\varphi_i = K\mathbf{f}\alpha_i$ and $\psi_i = K\mathbf{f}\beta_i$, $i = 1, 2$. Then according to the proof of Lemma 14, we have

$$\varphi_1 \ll \psi_1 \ll \varphi_2 \ll \psi_2 \tag{18}$$

$$\varphi_1 \ll A\varphi_1, \quad A\psi_1 \ll \psi_1, \quad \varphi_2 \ll A\varphi_2, \quad A\psi_2 \ll \psi_2. \tag{19}$$

Define $D_1 = \{u \in E : u \ll \psi_2\}$ and $D_2 = \{u \in E : u \gg \varphi_1\}$. Then by the proof of Theorem 15, the functional J has at least four critical points v_1, v_2, v_3 and v_4 such that $v_1 \in D_1 \cap D_2$, $v_2 \in D_1 \setminus \overline{D_2}^E$, $v_3 \in D_2 \setminus \overline{D_1}^E$ and $v_4 \in E \setminus (\overline{D_1}^E \cap \overline{D_2}^E)$.

Since P is a normal solid cone in E and (18) and (19) hold, Amann's three-solution theorem guarantees that J has at least three critical points v_5, v_6 and v_7 in the ordered interval $[\varphi_1, \psi_2]$

with $v_5 \ll v_6 \ll v_7$. Observing the locations of these seven critical points, J has at least six critical points. Consequently, BVP (1) has at least six solutions. The case of (H3.2′) holding can be proved in a similar way. □

The following corollary is obvious.

Corollary 20. *In addition to the conditions in Theorem 19, if $f(t,0) = 0$ for all $t \in [0,1]$, $\beta_1 < 0$ and $\alpha_2 > 0$, then BVP (1) possesses at least two positive solutions, two negative solutions, and a sign-changing solution.*

At the end of this paper, we present two simple examples to which Corollaries 18 and 20 can be applied respectively.

Example 21. Let

$$f(t,u) = \begin{cases} e^u - 1, & (t,u) \in [0,1] \times [0,+\infty), \\ u^3, & (t,u) \in [0,1] \times (-\infty,0]. \end{cases} \tag{20}$$

It is easy to verify that all conditions of Corollary 18 are satisfied. So Corollary 18 ensures that BVP (1) has at least one positive solution, one negative solution, and one sign-changing solution.

Example 22. The following nonlinearity is a sum of a sublinear function and a superlinear function.

$$f(t,u) = \begin{cases} u^{\frac{1}{2}} + u^2, & (t,u) \in [0,1] \times [0,+\infty), \\ u^{\frac{1}{3}} + u^3, & (t,u) \in [0,1] \times (-\infty,0]. \end{cases} \tag{21}$$

For suitable positive numbers M and δ, it is easy to see that $\alpha_1 = -Me < -\delta \sin \pi t = \beta_1$ and $\alpha_2 = \delta \sin \pi t < Me = \beta_2$ are two pairs of strict subsolutions and strict supersolutions, where $e(t)$ is the solution of the following linear problem

$$\begin{cases} e^{(4)}(t) = 1, & t \in [0,1] \\ e(0) = e(1) = 0, \\ e''(0) = e''(1) = 0. \end{cases}$$

Corollary 20 guarantees that BVP (1) has at least two positive solutions, two negative solutions and one sign-changing solution.

References

[1] H. Amann, Fixed point equations and nonlinear eigenvalue problems in order Banach spaces, SIAM Rev. 18 (1976) 620–709.

[2] Z. Bai, H. Wang, On positive solutions of some nonlinear fourth-order beam equations, J. Math. Anal. Appl. 270 (2002) 357–368.

[3] K.C. Chang, A variant mountain pass lemma, Sci. Sinica (Ser. A) 26 (1983) 1241–1255.

[4] J.M. Davis, P.W. Eloe, J. Henderson, Triple positive solutions and dependence on higher order derivatives, J. Math. Anal. Appl. 237 (1999) 710–720.

[5] J.M. Davis, J. Henderson, P.J.Y. Wong, General Lidstone problems: Multiplicity and symmetry of solutions, J. Math. Anal. Appl. 251 (2000) 527–548.

[6] J.R. Graef, C. Qian, B. Yang, Multiple symmetric positive solution of a class of boundary value problem for higher order ordinary differential equations, Proc. Amer. Math. Soc. 131 (2003) 577–585.

[7] D. Guo, Nonlinear Functional Analysis, second edn., Shandong Science and Technology Press, 2001 (in Chinese).

[8] H. Hofer, Variational and topological methods in partially ordered Hilbert spaces, Math. Ann. 261 (1982) 493–514.

[9] F. Li, Q. Zhang, Z. Liang, Existence and multiplicity of solutions of a kind of fourth-order boundary value problem, Nonlinear Anal. 62 (2005) 803–816.
[10] Y. Li, Positive solutions of fourth-order periodic boundary problems, Nonlinear Anal. 54 (2003) 1069–1078.
[11] Y. Li, Positive solutions of fourth-order boundary value problems with two parameters, J. Math. Anal. Appl. 281 (2003) 477–484.
[12] B. Liu, Positive solutions of fourth-order two point boundary value problems, Appl. Math. Comput. 148 (2004) 407–420.
[13] Z. Liu, J. Sun, Invariant sets of descending flow in critical point theory with applications to nonlinear differential equations, J. Differential Equations 172 (2001) 257–299.
[14] Q. Yao, Positive solutions for eigenvalue problems of fourth-order elastic beam equations, Appl. Math. Lett. 17 (2004) 237–243.
[15] B. Zhang, X. Liu, Existence of multiple symmetric positive solutions of higher order Lidstone problems, J. Math. Anal. Appl. 284 (2003) 672–689.