Algebraic Properties and Panconnectivity of Folded Hypercubes*

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Abstract This paper considers the folded hypercube FQ_n, as an enhancement on the hypercube, and obtain some algebraic properties of FQ_n. Using these properties the authors show that for any two vertices x and y in FQ_n with distance d and any integers $h \in \{d, n+1-d\}$ and l with $h \leq l \leq 2^n - 1$, FQ_n contains an xy-path of length l and no xy-paths of other length provided that l and h have the same parity.

Keywords: Path, Folded hypercube, Transitivity, Panconnectivity

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1 Introduction

It is well-known that a topological structure for an interconnection network can be modelled by a connected graph $G = (V, E)$ [14]. As a topology for an interconnection network of a multiprocessor system, the hypercube structure is a widely used and well-known interconnection model since it possesses many attractive properties [8, 14]. The n-dimensional hypercube Q_n is a graph with 2^n vertices, each vertex with a distinct binary string $x_1x_2\cdots x_n$ of length n on the set $\{0, 1\}$, and two vertices being linked by an edge if and only if their strings differ in exactly one bit.

As a variant of the hypercube, the n-dimensional folded hypercube FQ_n, proposed first by El-Amawy and Latifi [3], is a graph obtained from

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the hypercube Q_n by adding an edge between any two complementary vertices $x = (x_1x_2 \cdots x_n)$ and $\bar{x} = (\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n)$, where $\bar{x}_i = 1 - x_i$. We call these added edges complementary edges, to distinguish them from the edges, called regular edges, in Q_n.

From definitions, Q_n is a proper spanning subgraph of FQ_n, and so FQ_n has 2^n vertices. It has been shown that FQ_n is $(n+1)$-regular $(n+1)$-connected, and has diameter $\lceil \frac{n}{2} \rceil$, about half the diameter of Q_n [3]. Thus, the folded hypercube FQ_n is an enhancement on the hypercube Q_n and has recently attracted many researchers' attention [2, 4, 5, 7, 10, 12]. In this paper, we further investigate other topological properties of FQ_n, transitivity and panconnectivity.

A graph G is called to be vertex-transitive if for any $x, y \in V(G)$ there is some $\sigma \in \mathrm{Aut}\,(G)$, the automorphism group of G, such that $\sigma(x) = y$; and edge-transitive if for any $xy, uv \in E(G)$ there is some $\phi \in \mathrm{Aut}\,(G)$ such that $\{\phi(x), \phi(y)\} = \{u, v\}$. It has been known that Q_n is both vertex-transitive and edge-transitive [1]. However, the transitivity of FQ_n has not been proved straightforwardly in the literature. In this paper, we will study some algebraic properties of FQ_n. Using these properties, we give another proof of a known result that FQ_n is vertex and edge-transitive.

A graph G is panconnected if for any two different vertices x and y in G and any integer l with $d_G(x, y) \leq l \leq |V(G)| - 1$ there exists an xy-path of length l, where $d_G(x, y)$ is the distance between x and y in G [11]. It is easy to see that any bipartite graph with at least three vertices is not panconnected. For this reason, Li *et al* [6] suggested the concept of bipanconnected bipartite graphs. A bipartite graph G is called to be bipanconnected if for any two different vertices x and y in G and any integer l with $d_G(x, y) \leq l \leq |V(G)| - 1$ such that l and $d_G(x, y)$ have the same parity there exists an xy-path of length l. Li *et al* [6] have shown that Q_n is bipanconnected. In this paper, we show that for any two vertices x and y in FQ_n with distance d, FQ_n contains an xy-path of length l with $h \leq l \leq 2^n - 1$ such that l and h have the same parity, where $h \in \{d, n+1-d\}$. Hence, FQ_n is bipanconnected if n is odd.

The proofs of our results are in Section 2 and Section3, respectively.

2 Algebraic Properties

In this section, we study some algebraic properties of FQ_n, and as applications, show that FQ_n is vertex and edge-transitive.

The following notations will be used in the proofs of our main results. The symbol $H(x, y)$ denotes the Hamming distance between two vertices x and y in Q_n, that is, the number of different bits in the corresponding strings of both vertices. Clearly, $H(x, y) = d_{Q_n}(x, y)$. It is also clear that

$d_{FQ_n}(x,y)=i$ if and only if $H(x,y)=i$ or $n+1-i$. Let $x=0u$ and $y=1v$ be two vertices in FQ_n. It is easy to count that

$$\begin{aligned} H(0u,1\bar{v}) = & \; H(0u,1u)+H(1u,1\bar{v}) \\ = & \; 1+[(n-1)-H(1u,1v)] \\ = & \; n+1-H(0u,1v). \end{aligned} \tag{1}$$

Let Γ be a non-trivial finite group, S be a non-empty subset of Γ without the identity of Γ and with $S^{-1}=S$. The Cayley graph $C_\Gamma(S)$ of Γ with respect to S is defined as follows.

$$V=\Gamma; \quad (x,y)\in E \Leftrightarrow x^{-1}y\in S, \text{ for any } x,y\in\Gamma.$$

It has been proved that any Cayley graph is vertex-transitive (see, for example, Theorem 2.2.15 in [14]).

As we have known that the hypercube Q_n is the Cayley graph $C_{Z_2^n}(S)$, where Z_2 denotes the additive group of residue classes modulo 2 on the set $\{0,1\}$, $Z_2^n=Z_2\times Z_2\times\cdots\times Z_2$, and $S=\{(10\cdots0),(010\cdots0),\cdots,(0\cdots010\cdots0),\cdots,(0\cdots01)\}$ (see, for example, Example 2 in p89 in [14]). The following theorem shows that FQ_n is also a Cayley graph.

Theorem 2.1 The folded hypercube $FQ_n\cong C_{Z_2^n}(S\cup\{(11\cdots1)\})$.

Proof Clearly, $V(FQ_n)=Z_2^n$. Define a natural mapping

$$\begin{array}{rcl} \varphi: \quad V(FQ_n) & \to & Z_2^n \\ x & \mapsto & \varphi(x)=x. \end{array}$$

Let x and y be any two vertices in FQ_n. Since $(x,y)\in E(FQ_n)$ if and only if $H(x,y)=1$ or n. Note that $x^{-1}=x$ for any $x\in Z_2^n$. It follows that $H(x,y)=1$ if and only if $x^{-1}y\in S$; and $H(x,y)=n$ if and only if $x^{-1}y=(11\cdots1)$, whereby $(x,y)\in E(C_{Z_2^n}(S\cup\{(11\cdots1)\}))$. Thus, φ preserves the adjacency of vertices, which implies that φ is an isomorphism between FQ_n and $C_{Z_2^n}(S\cup\{(11\cdots1)\})$, and so $FQ_n\cong C_{Z_2^n}(S\cup\{(11\cdots1)\})$. ∎

Corollary 2.2 The folded hypercube FQ_n is vertex-transitive.

For convenience, we express FQ_n as $FQ_n=L\otimes R$, where L and R are the two $(n-1)$-dimensional subcubes of Q_n induced by the vertices with the leftmost bit is 0 and 1, respectively. A vertex in L will be denoted by $0u$ and a vertex in R denoted by $1v$, where u and v are any two vertices in Q_{n-1}. Between L and R, apart from the regular edges, there exists a complementary edge joining $0u$ and $1\bar{u}\in R$ for any $0u\in L$.

Theorem 2.3 Let σ be a mapping from $V(FQ_n)$ to itself defined by

$$\left\{\begin{array}{l} \sigma(0u)=0u \\ \sigma(1u)=1\bar{u} \end{array}\right. \quad \text{for any } u\in V(Q_{n-1}). \tag{2}$$

Then $\sigma \in \mathrm{Aut}\,(FQ_n)$. Moreover, for an edge (x, y) between L and R in FQ_n, $(\sigma(x), \sigma(y))$ is complementary if and only if (x, y) is regular.

Proof Clearly, σ is a permutation on $V(FQ_n)$. To show $\sigma \in \mathrm{Aut}\,(FQ_n)$, it is sufficient to show that σ preserves adjacency of vertices in FQ_n, that is, to show that any pair of vertices x and y in FQ_n satisfies the following condition.

$$(x, y) \in E(FQ_n) \Leftrightarrow (\sigma(x), \sigma(y)) \in E(FQ_n). \tag{3}$$

Let $FQ_n = L \otimes R$, u and v be any two distinct vertices in Q_{n-1}. Because of vertex-transitivity of FQ_n by Theorem 2.1, without loss of generality, suppose $x = 0u \in L$. We consider two cases according to the location of y.

Case 1 $y \in L$. In this case, let $y = 0v$. Since σ is the identical permutation on $L \cong Q_{n-1}$, it is clear that

$$(0u, 0v) \in E(FQ_n) \Leftrightarrow (\sigma(0u), \sigma(0v)) = (0u, 0v) \in E(FQ_n).$$

Case 2 $y \in R$. In this case, let $y = 1v$. By the definition of FQ_n, $(0u, 1v) \in E(FQ_n) \Leftrightarrow v = u$ or $\bar{u}$. Since $(0u, 1u), (0u, 1\bar{u}) \in E(FQ_n)$ by the definition of FQ_n, it follows that

$$\begin{array}{ll} (0u, 1u) \in E(FQ_n) \Leftrightarrow (\sigma(0u), \sigma(1u)) = (0u, 1\bar{u}) \in E(FQ_n) & \text{if } v = u \\ (0u, 1\bar{u}) \in E(FQ_n) \Leftrightarrow (\sigma(0u), \sigma(1\bar{u})) = (0u, 1u) \in E(FQ_n) & \text{if } v = \bar{u}. \end{array}$$

From the above arguments, we have shown $\sigma \in \mathrm{Aut}\,(FQ_n)$.

We now show the remaining part of the theorem. Without loss of generality, we may suppose $x = 0u$ since FQ_n is vertex-transitive. By (2), we have $\sigma(x) = \sigma(0u) = 0u$.

Suppose that (x, y) is a regular edge between L and R in FQ_n. Then $y = 1u$ and $\sigma(y) = \sigma(1u) = 1\bar{u}$ by (2). By (3) $(0u, 1\bar{u}) \in E(FQ_n)$, which is a complementary edge.

Conversely, suppose that (x, y) is a complementary edge in FQ_n. Then $y = 1\bar{u}$ and $\sigma(1\bar{u}) = 1u$ by (2), and $(0u, 1u) \in E(FQ_n)$ by (3), which is a regular edge.

The lemma follows. ∎

Theorem 2.4 $\mathrm{Aut}\,(Q_n)$ is a proper subgroup of $\mathrm{Aut}\,(FQ_n)$. Moreover, for any $\sigma \in \mathrm{Aut}\,(Q_n)$, (x, y) is a complementary edge if and only if $(\sigma(x), \sigma(y))$ is also a complementary edge in FQ_n.

Proof For any element $\sigma \in \mathrm{Aut}\,(Q_n)$, we will prove $\sigma \in \mathrm{Aut}\,(FQ_n)$.

It is clear that σ is a permutation on $V(FQ_n)$ since Q_n is a spanning subgraph of FQ_n. We only need to show that σ preserves adjacency of vertices in FQ_n, that is, to check that (3) holds for any pair of vertices x and y in FQ_n. In fact, since

$$H(x, y) = d_{Q_n}(x, y) = d_{Q_n}(\sigma(x), \sigma(y)) = H(\sigma(x), \sigma(y))$$

and

$$(x,y) \in E(FQ_n) \Leftrightarrow H(x,y) = 1 \text{ or } n,$$

we have

$$\begin{aligned}(x,y) \in E(FQ_n) &\Leftrightarrow H(x,y) = 1 \text{ or } n \\ &\Leftrightarrow H(\sigma(x),\sigma(y)) = 1 \text{ or } n \\ &\Leftrightarrow (\sigma(x),\sigma(y)) \in E(FQ_n).\end{aligned}$$

Thus, $\mathrm{Aut}\,(Q_n) \subseteq \mathrm{Aut}\,(FQ_n)$. It is clear that the automorphism σ defined by (2) is not in $\mathrm{Aut}\,(Q_n)$ by Theorem 2.3. Therefore, $\mathrm{Aut}\,(Q_n)$ is a proper subgraph of $\mathrm{Aut}\,(FQ_n)$.

By the definition of FQ_n, for any $\sigma \in \mathrm{Aut}\,(Q_n)$, it is clear that (x,y) is a complementary edge in FQ_n if and only if $n = H(x,y) = H(\sigma(x),\sigma(y))$, if and only if $\sigma(x,y) = (\sigma(x),\sigma(y))$ is a complementary edge in FQ_n.

The theorem follows. ∎

Corollary 2.5 The folded hypercube FQ_n is edge-transitive.

Proof For any two edges (x,y) and (x',y') in FQ_n, we will show there is an element $\sigma \in \mathrm{Aut}\,(FQ_n)$ such that $\{\sigma(x),\sigma(y)\} = \{x',y'\}$. Since FQ_n is vertex-transitive, we may assume $x = x'$. We only need to find $\sigma \in \mathrm{Aut}\,(FQ_n)$ that takes y to y' and fixes x. Since for any two vertices z and t in FQ_n, $(z,t) \in E(FQ_n)$ if and only if $H(z,t) = 1$ or n. Without loss of generality, we may suppose that $H(x,y) = 1$, that is, (x,y) is a regular edge in FQ_n.

If $H(x,y') = 1$, then (x,y') is a regular edge. Since Q_n is edge-transitive, there is an element $\sigma \in \mathrm{Aut}\,(Q_n)$ such that $\{\sigma(x),\sigma(y)\} = \{x,y'\}$. By Theorem 2.4, $\sigma \in \mathrm{Aut}\,(FQ_n)$, which satisfies our requirement.

If $H(x,y') = n$, then $y' = \bar{x}$ and (x,y') is a complementary edge in FQ_n. Without loss of generality, we may suppose that $x = 0u$. Then $y' = 1\bar{u}$. Let $z = 1u$. Then the automorphism σ defined in (2) can take z to y' and fixes x. If $y = z$, then the σ satisfies our requirement. If $y \neq z$, then there is $\phi \in \mathrm{Aut}\,(Q_n) \subset \mathrm{Aut}\,(FQ_n)$ such that ϕ takes y to z and fixes x. Thus, $\sigma\phi(y) = \sigma(\phi(y)) = \sigma(z) = y'$ and $\sigma\phi(x) = \sigma(\phi(x)) = \sigma(x) = x$, and so $\sigma\phi$ satisfies our requirement.

The corollary follows. ∎

3 Panconnectivity

In this section, we investigate the panconnectivity of FQ_n. The proof of the main theorem in this section is strongly dependent on the following lemmas.

Lemma 3.1 [6] If $n \geq 2$, then Q_n is bipanconnected, that is, for any two vertices x and y in Q_n there exists an xy-path of length l with $H(x,y) \leq l \leq 2^n - 1$ such that l and $H(x,y)$ have the same parity.

Lemma 3.2 [13] FQ_n is a bipartite graph if and only if n is odd. Moreover, if n is even, then the length of the shortest odd cycle in FQ_n is $n+1$.

Theorem 3.3 For any two distinct vertices x and y in FQ_n with distance d, FQ_n contains an xy-path of length l with $h \leq l \leq 2^n - 1$ such that l and h have the same parity, where $h \in \{d, n+1-d\}$.

Proof If $n = 1$, the theorem is true clearly since $FQ_1 = K_2$. Assume $n \geq 2$ below. Without loss of generality, we may assume $x = 0u, y = 1v$ since $d \geq 1$ and FQ_n is vertex-transitive by Corollary 2.2. We first deduce two conclusions from Lemma 3.2 and Theorem 2.3.

(a) By Lemma 3.1, Q_n contains an xy-path P of length l with $H(x,y) \leq l \leq 2^n - 1$ such that l and $H(x,y)$ have the same parity. Since Q_n is a spanning subgraph of FQ_n, P is an xy-path of length l in FQ_n.

(b) Consider the vertex $z = 1\bar{v}$. By Lemma 3.1, Q_n contains an xz-path R of length l' with $H(x,z) \leq l' \leq 2^n - 1$ such that l' and $H(x,z)$ have the same parity. Since $H(x,z) = n+1-H(x,y)$ by (1), l' and $n+1-H(x,y)$ have the same parity. Let $\sigma \in \mathrm{Aut}\,(FQ_n)$ defined in (2). Then $P' = \sigma(R)$ is an xy-path of length l' with $n+1-H(x,y) \leq l' \leq 2^n - 1$ such that l' and $n+1-H(x,y)$ have the same parity.

To prove the theorem, it is sufficient to check that $H(x,y) = d$ or $n+1-d$. In fact, it is clear that if $H(x,y) \leq \lceil \frac{n}{2} \rceil$ then $d = H(x,y)$; if $H(x,y) > \lceil \frac{n}{2} \rceil$ then $H(x,y) = n-d+1$. The theorem is proved. ∎

Corollary 3.4 If n is odd, then FQ_n is bipanconncted.

Proof If n is odd, then FQ_n is a bipartite graph by Lemma 3.2. Let x and y be any two vertices in FQ_n with distance d. Since n is odd, the condition that l and $n+1-d$ have the same parity implies that l and d have the same parity. Note that $d \leq n+1-d$ since $d \leq \lceil \frac{n}{2} \rceil$. By Theorem 3.3, FQ_n contains an xy-path of length l with $d \leq l \leq 2^n - 1$ such that l and d have the same parity, and so FQ_n is bipanconncted. ∎

Corollary 3.5 If n is even then for any two different vertices x and y with $d_{FQ_n}(x,y) = d$ in FQ_n, there is an xy-path of length l for each l satisfying $n-d+1 \leq l \leq 2^n - 1$ and there is also an xy-path of length l' for each l' satisfying $d \leq l' \leq n-d$ such that l' and d have the same parity; there is no xy-path of other length.

Proof If n is even, then d and $n-d+1$ have different parity. Thus, for any integer l, either l and d have the same parity, or l and $n-d+1$

have the same parity. Since $d \leq \frac{n}{2}$, $d < n-d+1$. By Theorem 3.3, there is an xy-path of length l with $n-d+1 \leq l \leq 2^n-1$ in FQ_n.

Since the length of the shortest odd cycle in FQ_n is $n+1$ by Lemma 3.2, FQ_n contains no xy-path of length l with $d < l \leq n-d$ if l and d have different parity. In other words, the length l of the second shortest path between x and y with distance d is certainly $n-d+1$ if l and d have different parity. It follows from Theorem 3.3 that there is an xy-path of length l' with $d \leq l' \leq n-d$ provided l' and d have the same parity.

The corollary is proved. ∎

A graph is called to be hamiltonian connected if there is a hamiltonian path between any two vertices. It is easy to see that any bipartite graph with at least three vertices is not hamiltonian connected. For this reason, Simmons [9] introduces the concept of hamiltonian laceable for hamiltonian bipartite graphs. A hamiltonian bipartite graph is hamiltonian laceable if there is a hamiltonian path between any two vertices in different bipartite sets. It is clear that if a bipartite graph is bipanconnected then it is certainly hamiltonian laceable. It follows from Corollary 3.4 and Corollary 3.5 that the following result is true clearly.

Corollary 3.6 FQ_n is hamiltonian laceable if n is odd, and hamiltonian connected if n is even. ∎

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