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# A rational high-order compact ADI method for unsteady convection-diffusion equations 

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#### Abstract

Based on a fourth-order compact difference formula for the spatial discretization, which is currently proposed for the one-dimensional (1D) steady convection-diffusion problem, and the Crank-Nicolson scheme for the time discretization, a rational high-order compact alternating direction implicit (ADI) method is developed for solving two-dimensional (2D) unsteady convection-diffusion problems. The method is unconditionally stable and second-order accurate in time and fourth-order accurate in space. The resulting scheme in each ADI computation step corresponds to a tridiagonal matrix equation which can be solved by the application of the 1D tridiagonal Thomas algorithm with a considerable saving in computing time. Three examples supporting our theoretical analysis are numerically solved. The present method not only shows higher accuracy and better phase and amplitude error properties than the standard second-order Peaceman-Rachford ADI method in Peaceman and Rachford (1959) [4], the fourth-order ADI method of Karaa and Zhang (2004) [5] and the fourth-order ADI method of Tian and Ge (2007) [23], but also proves more effective than the fourth-order Padé ADI method of You (2006) [6], in the aspect of computational cost. The method proposed for the diffusion-convection problems is easy to implement and can also be used to solve pure diffusion or pure convection problems.


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## 1. Introduction

This paper is devoted to the numerical computation the 2 D unsteady convection-diffusion equation
$\frac{\partial u}{\partial t}-a \frac{\partial^{2} u}{\partial x^{2}}-b \frac{\partial^{2} u}{\partial y^{2}}+p \frac{\partial u}{\partial x}+q \frac{\partial u}{\partial y}=0, \quad(x, y, t) \in \Omega \times(0, T]$
for unknown function $u(x, y, t)$ in a rectangular domain $\Omega$ with the Dirichlet boundary condition
$u(x, y, t)=g(x, y, t), \quad(x, y, t) \in \partial \Omega \times(0, T]$
and the initial condition
$u(x, y, 0)=\varphi(x, y), \quad(x, y) \in \Omega$
where $\partial \Omega$ is the boundary of $\Omega,(0, T]$ is the time interval, $\varphi$ and $g$ are given sufficiently smooth functions. In Eq. (1), the constant coefficients $a$ and $b$ are nonnegative diffusion coefficients and $p$ and $q$ are convective velocities in the $x$ - and $y$-directions, respectively.

[^0]Eq. (1), which may be regarded as a simplified version of the Navier-Stokes equations and plays an important role in computational fluid dynamics (CFD), can describe the convection and diffusion of various physical quantities, e.g., mass, momentum and energy, etc. Eq. (1) is also encountered in many fields of science and engineering, such as heat transfer, fluid flows, the groundwater pollution problems and chemical separation processes [1-3,14,15]. Therefore, developing accurate and stable difference methods for approximating the convection-diffusion equations is of vital importance.

A great deal of effort has been devoted to the development of finite difference (FD) methods for the numerical approximation of transport problems involving convective and diffusive processes [4-9,11,12,17-23]. It is well known that the standard alternating direction implicit (ADI) method developed by Peaceman and Rachford [4] has been popular due to its computational costeffectiveness. However, the Peaceman and Rachford ADI (PR-ADI) scheme, which is second-order in space and often produces significant dissipation and phase errors [5,6], is not ideally suited to deal with the spatial discretization of convection-dominated transport problems. To obtain accurate solution, it is desirable to use higher order spatial methods.

In the last few years, higher order compact (HOC) schemes, which feature high-order accuracy and smaller stencils, have been utilized for spatial approximations. For 2D unsteady convectiondiffusion problems, Hirsh [7] and Ciment et al. [8] have discussed
the compact FD schemes which are spatially fourth order and temporally second order and conditionally stable. In [24], Noye and Tan developed a third-order nine-point HOC implicit scheme for the 2D unsteady problem with constant coefficients. The scheme is spatially third-order accurate and temporally second-order accurate, and has a large stability region. Based on the time-splitting difference techniques, Dehghan [16] developed and discussed several different computational LOD procedures for the 2D transport equation. The LOD procedure is simple to implement and economical to use. In [13], based on the modified equivalent partial differential equation as described by Warming and Hyett [30], Dehghan also developed several numerical techniques for the 3D unsteady convection-diffusion equation. The proposed numerical schemes solved their model quite satisfactory. Based on Spotz and Carey's work [25], Kalita et al. [19] derived an implicit HOC scheme for the 2D unsteady problem with variable convection coefficients. In [5], Karra and Zhang developed a high-order compact ADI (HOC-ADI) method for the solution of 2D unsteady convection diffusion problems. The method, in which the high-order accuracy of the HOC scheme and computational efficiency of the ADI approach were combined, is unconditionally stable and spatially order 4 and temporally order 2 . Based on an exponential fourth-order compact difference formula for the spatial discretization and the CrankNicolson scheme for the time discretization, Tian and Ge [23] proposed a fourth-order compact ADI (EHOC-ADI) method for 2D unsteady convection-diffusion problems. The method is unconditionally stable and of second-order in time and of fourth-order in space. Recently, You [6] proposed a high-order Padé ADI (PDE-ADI) method for unsteady convection-diffusion equations. The method not only is second order accurate in temporal and fourth order accurate in spatial and unconditionally stable, but also shows higher accuracy and better phase and amplitude error characteristics than the standard PR-ADI method in [4] and the HOC-ADI method in [5]. However, a disadvantage of the PDE-ADI scheme is the higher computational cost due to the increased number of factorizations of the governing equation.

The main advantage of the ADI methods is their high efficiency for solving parabolic and hyperbolic initial-boundary value problems [4,5,10,12,23,26] and elliptic boundary value problems [22]. The efficiency of the ADI methods, as was shown in [26], is based on reducing problems in several space variables to the collections of one-dimensional problems and only requiring to solve tridiagonal matrices. In addition, many full implicit and semi-implicit algorithms for the solution of the Navier-Stokes equations also utilize the computational efficiency of ADI-type technique (e.g., [6]). Therefore, the solutions of the Navier-Stokes equations will be one of the most promising applications of the ADI methods with highorder spatial accuracy [6].

In this paper, we first develop a new HOC scheme for the 1D steady convection-diffusion equation. The scheme is measured using wave number analysis and shows its superiority over the existing HOC schemes [5,23]. A new HOC scheme-based ADI method is then proposed for solving 2 D unsteady convection-diffusion equations. The derivation of the present HOC-ADI method is based on the proposed fourth-order compact difference operator for the spatial approximation and an exponential difference operator for the temporal approximation.

The rest of this paper is organized into three sections. In Section 2, we outline the HOC-ADI scheme and issues related to it. And, the linear of Fourier (or von Neumann) stability of the proposed HOC-ADI method is also analyzed. We present some numerical results and comparisons in Section 3. Concluding remarks are included in Section 4.

## 2. The new HOC-ADI method

### 2.1. Development of the RHOC scheme for steady 1D case

To describe the new HOC-ADI method, we will start from the elementary 1D steady convection diffusion equation
$-a u_{x x}+p u_{x}=f$
where $a$ is the nonnegative constant conductivity, $p$ the constant convective velocity, and $f$ a sufficiently smooth function of $x$. Suppose that the starting second-order finite difference scheme for Eq. (4) with constant convection coefficient at a grid point $x_{i}$ is
$-a \delta_{x}^{2} u_{i}+p \delta_{x} u_{i}=f_{i}$
where $\delta_{x}^{2}$ and $\delta_{x}$ are the second-order central difference operators for the second and first derivatives. The truncation error of Eq. (5) is given in

$$
\begin{align*}
0 & =-a u_{x x}+p u_{x}-f \\
& =-a \delta_{x}^{2} u_{i}+p \delta_{x} u_{i}-f+\left(a \delta_{x}^{2} u_{i}-p \delta_{x} u_{i}-a u_{x x}+p u_{x}\right) \\
& =-a \delta_{x}^{2} u_{i}+p \delta_{x} u_{i}-f+\frac{a h_{x}^{2}}{12} u_{x x x x}-\frac{p h_{x}^{2}}{6} u_{x x x}+O\left(h_{x}^{4}\right) \tag{6}
\end{align*}
$$

where $h_{x}$ is the mesh size. To obtain the fourth-order compact scheme to (4), we need to apply second order approximations to each of the derivative terms in (6). To this end, we differentiate both sides of (4) with respect to $x$ and obtain
$u_{x x x}=\frac{p}{a} u_{x x}-\frac{1}{a} f_{x}$
Differentiating both sides of (7), we get
$u_{x x x x}=\frac{p}{a} u_{x x x}-\frac{1}{a} f_{x x}$
Combining (7) and (8) yields
$u_{x x x x}=\left(\frac{p}{a}\right)^{2} u_{x x}-\frac{p}{a^{2}} f_{X}-\frac{1}{a} f_{x x}$
Substituting (7) and (9) into (6), and rearranging it, we have

$$
\begin{align*}
& -a \delta_{x}^{2} u_{i}+p \delta_{x} u_{i}-\frac{a P e_{x}^{2}}{12} u_{x x} \\
& \quad=f-\frac{P e_{x} h_{x}}{12} f_{x}+\frac{h_{x}^{2}}{12} f_{x x}+O\left(h_{x}^{4}\right) \tag{10}
\end{align*}
$$

where $P e_{x}$ is the cell Reynolds number in the $x$-direction and $P e_{x}=$ $p h_{x} / a$.

Using the second-order central difference formulas to approximate $u_{x x}, f_{x}$ and $f_{x x}$ in Eq. (10), we get
$-a\left(1+\frac{P e_{x}^{2}}{12}\right) \delta_{x}^{2} u_{i}+p \delta_{x} u_{i}=\left(1-\frac{P e_{x} h_{x}}{12} \delta_{x}+\frac{h_{x}^{2}}{12} \delta_{x}^{2}\right) f_{i}$
Eq. (11), which is called as the HOC scheme for Eq. (4), has been derived by other authors [5] using the same approach. Actually, to derive the HOC-FD approximations for convection and diffusion problems, this technique has been outlined by several authors [21,25].

It is easily found that the HOC scheme (11) is derived by replacing the fourth- and third-order derivatives in (6) with the second-order derivative. This shows that the HOC scheme (11) is a dissipation-dominant scheme. However, as was shown in [6], the HOC scheme (11) produces significantly enhanced dissipation at high cell Reynolds numbers and becomes singular for pure convective problems $(a=0)$. Excessive dissipation degrades the numerical
resolution. Fortunately, we have noticed that the converted term of the truncation error $\frac{a h_{x}^{2}}{12} u_{x x x x}-\frac{p h_{x}^{2}}{6} u_{x x x}$ leads to the flaws associated with the HOC scheme (11), in which both the fourth- and third-order derivatives are replaced with the second-order derivative.

In order to circumvent the flaws associated with the HOC scheme (11), we only need to replace the fourth-order derivative in the truncation error with the first-order derivative. Applying (4) to Eq. (9) gets
$u_{x x x x}=\left(\frac{p}{a}\right)^{3} u_{x}-\frac{p^{2}}{a^{3}} f-\frac{p}{a^{2}} f_{x}-\frac{1}{a} f_{x x}$
New, substituting (7) and (12) into (6), and rearranging it, we obtain

$$
\begin{align*}
& -a \delta_{x}^{2} u_{i}+p \delta_{x} u_{i}-\frac{a P e_{x}^{2}}{6} u_{x x}+\frac{p P e_{x}^{2}}{12} u_{x} \\
& \quad=\left(1+\frac{P e_{x}^{2}}{12}\right) f-\frac{P e_{\chi} h_{x}}{12} f_{x}+\frac{h_{x}^{2}}{12} f_{x x}+O\left(h_{x}^{4}\right) \tag{13}
\end{align*}
$$

Using the second-order central difference formulas to approximate $u_{x}$ and $u_{x x}$ in Eq. (13), and retaining the leading truncation errors, we have

$$
\begin{align*}
& -a\left(1-\frac{P e_{x}^{2}}{12}\right) \delta_{x}^{2} u_{i}+p\left(1-\frac{P e_{x}^{2}}{6}\right) \delta_{x} u_{i} \\
& \quad-\frac{a P e_{x}^{2} h_{x}^{2}}{144} u_{x x x x}+\frac{p P e_{x}^{2} h_{x}^{2}}{36} u_{x x x} \\
& \quad=\left(1-\frac{P e_{x}^{2}}{6}\right) f-\frac{P e_{x} h_{x}}{12} f_{x}+\frac{h_{x}^{2}}{12} f_{x x}+O\left(h_{x}^{4}\right) \tag{14}
\end{align*}
$$

Again, substituting (8) and (9) into (14) and approximating $u_{x}$ and $u_{x x}$ by the second-order central difference formulas, and rearranging it, yield

$$
\begin{align*}
& -a\left(1-\frac{P e_{x}^{2}}{12}+\frac{P e_{x}^{4}}{144}\right) \delta_{x}^{2} u_{i}+p\left(1-\frac{P e_{x}^{2}}{6}+\frac{P e_{x}^{4}}{36}\right) \delta_{x} u_{i} \\
& \quad+\frac{a P e_{x}^{4} h_{x}^{2}}{1728} u_{x x x x}-\frac{p P e_{x}^{4} h_{x}^{2}}{216} u_{x x x} \\
& =\left(1-\frac{P e_{x}^{2}}{6}+\frac{P e_{x}^{4}}{36}\right) f-\frac{P e_{x} h_{x}}{12}\left(1-\frac{P e_{x}^{2}}{4}\right) f_{x} \\
& \quad+\frac{h_{x}^{2}}{12}\left(1-\frac{P e_{x}^{2}}{12}\right) f_{x x}+O\left(h_{x}^{4}\right) \tag{15}
\end{align*}
$$

Applying Eq. (4) to (15), we get

$$
\begin{align*}
-a( & \left.1-\frac{P e_{x}^{2}}{12}+\frac{P e_{x}^{4}}{144}\right) \delta_{x}^{2} u_{i}+p\left(1-\frac{P e_{x}^{2}}{6}+\frac{P e_{x}^{4}}{36}\right) \delta_{x} u_{i} \\
& -\frac{7 a P e_{x}^{4} h^{2}}{1728} u_{x x x x} \\
= & \left(1-\frac{P e_{x}^{2}}{6}+\frac{P e_{x}^{4}}{36}\right) f-\frac{P e_{x} h_{x}}{12}\left(1-\frac{P e_{x}^{2}}{4}\right) f_{x} \\
& +\frac{h_{x}^{2}}{12}\left(1-\frac{P e_{x}^{2}}{12}+\frac{P e_{x}^{4}}{18}\right) f_{x x}+O\left(h_{x}^{4}\right) \tag{16}
\end{align*}
$$

Approximating $f_{x}$ and $f_{x x}$ in Eq. (16) by the second-order central difference formulas and neglecting the terms of fourth order, a new fourth-order compact scheme for the steady convective diffusion problem (4) is given by
$\left(-\alpha \delta_{x}^{2}+p \delta_{x}\right) u_{i}=\left(1+\alpha_{1} \delta_{x}+\alpha_{2} \delta_{x}^{2}\right) f_{i}$
in which
$\alpha=a\binom{1-\frac{P e_{x}^{2}}{12}+\frac{P e_{x}^{4}}{144}}{1-\frac{P e_{x}^{2}}{6}+\frac{P e_{x}^{4}}{36}}, \quad \alpha_{1}= \begin{cases}\frac{a-\alpha}{p}, & p \neq 0 \\ 0, & p=0\end{cases}$
$\alpha_{2}= \begin{cases}\frac{a(a-\alpha)}{p^{2}}+\frac{h_{x}^{2}}{6}, & p \neq 0 \\ \frac{h_{x}^{2}}{12}, & p=0\end{cases}$
The truncation error analysis shows that Eq. (17) with (18) is a fourth-order scheme for the convection-diffusion equation (4). The scheme (17) may be named as a high-order compact rational (RHOC) FD scheme; i.e., the influencing coefficients of the FD formulation are connected to the rational functions of the coefficients of the differential operator and mesh size. It is interesting to note that the RHOC scheme (17) for the model equation (4) becomes actually the standard fourth-order Padé scheme for pure convective problems $(a=0)$ or for pure diffusive problems $(p=0)$. In addition, the good resolution properties of the RHOC scheme will also be shown in Section 2.2.

### 2.2. Fourier analysis of the difference error

The classical truncation error analysis does not represent all the characteristics of a numerical discretization scheme. The Fourier analysis of numerical schemes is a useful tool (e.g., Refs. [6,27]) in which the resolution of the finite difference operator with its analytical counterpart can be compared. The Fourier analysis of a scheme, which allows one to assess how different frequency components of a harmonic function in a periodic domain are represented by the scheme, can provide additional information about its resolution properties. Consider the trial function, $u=e^{i k x}$ ( $i=\sqrt{-1}$ ) on a periodic domain. By application of this trial function to the differential equation (4), the exact characteristic is obtained as follows:
$\lambda_{\text {Exact }}=a k^{2}+i p k$
where $k$ is the wave number. Replacing the difference operators in the approximation (16) with the corresponding modified wave numbers, the characteristic function of the RHOC scheme is given by
$\lambda_{\mathrm{RHOC}}=\frac{\alpha k^{\prime \prime}+i p k^{\prime}}{\left(1-\alpha_{2} k^{\prime \prime}\right)+i \alpha_{1} k^{\prime}}$
in which $k^{\prime}=\frac{\sin k h_{x}}{h_{x}}$ and $k^{\prime \prime}=\frac{2-2 \cos k h_{x}}{h_{x}^{2}}$.
For comparison, the Fourier analysis is also performed for the fourth-order Padé (PDE) [6] scheme, the fourth-order compact (HOC) [5] scheme, the standard second-order central difference (CD) scheme and the exponential fourth-order compact (EHOC) [23] scheme and the results are compared with (19) along with the RHOC scheme. The characteristic functions for the HOC, the PDE and the CD schemes, given by [6], are
$\lambda_{\mathrm{HOC}}=\frac{\hat{\alpha} k^{\prime \prime}+i p k^{\prime}}{\left(1-\hat{\alpha}_{2} k^{\prime \prime}\right)+i \hat{\alpha}_{1} k^{\prime}}$
where $\hat{\alpha}=a\left(1+\frac{P e_{x}^{2}}{12}\right), \hat{\alpha}_{1}=\frac{a-\hat{\alpha}}{p}$ and $\hat{\alpha}_{2}=\frac{a(a-\hat{\alpha})}{p^{2}}+\frac{h_{\alpha}^{2}}{6}$.
$\lambda_{\mathrm{PDE}}=a \hat{k}^{\prime \prime}+i p \hat{k}^{\prime}$
where $\hat{k}^{\prime}=\frac{3 \sin k h_{x}}{h_{x}\left(2+\cos k h_{x}\right)}$ and $\hat{k}^{\prime \prime}=\frac{12\left(1-\cos k h_{x}\right)}{h_{x}^{2}\left(5+\cos k h_{x}\right)}$, and
$\lambda_{\mathrm{CD}}=a k^{\prime \prime}+i p k^{\prime}$
respectively. $k^{\prime}$ and $k^{\prime \prime}$ in above expressions are the same as the ones given in (20).

(b)

Fig. 1. The nondimensional real part of $\lambda$ for four numerical schemes at different cell Reynolds numbers: (a) $P e_{x}=0.1$; (b) $P e_{x}=10$; (c) $P e_{x}=100$; (d) $P e_{x}=1000$. Horizontal coordinate represents $k h_{x}$.

The characteristic function of the EHOC scheme in [23] is
$\lambda_{\text {енос }}=\frac{\bar{\alpha} k^{\prime \prime}+i p k^{\prime}}{\left(1-\bar{\alpha}_{2} k^{\prime \prime}\right)+i \bar{\alpha}_{1} k^{\prime}}$
where
$\bar{\alpha}=\left\{\begin{array}{ll}a \frac{P e_{x}}{2} \operatorname{coth}\left(\frac{P e_{x}}{2}\right), & p \neq 0, \\ a, & p=0,\end{array} \quad \bar{\alpha}_{1}= \begin{cases}\frac{a-\alpha}{p}, & p \neq 0 \\ 0, & p=0\end{cases}\right.$
and
$\bar{\alpha}_{2}= \begin{cases}\frac{a(a-\alpha)}{p^{2}}+\frac{h_{x}^{2}}{6}, & p \neq 0 \\ \frac{h_{x}^{2}}{12}, & p=0\end{cases}$
The nondimensional real and imaginary parts $\operatorname{Re}(\lambda) h_{\chi}^{2} / a$ and $\operatorname{Im}(\lambda) h_{\chi} / p$ of $\lambda$ as the functions of $k h_{x}$ are shown in Figs. 1 and 2 with their exact counterparts (19) at four different cell

Reynolds numbers $P e_{x}=0.1,10,100$ and 1000, respectively. For the case of $P e_{x}=0.1$, shown in Fig. 1(a), the HOC, the EHOC, the PDE and the RHOC schemes are almost indistinguishably showing identical dissipation errors. In contrast to the three fourthorder compact schemes, the CD scheme has much larger dissipation error $\left(\operatorname{Re}(\lambda) h_{x}^{2} / a\right)$. Figs. $1(\mathrm{~b})$, (c) and (d) show that when the cell Reynolds number is increased to 10,100 and 1000 , for the convection-dominated cases, the HOC and the EHOC schemes have dramatically increased dissipation errors, while the RHOC scheme shows better resolution characteristic, and the PDE and the CD schemes do not alter their nondissipative properties. The dispersive errors for the five schemes are shown in Fig. 2. At the cell Reynolds number of 0.1 , the imaginary parts $\operatorname{Im}(\lambda) h_{x} / p$ for the HOC, the EHOC and the RHOC schemes are basically the same, while the PDE scheme shows much smaller dispersive error than the HOC, the EHOC and the RHOC schemes at the higher wavenumbers. When the cell Reynolds number is $10, \operatorname{Im}(\lambda) h_{x} / p$ of the HOC

(b)

Fig. 2. Nondimensional imaginary parts of $\lambda$ for four numerical schemes at different cell Reynolds numbers: (a) $P e_{x}=0.1$; (b) $P e_{x}=10$; (c) $P e_{x}=100$; (d) $P e_{x}=1000$. Horizontal coordinate represents $k h_{x}$.
scheme produces overshoot, the EHOC scheme depicts much less dispersive error than the RHOC and the PDE schemes. If the cell Reynolds number is increased to 100 and $1000, \operatorname{Im}(\lambda) h_{x} / p$ of the HOC scheme produces significant overshoot, while the RHOC and the PDE schemes give better resolutions than the other schemes and are almost indistinguishable. Figs. 1(c) and (d) also show that, at $P e_{x}=100$ and 1000 , although $\operatorname{Im}(\lambda) h_{x} / p$ of the EHOC scheme produces overshoot, the EHOC scheme is far better in resolution than the HOC, the CD schemes and is almost the same as the resolution of the RHOC and the PDE schemes.

### 2.3. The RHOC-ADI scheme for the 2D unsteady case

The RHOC scheme (17) discussed in the previous section shows a good wave resolution property. In this section, we will utilize (17) to establish a new high-order ADI scheme for the 2D unsteady convection-diffusion equation (1).

Eq. (17) can be rewritten symbolically as
$\left(1+\alpha_{1} \delta_{x}+\alpha_{2} \delta_{x}^{2}\right)^{-1}\left(-\alpha \delta_{x}^{2}+p \delta_{x}\right) u_{i}=f_{i}$
where the operator $\left(1+\alpha_{1} \delta_{x}+\alpha_{2} \delta_{x}^{2}\right)^{-1}$ has symbolic meaning only.

In order to derive HOC schemes for numerical solution of transport problems involving convective and diffusive processes, the symbolic HOC operator approximation technique has been used by several authors [5,7,20,23,28]. An analogous symbolic RHOC approximation operator can also be given for the variable $y$.

For convenience, several FD operators are defined as follows:
$\begin{array}{ll}L_{x}=1+\alpha_{1} \delta_{x}+\alpha_{2} \delta_{x}^{2}, & A_{x}=-\alpha \delta_{x}^{2}+p \delta_{x} \\ L_{y}=1+\beta_{1} \delta_{y}+\beta_{2} \delta_{y}^{2}, & A_{y}=-\beta \delta_{y}^{2}+q \delta_{y}\end{array}$
in which $\delta_{y}$ and $\delta_{y}^{2}$ are the second-order central difference operators for the first and second derivatives in the $y$-direction, and
$\beta=b\left(\frac{1-\frac{P e_{y}^{2}}{12}+\frac{P e_{y}^{4}}{144}}{1-\frac{P e_{y}^{2}}{6}+\frac{P e_{y}^{4}}{36}}\right), \quad \beta_{1}= \begin{cases}\frac{b-\beta}{q}, & q \neq 0 \\ 0, & q=0\end{cases}$
$\beta_{2}= \begin{cases}\frac{b(b-\beta)}{q^{2}}+\frac{h_{y}^{2}}{6}, & q \neq 0 \\ \frac{h_{y}^{2}}{12}, & q=0\end{cases}$
where $P e_{y}=q h_{y} / b$, while $h_{y}$ is the mesh size in the $y$-direction.
Applying the rational fourth-order FD operators $L_{x}^{-1} A_{x}$ and $L_{y}^{-1} A_{y}$ to the 2D unsteady convection diffusion equation in (1) yields the following rational fourth-order compact approximation:
$\left(\frac{\partial u^{n}}{\partial t}\right)_{i j}=-\left(L_{x}^{-1} A_{x}+L_{y}^{-1} A_{y}\right) u_{i j}^{n}+O\left(h^{4}\right)$
in which $O\left(h^{4}\right)$ represents the $O\left(h_{x}^{4}\right)+O\left(h_{y}^{4}\right)$ term, $i j$ the spatial position of ( $x_{i}, y_{j}$ ) and $u^{n}$ the approximate solution at time level $t^{n}=n \Delta t, n$ the time level, and $\Delta t=t^{n+1}-t^{n}$ the temporal step size.

Eq. (27) is a fourth-order semi-discrete formula for the 2D unsteady convection-diffusion problem (1). This semi-discrete approximation approach has been used in [5,23]. In the following, $u^{n}$ will be written in short for $u_{i j}^{n}$ if there is no confusion about the notations. By the application of the forward Taylor series development, we have

$$
\begin{align*}
u^{n+1} & =\left(1+\Delta t \frac{\partial}{\partial t}+\frac{1}{2!} \Delta t^{2} \frac{\partial^{2}}{\partial t^{2}}+\frac{1}{3!} \Delta t^{3} \frac{\partial^{3}}{\partial t^{3}}+\cdots\right) u^{n} \\
& =\exp \left(\Delta t \frac{\partial}{\partial t}\right) u^{n} \tag{28}
\end{align*}
$$

whose equivalent equation is
$\exp \left(-\frac{\Delta t}{2} \frac{\partial}{\partial t}\right) u^{n+1}=\exp \left(\frac{\Delta t}{2} \frac{\partial}{\partial t}\right) u^{n}$
When applied to the right-hand side of (29) with (27), we obtain a rational fourth-order FD approximation of Eq. (1)

$$
\begin{align*}
& \exp \left(\frac{\Delta t}{2}\left(L_{x}^{-1} A_{x}+L_{y}^{-1} A_{y}\right)\right) u^{n+1} \\
& \quad=\exp \left(-\frac{\Delta t}{2}\left(L_{x}^{-1} A_{x}+L_{y}^{-1} A_{y}\right)\right) u^{n} \tag{30}
\end{align*}
$$

Noting that the commutativity of the difference operators $A_{x}, L_{x}$, $A_{y}$ and $L_{y}$ gives

$$
\begin{align*}
& \exp \left(\frac{\Delta t}{2} L_{x}^{-1} A_{x}\right) \exp \left(\frac{\Delta t}{2} L_{y}^{-1} A_{y}\right) u^{n+1} \\
& \quad=\exp \left(\frac{-\Delta t}{2} L_{x}^{-1} A_{x}\right) \exp \left(\frac{-\Delta t}{2} L_{y}^{-1} A_{y}\right) u^{n} \tag{31}
\end{align*}
$$

By using the Taylor expansions, we have

$$
\begin{align*}
(1+ & \left.\frac{\Delta t}{2} L_{x}^{-1} A_{x}\right)\left(1+\frac{\Delta t}{2} L_{y}^{-1} A_{y}\right) u^{n+1} \\
= & \left(1-\frac{\Delta t}{2} L_{x}^{-1} A_{x}\right)\left(1-\frac{\Delta t}{2} L_{y}^{-1} A_{y}\right) u^{n}+O\left(\Delta t^{3}\right) \\
& +O\left(\Delta t h^{4}\right) \tag{32}
\end{align*}
$$

Applying to both sides of Eq. (32) with the difference operator $L_{x} L_{y}$ and neglecting $O\left(\Delta t^{3}\right)+O\left(\Delta t h^{4}\right)$, we obtain

$$
\begin{align*}
& \left(L_{x}+\frac{\Delta t}{2} A_{x}\right)\left(L_{y}+\frac{\Delta t}{2} A_{y}\right) u^{n+1} \\
& \quad=\left(L_{x}-\frac{\Delta t}{2} A_{x}\right)\left(L_{y}-\frac{\Delta t}{2} A_{y}\right) u^{n} \tag{33}
\end{align*}
$$

The resulting approximation (33) is temporally second order and spatially fourth order. Introducing an intermediate variable $u^{*}$, Eq. (33) can be solved in two steps as
$\left(L_{x}+\frac{\Delta t}{2} A_{x}\right) u^{*}=\left(L_{x}-\frac{\Delta t}{2} A_{x}\right)\left(L_{y}-\frac{\Delta t}{2} A_{y}\right) u^{n}$
$\left(L_{y}+\frac{\Delta t}{2} A_{y}\right) u^{n+1}=u^{*}$
The solutions to the resulting RHOC-ADI schemes (34a) and (34b) can be computed by applying the one-dimensional tridiagonal Thomas algorithm with a considerable saving in computing time. The intermediate variable values of $u^{*}$ at the boundary in first ADI scheme above is explicitly given in terms of the central difference of $g^{n+1}$ with respect to $y$ from Eq. (2):
$u^{*}=\left(L_{y}+\frac{\Delta t}{2} A_{y}\right) g^{n+1}$

### 2.4. Stability analysis

In this section, we study the stability of the FD scheme (33) using the von Neumann method for linear stability analysis. We assume that the numerical solution can be expressed by virtue of a Fourier series, whose typical term is
$u_{i j}^{n}=\eta^{n} \exp \left\{\mathrm{i}\left(k_{x} x_{i}+k_{y} y_{j}\right)\right\}$
where $\mathrm{i}=\sqrt{-1}, \eta^{n}$ is the amplitude at time level $n, x_{i}=i h_{x}$ and $y_{j}=j h_{y}$, and the wavenumbers $k_{x}$ and $k_{y}$ in the $x$ - and $y$ directions, respectively. Exploiting the discrete Fourier mode (36) in both sides of Eq. (33), the amplification factor $G\left(\theta_{x}, \theta_{y}\right)=$ $\eta^{n+1} / \eta^{n}$ is found to be
$\left|G\left(\theta_{x}, \theta_{y}\right)\right|=\left|g\left(\theta_{x}\right)\right|\left|g\left(\theta_{y}\right)\right|$
in which $\theta_{x}=k_{x} h_{x}$ and $\theta_{y}=k_{y} h_{y}$, and $g\left(\theta_{y}\right)$ is given by
$g\left(\theta_{y}\right)=\frac{\left(\gamma_{1}-\gamma_{2}\right)+\mathrm{i}\left(\gamma_{3}-\gamma_{4}\right)}{\left(\gamma_{1}+\gamma_{2}\right)+\mathrm{i}\left(\gamma_{3}+\gamma_{4}\right)}$
with
$\gamma_{1}=1-\frac{4 \beta_{2}}{h_{y}^{2}} \sin ^{2} \frac{\theta_{y}}{2}, \quad \gamma_{2}=\frac{2 \beta \Delta t}{h_{y}^{2}} \sin ^{2} \frac{\theta_{y}}{2}$
$\gamma_{3}=\frac{\beta_{1}}{h_{y}} \sin \theta_{y}, \quad \gamma_{4}=\frac{q \Delta t}{2 h_{y}} \sin \theta_{y}$
and the other similar term $g\left(\theta_{x}\right)$ can be obtained by replacing $y$ with $x$, and $q, \beta, \beta_{1}$ and $\beta_{2}$ with $p, \alpha, \alpha_{1}$ and $\alpha_{2}$, respectively in the above expression.

For stability it is sufficient that $\left|g\left(\theta_{x}\right)\right|^{2} \leqslant 1$ and $\left|g\left(\theta_{y}\right)\right|^{2} \leqslant 1$. It is easy to verify that $\gamma_{1} \gamma_{2}+\gamma_{3} \gamma_{4} \geqslant 0$ as a necessary and sufficient condition for $\left|g\left(\theta_{y}\right)\right|^{2} \leqslant 1$. Simple calculation of $\gamma_{1} \gamma_{2}+\gamma_{3} \gamma_{4}$ gives that

$$
\begin{align*}
& \gamma_{1} \gamma_{2}+\gamma_{3} \gamma_{4} \\
& =\frac{2 \beta \Delta t}{h_{y}^{2}}\left(1-\frac{4 \beta_{2}}{h_{y}^{2}} \sin ^{2} \frac{\theta_{y}}{2}\right) \sin ^{2} \frac{\theta_{y}}{2}+\frac{q \beta_{1} \Delta t}{2 h_{y}^{2}} \sin ^{2} \theta_{y} \\
& = \\
& \frac{2 \beta \Delta t}{h_{y}^{2}}\left(1-\frac{4 \beta_{2}}{h_{y}^{2}} \sin ^{2} \frac{\theta_{y}}{2}\right) \sin ^{2} \frac{\theta_{y}}{2}  \tag{38}\\
& \\
& \quad+\frac{2 q \beta_{1} \Delta t}{h_{y}^{2}}\left(1-\sin ^{2} \frac{\theta_{y}}{2}\right) \sin ^{2} \frac{\theta_{y}}{2}
\end{align*}
$$

From Eq. (26), it is easy to find that $\beta=\beta_{1}=0$ if $b=0$. Substituting $\beta=\beta_{1}=0$ into Eq. (38), we get $\gamma_{1} \gamma_{2}+\gamma_{3} \gamma_{4}=0$ for all $\theta_{y} \in[-\pi, \pi]$.

Suppose now that $b>0$, we will verify that $\gamma_{1} \gamma_{2}+\gamma_{3} \gamma_{4} \geqslant 0$. First assume that $q=0$, then
$\beta=b, \quad \beta_{1}=0, \quad \beta_{2}=\frac{h_{y}^{2}}{12}$
Substituting (39) into (38), we have
$\gamma_{1} \gamma_{2}+\gamma_{3} \gamma_{4}=\frac{2 b \Delta t}{h_{y}^{2}}\left(1-\frac{1}{3} \sin ^{2} \frac{\theta_{y}}{2}\right) \sin ^{2} \frac{\theta_{y}}{2}$
Notice that $b>0$ and $0 \leqslant \sin ^{2} \frac{\theta_{y}}{2} \leqslant 1$, we conclude that $\gamma_{1} \gamma_{2}+$ $\gamma_{3} \gamma_{4} \geqslant 0$ for all $\theta_{y} \in[-\pi, \pi]$. Assuming $q \neq 0$, from Eq. (38), we obtain

$$
\begin{align*}
& \gamma_{1} \gamma_{2}+\gamma_{3} \gamma_{4} \\
& =\frac{2 \Delta t}{h_{y}^{2}}\left[\beta\left(1-\frac{4 \beta_{2}}{h_{y}^{2}}\right) \sin ^{2} \frac{\theta_{y}}{2}+b\left(1-\sin ^{2} \frac{\theta_{y}}{2}\right)\right] \sin ^{2} \frac{\theta_{y}}{2} \\
& = \\
& \frac{2 \Delta t}{h_{y}^{2}}\left\{\beta\left[\frac{1}{3}-\frac{4 b(b-\beta)}{q^{2} h_{y}^{2}}\right] \sin ^{2} \frac{\theta_{y}}{2}+b\left(1-\sin ^{2} \frac{\theta_{y}}{2}\right)\right\}  \tag{41}\\
& \quad \times \sin ^{2} \frac{\theta_{y}}{2}
\end{align*}
$$

Given that $1-\frac{z^{2}}{12}+\frac{z^{4}}{144}=\left(\frac{z^{2}}{12}-\frac{1}{2}\right)^{2}+\frac{3}{4}>0$ and $1-\frac{z^{2}}{6}+\frac{z^{4}}{36}=$ $\left(\frac{z^{2}}{6}-\frac{1}{2}\right)^{2}+\frac{3}{4}>0$ for all real $z$ and $b>0$, we have
$\beta=b\left(\frac{1-\frac{P e_{y}^{2}}{12}+\frac{P e_{y}^{4}}{144}}{1-\frac{P e_{y}^{2}}{6}+\frac{P e_{y}^{4}}{36}}\right)>0$
Note that

$$
\begin{align*}
\frac{1}{3}-\frac{4 b(b-\beta)}{q^{2} h_{y}^{2}} & =\frac{1}{3}-\left(\frac{-\frac{1}{3}+\frac{P e_{y}^{2}}{12}}{1-\frac{P e_{y}^{2}}{6}+\frac{P e_{y}^{4}}{36}}\right) \\
& =\frac{\frac{1}{3}\left(\frac{P e_{y}^{2}}{6}-\frac{5}{4}\right)^{2}+\frac{7}{48}}{\left(\frac{P e_{y}^{2}}{6}-\frac{1}{2}\right)^{2}+\frac{3}{4}} \tag{43}
\end{align*}
$$

and hence
$\frac{1}{3}-\frac{4 b(b-\beta)}{q^{2} h_{y}^{2}}>0$
Since $0 \leqslant \sin ^{2} \frac{\theta_{y}}{2} \leqslant 1$, the following inequality holds by means of (42) and (44):
$\beta\left[\frac{1}{3}-\frac{4 b(b-\beta)}{q^{2} h_{y}^{2}}\right] \sin ^{2} \frac{\theta_{y}}{2}+b\left(1-\sin ^{2} \frac{\theta_{y}}{2}\right)>0$
Combining (41) and (45) yields

$$
\begin{align*}
& \gamma_{1} \gamma_{2}+\gamma_{3} \gamma_{4} \\
& =\frac{2 \Delta t}{h_{y}^{2}}\left\{\beta\left[\frac{1}{3}-\frac{4 b(b-\beta)}{q^{2} h_{y}^{2}}\right] \sin ^{2} \frac{\theta_{y}}{2}+b\left(1-\sin ^{2} \frac{\theta_{y}}{2}\right)\right\} \\
& \quad \times \sin ^{2} \frac{\theta_{y}}{2} \geqslant 0 \tag{46}
\end{align*}
$$

which follows that $\left|g\left(\theta_{y}\right)\right|^{2} \leqslant 1$ for all $\theta_{y} \in[-\pi, \pi]$. Similarly, we can find that $\left|g\left(\theta_{x}\right)\right|^{2} \leqslant 1$. Thus the presented method, when applied to the 2D unsteady linear convection diffusion equation, is unconditionally stable.

Table 1
$L^{2}$ norm errors and the convergence rate with $\Delta t=h^{2}, T=0.25$.

| Grid | PR-ADI method |  |  | RHOC-ADI method |  |  |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- |
|  | $L^{2}$ norm error |  | Rate |  | $L^{2}$ norm error | Rate |
| $11 \times 11$ | $1.42176 \times 10^{-4}$ | - |  | $1.45398 \times 10^{-5}$ | - |  |
| $21 \times 21$ | $3.53994 \times 10^{-5}$ | 2.006 |  | $8.47123 \times 10^{-7}$ | 4.101 |  |
| $41 \times 41$ | $8.94131 \times 10^{-6}$ | 1.985 |  | $5.26653 \times 10^{-8}$ | 4.008 |  |

Table 2
$L^{2}$ norm errors at $h_{x}=h_{y}=0.05, T=0.25$ with different time steps.

| $\Delta t$ | PR-ADI method |  |  | RHOC-ADI method |  |  |
| :--- | :--- | :--- | :--- | :--- | :---: | :---: |
|  | $L^{2}$ norm error |  | $L^{2}$ norm error | Rate |  |  |
| 0.005 | $3.14549 \times 10^{-5}$ |  | $3.38630 \times 10^{-6}$ | - |  |  |
| 0.0025 | $3.40370 \times 10^{-5}$ |  | $8.14478 \times 10^{-7}$ | 2.056 |  |  |
| 0.00125 | $3.46825 \times 10^{-5}$ |  | $1.71440 \times 10^{-7}$ | 2.248 |  |  |

## 3. Numerical experiments

In this section, we present the numerical results of the proposed rational higher order compact ADI (RHOC-ADI) method on three test problems possessing exact solution. To illustrate the validity and effectiveness, we compare with the numerical results of some other available methods involving the Karaa and Zhang ADI (HOC-ADI) scheme [5], the Tian and Ge ADI (EHOC-ADI) scheme [23], the Peaceman-Rachford ADI (PR-ADI) scheme [4] and the Padé ADI (PDE-ADI) scheme [6]. The ADI methods used were performed by repeatedly solving a series of triangular linear systems. We conduct our computations using double precision arithmetic on a SONY PCG-V505MCP machine.

### 3.1. Problem 1

Consider Eq. (1) with the coefficients $a=b=1$ and $p=q=0$ in the unit square domain $0 \leqslant x, y \leqslant 1$. The equation is a pure diffusion equation, whose analytical solution is given by
$u(x, y, t)=e^{-2 \pi^{2} t} \sin (\pi x) \sin (\pi y)$
The boundary and initial conditions can be taken easily from (47). This test problem was used in $[5,23]$.

The numerical results obtained for Problem 1 using the RHOCADI scheme and the PR-ADI scheme [4] under a uniform grids ( $h=h_{x}=h_{y}$ ) with different mesh sizes and their accuracies compared under the $L^{2}$ norm error of the numerical solution with respect to the analytical solution are presented in Tables 1 and 2. We see that the RHOC-ADI method proposed in this paper has more accurate results in comparison with the PR-ADI method. In Table $1, \Delta t=h^{2}$ and the final time $T=0.25$ are chosen for the verifications of spatial fourth-order accuracy and temporal secondorder accuracy. The use of the $\ln _{2}$ (err1/err2) estimates the rate of convergence, where err 1 and err2 are $L^{2}$ norm errors with the grid sizes $h$ and $h / 2$, respectively. These values are approximately 4 and 2 for the RHOC-ADI method and the PR-ADI method respectively. $L^{2}$ norm errors at $h=1 / 20$ and $t=0.25$ with various time steps are depicted in Table 2 for different schemes. Table 2 shows that the results of the RHOC-ADI method become more and more accurate with the reduction in time step, while the ones of the PR-ADI method are almost invariable. Also Tables 1 and 2 show the superiority of the RHOC-ADI scheme over the PR-ADI scheme. It should be pointed out that, for pure diffusion ( $p=q=0$ ) problems, the HOC-ADI scheme [5], the EHOC-ADI scheme [23] and the RHOCADI scheme are the same.

(b)
(d)

Fig. 3. Contour lines of the pulse in the sub-region $1.2 \leqslant x, y \leqslant 1.8$ at the final time $T=1.0$ : (a) exact and the present ADI, (b) exact and the PDE ADI, (c) exact and the EHOC ADI, (d) exact and the HOC ADI, (e) exact and the PR ADI, and (f) exact [ $\mathrm{Pe}=2.0, \Delta t=2.5 \times 10^{-3}$ ]. Dash-dot contour lines in (a)-(f) correspond to exact solution.

### 3.2. Problem 2

Consider Eq. (1) in the square domain $0 \leqslant x, y \leqslant 2$, which is a special problem with an analytical solution given, as in [9], by
$u(x, y, t)$

$$
\begin{equation*}
=\frac{1}{4 t+1} \exp \left[-\frac{(x-p t-0.5)^{2}}{a(4 t+1)}-\frac{(y-q t-0.5)^{2}}{b(4 t+1)}\right] \tag{48}
\end{equation*}
$$

The boundary and initial conditions can be taken easily from (48). In this study, four cell Reynolds numbers $P e=2, P e=20, P e=$ 200 and $P e=2000$ corresponding to the convective velocities $p=$ $q=1, p=q=10, p=q=100$ and $p=q=1000$, respectively, are considered. The values $a=b=0.01$ of viscosity coefficient are kept unchanged. We choose times step sizes of $\Delta t=2.5 \times 10^{-3}$, $2.5 \times 10^{-4}, 2.5 \times 10^{-5}$ and $2.5 \times 10^{-6}$ for $\mathrm{Pe}=2, \mathrm{Pe}=20, \mathrm{Pe}=200$


Fig. 4. Contour lines of the pulse in the sub-region $1.2 \leqslant x, y \leqslant 1.8$ at the final time $T=0.1$ : (a) exact and the present ADI, (b) exact and the PDE ADI, (c) exact and the EHOC ADI, (d) exact and the HOC ADI, (e) exact and the PR ADI, and (f) exact $\left[\mathrm{Pe}=20, \Delta t=2.5 \times 10^{-4}\right]$. Dash-dot contour lines in (a)-(f) correspond to exact solution.
and $P e=2000$, respectively. The space step sizes of $h_{x}=h_{y}=0.02$ are used to compare the accuracy of the numerical solution.

Figs. 3-6 contain contour curves for the analytical and computed pulses in the sub-region $1.2 \leqslant x, y \leqslant 1.8$ for each test carried out. For $P e=2$, the solutions obtained from the RHOC-ADI scheme (Fig. 3(a)), the PDE-ADI scheme (Fig. 3(b)) and the EHOCADI scheme (Fig. 3(c)) as well as the HOC-ADI scheme (Fig. 3(d)) capture very well the moving pulse, yielding pulses centered at
$(1.5,1.5)$ and almost indistinguishable from the exact one. However, the PR-ADI scheme produces a pulse distorted in both the $x$ and $y$-directions (Fig. 3(e)). As is observed in Figs. 1(a) and 2(a), this is due to the fact the second-order error terms of method is related to the wave numbers in both directions. In the high cell Reynolds number case ( $\mathrm{Pe}=20,200$ and 2000), the superiority of the RHOC-ADI and the PDE-ADI [6] schemes are more clearly exhibited. The present RHOC-ADI and the PDE-ADI [6] schemes

(b)
(d)


Fig. 5. Contour lines of the pulse in the sub-region $1.2 \leqslant x, y \leqslant 1.8$ at the final time $T=0.01$ : (a) exact and the present ADI, (b) exact and the PDE ADI, (c) exact and the EHOC ADI, ( d ) exact and the HOC ADI, (e) exact and the PR ADI, and (f) exact [ $\mathrm{Pe}=200, \Delta t=2.5 \times 10^{-5}$ ]. Dash-dot contour lines in (a)-(f) correspond to exact solution.
produce the solutions in good agreement with the analytical solution in terms of amplitude and phase (see Figs. 4-6(a), (b)). However, noticeable dissipated solutions in the HOC-ADI and the PR-ADI schemes, which are also highly distorted and oscillations are clearly observed from Figs. 4-6(d), (e). In particular, as pointed out in [6], the enhanced numerical dissipation makes the HOCADI scheme unattractive for direct numerical simulations (DNS) or large eddy simulations (LES) of turbulent flows. From Figs. 5(d), (e)
and 6(d), (e), we note that the solutions given with the PR-ADI and the HOC-ADI schemes produce the distortions and oscillations in opposite-direction. This feature can be explained by the characteristics of the schemes used in spatial directions (see Figs. 1(c), (d) and 2(c), (d)).

Figs. 4(c), 5(c) and 6(c) give the contour curves computed using the EHOC-ADI scheme [23] at $P e=20, P e=200$ and $P e=2000$, respectively. As expected, the EHOC-ADI solution is nonoscillatory

(d)
(f)

Fig. 6. Contour lines of the pulse in the sub-region $1.2 \leqslant x, y \leqslant 1.8$ at the final time $T=0.001$ : (a) exact and the present ADI, (b) exact and the PDE ADI, (c) exact and the EHOC ADI, ( d ) exact and the HOC ADI, (e) exact and the PR ADI, and (f) exact $\left[\mathrm{Pe}=2000, \Delta t=2.5 \times 10^{-6}\right]$. Dash-dot contour lines in (a)-(f) correspond to exact solution.
for all the chosen Pe , but dissipated solutions at $(1.5,1.5)$ surroundings may been clearly observed from Figs. 4(c), 5(c) and 6(c).

In Table 3, the $L^{\infty}$ norms errors, the $L^{2}$ norms errors and the CPU time used for $\mathrm{Pe}=20, \mathrm{Pe}=200$ and $\mathrm{Pe}=2000$, using the PRADI scheme [4], the HOC-ADI scheme [5], the EHOC-ADI scheme [23], the PDE-ADI scheme [6] and the RHOC-ADI scheme respectively, are given. The errors of the PDE-ADI scheme and the RHOC-

ADI method are almost identical and are distinctly lower than those of other ADI methods as can be seen from Table 3. We also notice that the RHOC-ADI, the HOC-ADI, the EHOC-ADI and the PR-ADI methods exhibit less CPU time than that of the PDE-ADI method from Table 3. The execution CPU time of the RHOC-ADI method is more than 2 times shorter than that of the PDE-ADI method. This clearly shows that the RHOC-ADI method is the most effective in view of accuracy and time consumption.

### 3.3. Problem 3

Consider a pure convective equation ( $p=q=1$ and $a=b=0$ ) with the periodic boundary condition in the square domain $0 \leqslant x$, $y \leqslant 2$. The initial condition is given by
$u(x, y)=\sin (\pi(x+y))$
This problem, which was used as a test one in [29], is a periodic flow of sin-surface. Computations, using the RHOC-ADI scheme and the PR-ADI scheme [4], are carried out until the final time $T=1.0$ on uniform grids of sizes $41 \times 41$ with a time step $\Delta t=0.01$. The evolution of solution under uniform grids $h=h_{x}=h_{y}=0.05$, is

Table 3
Errors and CPU times of four ADI methods with $h_{x}=h_{y}=0.02$, Problem 2.

| Method | $L^{\infty}$ norm error | $L^{2}$ norm error | CPU time (s) |
| :--- | :---: | :---: | :--- |
| (a) $t=0.1$ for $\mathrm{Pe}=20$ with $\Delta t=2.5 \times 10^{-4}$ |  |  |  |
| PR-ADI [4] | $1.596 \times 10^{-1}$ | $1.108 \times 10^{-2}$ | 2.00 |
| HOC-ADI [5] | $5.998 \times 10^{-2}$ | $3.048 \times 10^{-3}$ | 2.11 |
| EHOC-ADI [23] | $2.491 \times 10^{-2}$ | $1.200 \times 10^{-3}$ | 2.13 |
| PDE-ADI [6] | $2.936 \times 10^{-3}$ | $1.751 \times 10^{-4}$ | 4.47 |
| RHOC-ADI | $3.128 \times 10^{-3}$ | $1.748 \times 10^{-4}$ | 2.12 |
| (b) $t=0.01$ for $P e=200$ with $\Delta t=2.5 \times 10^{-5}$ |  |  |  |
| PR-ADI [4] | $2.827 \times 10^{-1}$ |  |  |
| HOC-ADI [5] | $1.691 \times 10^{-1}$ | $9.898 \times 10^{-2}$ | 2.00 |
| EHOC-ADI [23] | $5.802 \times 10^{-2}$ | $2.456 \times 10^{-3}$ | 2.11 |
| PDE-ADI [6] | $7.514 \times 10^{-3}$ | $3.752 \times 10^{-4}$ | 2.13 |
| RHOC-ADI | $7.462 \times 10^{-3}$ | $3.751 \times 10^{-4}$ | 4.47 |
|  |  | 2.12 |  |
| (c) $t=0.001$ for $P e=2000$ with $\Delta t=2.5 \times 10^{-6}$ |  |  |  |
| PR-ADI [4] | $3.017 \times 10^{-1}$ | $2.018 \times 10^{-2}$ | 2.00 |
| HOC-ADI [5] | $1.884 \times 10^{-1}$ | $1.094 \times 10^{-2}$ | 2.11 |
| EHOC-ADI [23] | $6.377 \times 10^{-2}$ | $2.662 \times 10^{-3}$ | 2.13 |
| PDE-ADI [6] | $8.400 \times 10^{-3}$ | $4.114 \times 10^{-4}$ | 4.47 |
| RHOC-ADI | $8.394 \times 10^{-3}$ | $4.113 \times 10^{-4}$ | 2.12 |




Table 5
Errors and the convergence rate at $h_{x}=h_{y}=0.05, T=0.5$ with different time steps using RHOC-ADI method, Problem 3.

| $\Delta t$ | $L_{\infty}$ norm error | Rate | $L_{2}$ norm error | Rate |
| :--- | :--- | :--- | :--- | :--- |
| 0.08 | $6.02674 \times 10^{-2}$ | - | $1.17485 \times 10^{-2}$ | - |
| 0.04 | $1.52228 \times 10^{-2}$ | 1.983 | $2.97219 \times 10^{-3}$ | 1.985 |
| 0.02 | $3.98562 \times 10^{-3}$ | 1.933 | $7.85990 \times 10^{-4}$ | 1.919 |
| 0.01 | $1.02742 \times 10^{-3}$ | 1.956 | $2.02641 \times 10^{-4}$ | 1.956 |

Errors and the convergence rate with $\Delta t=h^{2}, T=1$, using RHOC-ADI method, Problem 3.

| Grid | $L^{\infty}$ norm error | Rate | $L^{2}$ norm error | Rate |
| :--- | :--- | :--- | :--- | :--- |
| $11 \times 11$ | $4.08598 \times 10^{-2}$ | - | $9.55999 \times 10^{-3}$ | - |
| $21 \times 21$ | $3.12399 \times 10^{-3}$ | 3.709 | $6.67135 \times 10^{-4}$ | 3.841 |
| $41 \times 41$ | $2.07656 \times 10^{-4}$ | 3.912 | $4.32371 \times 10^{-5}$ | 3.948 |
| $81 \times 81$ | $1.32346 \times 10^{-5}$ | 3.972 | $2.74398 \times 10^{-6}$ | 3.978 |

(b)
shown in Figs. 7 and 8. Note that the approximation solution with the RHOC-ADI method is more accurate than that with the PR-ADI method. It is shown that, for pure convective problems, the RHOCADI scheme can resolve accurately the evolution of solution, while the PR-ADI scheme gives only very poor results.

In Table 4, we choose $\Delta t=h^{2}$ and the final time $T=1$ for the verifications of fourth-order accuracy in space and second-order accuracy in time. The rate of convergence is estimated by using the $\ln _{2}$ (err1/err2), where err1 and err2 denote $L^{\infty}$ and $L^{2}$ norm errors with the grid sizes $h$ and $h / 2$, respectively. These values are approximately 4 for the RHOC-ADI method in space. Table 5 depicts, at $h=1 / 40$ and the final time $T=0.5, L^{\infty}$ norm errors and $L^{2}$ norm errors with various time steps for the RHOC-ADI scheme. The results of Table 5 clearly indicate that the RHOC-ADI method achieve the expected, second-order accuracy in time. It is worth

Table 4
$1.0272 \times 10^{-3}$

(d)

Fig. 7. Periodic flow of the sin-surface. Solution, obtained using the RHOC-ADI method at different times: (a) initial condition; (b) $t=0.4$; (c) $t=0.8$; and (d) $t=1.0$.


Fig. 8. Periodic flow of the sin-surface. Solution, obtained using the PR-ADI method at different times: (a) initial condition; (b) $t=0.4$; (c) $t=0.8$; and (d) $t=1.0$.
noticing that, for the pure convection ( $a=0$ ) problems, the HOCADI scheme cannot be used. In this case, as was shown in [6], it becomes singular.

## 4. Conclusion and remarks

In the present article, a rational high-order compact alternating direction implicit (RHOC-ADI) method has been described for the numerical solution of 2D unsteady convection-diffusion problems. The method is temporally second order and spatially fourth order and only involves 3-point stencil for each 1D operator which allows a considerable saving in computing time. It is shown through a discrete Fourier analysis that the proposed RHOC-ADI scheme is unconditionally stable. Numerical studies are carried out to demonstrate its high accuracy and efficiency and to show its superiority over the PDE-ADI method, the HOC-ADI method, the EHOCADI method and the classical PR-ADI method, in the aspects of accuracy and/or computational cost.

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## References

[1] J. Bear, Dynamics of Fluids of Porous Media, American Elsevier Publishing Company, New York, 1972.
[2] S.V. Patankar, Numerical Heat Transfer and Fluid Flows, McGraw-Hill, New York, 1980.
[3] P.J. Roache, Computational Fluid Dynamics, Hermosa, Albuquerque, NM, 1976.
[4] D.W. Peaceman, H.H. Rachford Jr., The numerical solution of parabolic and elliptic differential equations, Journal of the Society for Industrial and Applied Mathematics 3 (1959) 28-41.
[5] S. Karaa, J. Zhang, High order ADI method for solving unsteady convectiondiffusion problems, Journal of Computational Physics 198 (2004) 1-9.
[6] D. You, A high-order Padé ADI method for unsteady convection-diffusion equations, Journal of Computational Physics 214 (2006) 1-11.
[7] R.S. Hirsh, Higher order accurate difference solutions of fluid mechanics problems by a compact differencing technique, Journal of Computational Physics 19 (1975) 90-109.
[8] M. Ciment, S.H. Leventhal, B.C. Weinberg, The operator compact implicit method for parabolic equations, Journal of Computational Physics 28 (1978) 135-166.
[9] B.J. Noye, H.H. Tan, Finite difference methods for solving the two-dimensional advection-diffusion equation, International Journal for Numerical Methods in Fluids 9 (1988) 75-89.
[10] W.Z. Dai, R. Nassar, Compact ADI method for solving parabolic differential equations, Numerical Methods for Partial Differential Equations 18 (2000) 129142.
[11] M. Dehghan, A. Mohebbi, High-order compact boundary value method for the solution of unsteady convection-diffusion problems, Mathematics and Computers in Simulation 79 (2008) 683-699.
[12] M. Dehghan, A new ADI technique for two-dimensional parabolic equation with an integral condition, Computers and Mathematics with Application 43 (2002) 1477-1488.
[13] M. Dehghan, Numerical solution of the three-dimensional convection-diffusion equation, Applied Mathematics and Computation 150 (2004) 5-19.
[14] M. Dehghan, On the numerical solution of the one-dimensional convectiondiffusion equation, Mathematical Problems Engineering 2005 (2005) 61-74.
[15] M. Dehghan, Weighted finite difference techniques for the one-dimensional advection-diffusion equation, Applied Mathematics and Computation 147 (2004) 307-319.
[16] M. Dehghan, Time-splitting procedures for the solution of the two-dimensional transport equation, Kybernetes 36 (2007) 791-805.
[17] A. Mohebbi, M. Dehghan, High-order compact solution of the one-dimensional heat and advection-diffusion equations, Applied Mathematical Modelling 34 (2010) 3071-3084.
[18] S.C.R. Dennis, J.D. Hudson, Compact $h^{4}$ finite-difference approximations to operators of Navier-Stokes type, Journal of Computational Physics 85 (1989) 390416.
[19] J.C. Kalita, D.C. Dalal, A.K. Dass, A class of higher order compact schemes for the unsteady two-dimensional convection-diffusion equation with variable convection coefficients, International Journal for Numerical Methods in Fluids 38 (2002) 1111-1131.
[20] M. Li, T. Tang, B. Fornberg, A compact fourth-order finite difference scheme for the incompressible Navier-Stokes equations, International Journal for Numerical Methods in Fluids 20 (1995) 1137-1151.
[21] R.J. MacKinnon, R.W. Johnson, Differential equation based representation of truncation errors for accurate numerical simulation, International Journal for Numerical Methods in Fluids 13 (1991) 739-757.
[22] Z.F. Tian, S.Q. Dai, High-order compact exponential finite difference methods for convection-diffusion type problems, Journal of Computational Physics 220 (2007) 952-974.
[23] Z.F. Tian, Y.B. Ge, A fourth-order compact ADI method for solving twodimensional unsteady convection-diffusion problems, Journal of Computational and Applied Mathematics 198 (2007) 268-286.
[24] B.J. Noye, H.H. Tan, A third-order semi-implicit finite difference method for solving the one-dimensional convection-diffusion equation, International Journal for Numerical Methods in Engineering 26 (1988) 16151629.
[25] W.F. Spotz, G.F. Carey, High-order compact scheme for the steady streamfunction vorticity equations, International Journal for Numerical Methods in Engineering 38 (1995) 3497-3512.
[26] J.W. Thomas, Numerical Partial Differential Equations: Finite Difference Methods, Springer-Verlag, New York, 1995.
[27] S.K. Lele, Compact finite difference schemes with spectral-like resolution, Journal of Computational Physics 103 (1) (1992) 16-42.
[28] Z.F. Tian, Y.B. Ge, A fourth-order compact finite difference scheme for the steady streamfunction-vorticity formulation of the Navier-Stokes/Boussinesq equations, International Journal for Numerical Methods in Fluids 41 (2003) 495-518.
[29] Y. Sun, Z.J. Wang, Evaluation of discontinuous Galerkin and spectral volume methods for scalar and system conservation laws on unstructured grids, International Journal for Numerical Methods in Fluids 45 (2004) 819838.
[30] R.F. Warming, B.J. Hyett, The modified equation approach to the stability and accuracy analysis of finite difference methods, Journal of Computational Physics 14 (1974) 159-179.


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