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Analysis of the Optimal Relaxed Control to an Optimal Control Problem

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Abstract Relaxed controls are widely used to analyze the existence of optimal controls in the literature. Though there are many optimal control problems admitting no optimal control, rare examples were shown. This paper will solve a particular optimal control problem by analyzing the optimal relaxed controls, showing the ideas we used to study such kind of problems.

Keywords Optimal control · Existence · Non-existence · Relaxed control

1 Introduction

To establish the existence theories for calculus of variations, Young and McShane [18, 28] introduced generalized curves in the late 1930s. In optimal control theory, generalized curves were transformed as relaxed controls (see Gamkrelidze [10], McShane [19] and Warga [26]). As its archetype to existence theories for calculus of variations, relaxed control is an important tool to study the existence of optimal controls without Cesari type conditions. In this aspect, there are many positive results. Among them, we would like to refer the readers the papers by Artstein [1], Balder [2–4], Berliocchi–Lasry [5], Cesari [6], Colombo–Goncharov [7], Flores-Bazán–Perrotta [9], Lou [11–13], Marcellini [16], Mariconda [17], Mordukhovich [20], Olech [21], and Raymond [22–24] and Suryanarayana [25], etc.

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Usually, for most systems, optimal controls are also optimal relaxed controls. Thus, if we consider the relaxed problem and could get all optimal relaxed controls, then we would see whether the original optimal control problem admits optimal controls or not. However, rare results were shown on the non-existence of optimal controls. Among those counter examples given in the literature, the following example is a simple but typical one, which is mentioned in many books, see, for example, [27, Chap. 3, p. 246]. Other similar examples can be found in [6, Chap. 9, p. 321] and [8, Chap. 2, p. 51].

Example 1 Let $U=[-1,1]$,

$$\mathcal{U}=\Big\{v(\cdot):[0,1]\to U\Big|v(\cdot)\text{measurable}\Big\},$$

$$\frac{dy(t)}{dt}=u(t),\quad t\in[0,1], \tag{1.1}$$

and

$$I(y(\cdot),u(\cdot))=\int_0^1\Big(y^2(t)-u^2(t)\Big)\,dt. \tag{1.2}$$

Let $\mathcal{P}$ denote the set of all pairs $(y(\cdot),u(\cdot))$ satisfying (1.1) and $u(\cdot)\in\mathcal{U}$.

The optimal control problem is that

Problem (S) *Find a pair* $(\bar{y}(\cdot),\bar{u}(\cdot))\in\mathcal{P}$ *such that*

$$I(\bar{y}(\cdot),\bar{u}(\cdot))=\inf_{(y(\cdot),u(\cdot))\in\mathcal{P}}I(y(\cdot),u(\cdot)). \tag{1.3}$$

One can prove that Problem (S) admits no solution.

To see this, let

$$u_k(t)=\begin{cases}1, & t\in\bigcup_{j=0}^{k-1}\big[\frac{j}{k},\frac{j}{k}+\frac{1}{2k}\big),\\ -1, & t\in\bigcup_{j=0}^{k-1}\big[\frac{j}{k}+\frac{1}{2k},\frac{j}{k}+\frac{1}{k}\big),\end{cases}$$

and $y_k(\cdot)$ be the solution of (1.1) corresponding to $u_k(\cdot)$ with $y_k(0)=0$. Then $(y_k(\cdot),u_k(\cdot))\in\mathcal{P}$ and

$$|y_k(t)|\le\frac{1}{2k},\quad\forall t\in[0,1].$$

Consequently, we have

$$I(y_k(\cdot),u_k(\cdot))=\int_0^1\Big(y_k^2(t)-u_k^2(t)\Big)\,dt\le\Big(\frac{1}{2k}\Big)^2-1. \tag{1.4}$$

On the other hand, for any $(y(\cdot),u(\cdot))\in\mathcal{P}$,

$$I(y(\cdot),u(\cdot))\ge\int_0^1-u^2(t)\,dt\ge-1. \tag{1.5}$$

Thus combining (1.4) and (1.5) we get

$$\inf_{(y(\cdot),u(\cdot))\in\mathcal{P}} I(y(\cdot),u(\cdot)) = -1.$$

However, the infimum -1 could not be achieved. Otherwise, if for some $(\bar{y}(\cdot),\bar{u}(\cdot))\in\mathcal{P}$,

$$I(\bar{y}(\cdot),\bar{u}(\cdot)) = -1,$$

then, by the definition of I and U, we must have

$$\bar{y}(t)=0, \quad |\bar{u}(t)|=1, \quad \text{a.e. } t\in[0,1].$$

Since

$$\bar{y}(t)=0, \quad \text{a.e. } t\in[0,1]$$

implies

$$\bar{u}(t)=0, \quad \text{a.e. } t\in[0,1],$$

which arrives a contradiction.

That is, Problem (S) has no solution.

The proof in the above example depends deeply on that U and I are symmetric. If we replace $U=[-1,1]$ by $[a,1]$ (or $\{a,1\}$) with $a<0, a\neq -1$, then the problem becomes more difficult. In [15], such generalized cases were considered and the author got that the generalized problem admits no optimal solution if and only if $a\in(-\frac{4}{3},-\frac{3}{4})$. In [14], systems governed by partial differential equations were considered.

This paper will generalize the results in [15]. More precisely, we will replace $U=[-1,1]$ by $U=[a,b]$ and define the cost functional I by

$$I(y(\cdot),u(\cdot)) = \int_{t_0}^{T} \Big[Ay^2(t)+By(t)u(t)+Cu^2(t)\Big]\,dt.$$

For notational simplicity, hereafter, symbols used in Example 1, such as "U, $\mathcal{P}$", will be used in new (but similar) senses. We state the generalized problem as follows. Let $a<b$, $T>t_0$, $U=[a,b]$,

$$\mathcal{U} = \Big\{v(\cdot):[t_0,T]\to U \,\Big|\, v(\cdot)\text{measurable}\Big\},$$

$$\frac{dy(t)}{dt} = u(t), \quad t\in[t_0,T] \tag{1.6}$$

and

$$I(y(\cdot),u(\cdot)) = \int_{t_0}^{T} \Big(Ay^2(t)+By(t)u(t)+Cu^2(t)\Big)\,dt. \tag{1.7}$$

Denote by $\mathcal{P}$ the set of all pairs $(y(\cdot),u(\cdot))$ satisfying (1.6) and $u(\cdot)\in\mathcal{U}$.

The optimal control problem is that

Table 1 Existence of optimal control to Problem (P)

Question	$\xrightarrow{\text{No}}$	$\downarrow$ Yes
$\boxed{A^2+B^2>0?}$	Problem (P) is trivial	next question
$\boxed{A>0?}$	$\inf J=-\infty$	next question
$\boxed{C<0?}$	there exists at least one optimal control	next question
$\boxed{a<0<b?}$	there exists at least one optimal control	next question
$\boxed{B^2\ge A^2(T-t_0)^2-4A\lvert C\rvert\Big(1-\min\big(\frac{b}{\lvert a\rvert},\frac{\lvert a\rvert}{b}\big)\Big)?}$	there exists no optimal control	$\xrightarrow{\text{Yes}}$ there exists at least one optimal control

Problem (P) *Find a pair* $(\bar{y}(\cdot),\bar{u}(\cdot))\in\mathcal{P}$ *such that*

$$I(\bar{y}(\cdot),\bar{u}(\cdot))=\inf_{(y(\cdot),u(\cdot))\in\mathcal{P}} I(y(\cdot),u(\cdot)). \tag{1.8}$$

A pair $(\bar{y}(\cdot),\bar{u}(\cdot))\in\mathcal{P}$ satisfying (1.8) is called an optimal pair.

The existence problem for Problem (P) will be solved completely in this paper. It will be solved by using relaxed controls. More precisely, we have Table 1 showing whether an optimal control to Problem (P) exists or not.

There are two motivations in writing this article. Firstly, existence theory of optimal controls have been studied for about 40 years. However, we have rarely seen "non-trivial" counterexamples in this aspect. Secondly, to seek conditions sufficient but nearly necessary for the existence of optimal controls is a central problem in the existence theory. In the literature, usually only sufficient conditions are considered. We think that results of this article will help researchers realizing more crucial information in studying existence theory. To show our ideas well, the optimal problems considered in this article looks quite simple.

2 State of Optimal Relaxed Control Problems

Now, we introduce the relaxed problem corresponding to Problem (P). We denote by $\mathcal{M}_+^1$ the set of all probability measures in U, by $\mathcal{R}$ the set of all measurable $\mathcal{M}_+^1$-

valued functions on $[t_0, T]$. An element in $\mathcal{R}$ is called a relaxed control. The relaxed state equation and cost functional are

$$\frac{dy(t)}{dt} = \int_U v\sigma(t)(dv), \quad t \in [t_0, T] \tag{2.1}$$

and

$$I(y(\cdot), \sigma(\cdot)) = \int_{t_0}^{T} dt \int_U \Big(Ay^2(t) + By(t)v + Cv^2\Big)\sigma(t)(dv). \tag{2.2}$$

Let $\mathcal{RP}$ denote the set of all pairs $(y(\cdot), \sigma(\cdot))$ satisfying (2.1) and $\sigma(\cdot) \in \mathcal{U}$.

The optimal relaxed control problem corresponding to Problem (P) is

Problem (R) *Find a pair* $(\bar{y}(\cdot), \bar{\sigma}(\cdot)) \in \mathcal{RP}$ *such that*

$$I(\bar{y}(\cdot), \bar{\sigma}(\cdot)) = \inf_{(y(\cdot),\sigma(\cdot)) \in \mathcal{RP}} I(y(\cdot), \sigma(\cdot)). \tag{2.3}$$

A pair $(\bar{y}(\cdot), \bar{\sigma}(\cdot)) \in \mathcal{RP}$ satisfying (2.3) is called an optimal relaxed pair (of Problem (R), or of Problem (P)). We would like to mention that $\mathcal{U}$ can be embedded into $\mathcal{R}$ by identifying each $u(\cdot) \in \mathcal{U}$ with the Dirac measure-valued function $\delta_{u(\cdot)} \in \mathcal{R}$. Moreover, $I(\delta_{u(\cdot)})$ defined by (2.2) coincides with $I(u(\cdot))$ defined by (1.7). It is known that $\mathcal{U}$ is dense in $\mathcal{R}$ (see for example, [27]), namely, for any $\sigma(\cdot) \in \mathcal{R}$, there exists a sequence $u_k(\cdot)$ in $\mathcal{U}$, such that

$$\int_{t_0}^{T} f(t, u_k(t))dt \to \int_{t_0}^{T} dt \int_U f(t, v)\sigma(t)(dv), \quad \forall f \in L^1([t_0, T]; C(U)).$$

By the denseness of $\mathcal{U}$ in $\mathcal{R}$, one can easily get that

$$\inf_{(y(\cdot),\sigma(\cdot)) \in \mathcal{RP}} I(y(\cdot), \sigma(\cdot)) = \inf_{(y(\cdot),u(\cdot)) \in \mathcal{P}} I(y(\cdot), u(\cdot)). \tag{2.4}$$

Thus, a solution to Problem (P) must be a solution to Problem (R). Furthermore, if $(\bar{y}(\cdot), \bar{\sigma}(\cdot)) \in \mathcal{RP}$ is a solution to Problem (R) and

$$\bar{\sigma}(t) = \delta_{\bar{u}(t)}, \quad \text{a.e. } t \in [t_0, T],$$

then $(\bar{y}(\cdot), \bar{u}(\cdot)) \in \mathcal{P}$ and it must be a solution to Problem (P). In this case, we simply say that $(\bar{y}(\cdot), \bar{\sigma}(\cdot)) \in \mathcal{P}$ and it is an optimal pair.

When $A > 0$, we can see that for any $(y(\cdot), \sigma(\cdot)) \in \mathcal{RP}$,

$$\begin{aligned} I(y(\cdot), \sigma(\cdot)) &= \int_{t_0}^{T} dt \int_U \Big(Ay^2(t) + By(t)v + Cv^2\Big)\sigma(t)(dv) \\ &\geq \int_{t_0}^{T} dt \int_U \Big(-\frac{B^2v^2}{4A} + Cv^2\Big)\sigma(t)(dv) \\ &\geq -\Big(\frac{B^2}{4A} + |C|\Big)(T - t_0)\max(|a|^2, |b|^2). \end{aligned}$$

Thus, by Theorem IV.2.1 in Warga [27], p. 272, we have

Lemma 2.1 *Let $A > 0$. Then Problem (R) admits at least one solution.*

Moreover, it is easy to prove that

Proposition 2.2 *Let $A > 0$ and $(\bar{y}(\cdot), \bar{\sigma}(\cdot)) \in \mathcal{RP}$ be an optimal relaxed pair of Problem (R). Define $\bar{\psi}(\cdot)$ to be the solution of*

$$\begin{cases} \frac{d}{dt}\bar{\psi}(t) = 2A\bar{y}(t) + B\int_U v\bar{\sigma}(t)(dv), & \text{in } [t_0, T], \\ \bar{\psi}(t_0) = \bar{\psi}(T) = 0, \end{cases} \tag{2.5}$$

and

$$\mathcal{H}(y, v, \psi) \equiv v\psi - (Ay^2 + Byv + Cv^2). \tag{2.6}$$

Then we have

$$\operatorname{supp}\bar{\sigma}(t) \subseteq \left\{ v \in U \,\middle|\, \mathcal{H}\Big(\bar{y}(t), v, \bar{\psi}(t)\Big) = \max_{w \in U} \mathcal{H}\Big(\bar{y}(t), w, \bar{\psi}(t)\Big) \right\}. \tag{2.7}$$

3 Simplification of the Problem

In this section, we will give some simple observations on the problem. Then we would focus on some typical cases.

First, we set

$$\bar{\varphi}(\cdot) = \bar{\psi}(\cdot) - B\bar{y}(\cdot). \tag{3.1}$$

Then, by (3.4) and the state equation (1.1), we have

$$\frac{1}{2A}\frac{d^2\bar{\varphi}(t)}{dt^2} = \int_U v\bar{\sigma}(t)(dv) \tag{3.2}$$

and $\bar{\varphi}(\cdot)$ is differentiable. Moreover, $\bar{\varphi}'(\cdot)$ is differentiable almost everywhere.

We have the following observations.

I. We need only to consider $t_0 = 0$ and $T = 1$. In fact, a simple transformation can cover the other cases. Thus, hereafter, we assume that $t_0 = 0$, $T = 1$.

II. If $A \le 0$, $A^2 + B^2 \ne 0$, then it is easy to see that

$$\inf_{(y(\cdot),u(\cdot)) \in \mathcal{P}} I(y(\cdot), u(\cdot)) = -\infty.$$

If $A = B = 0$, then the problem is trivial. Thus, hereafter, we always assume that $A > 0$.

III. If $C \ge 0$, then classical result shows that Problem (P) admits at least one solution. Thus, we will now assume that $C < 0$. Moreover, without loss of generality, we assume that $C = -1$.

IV. If $ab \geq 0$, then one can prove that Problem (P) admits at least one solution. To see this, recall that by now it has been supposed that $A > 0, C = -1, t_0 = 0$ and $T = 1$. Let $(\bar{y}(\cdot), \bar{\sigma}(\cdot)) \in \mathcal{RP}$ be an optimal relaxed pair. Then by Proposition 2.2 and (3.1), for almost all $t \in \{\bar{\varphi} \neq -(b+a)\}$, $\operatorname{supp} \bar{\sigma}(t)$ is a singleton, or equivalently, $\bar{\sigma}(t)$ is a Dirac measure. On the other hand, by (3.2),

$$\int_U v\bar{\sigma}(t)(dv) = 0, \quad \text{a.e. } t \in \{\bar{\varphi} = -(b+a)\}. \tag{3.3}$$

(i) If $ab > 0$, then

$$\int_U v\bar{\sigma}(t)(dv) \neq 0, \quad \forall t \in [0,1].$$

Therefore, (3.3) implies $\{\bar{\varphi} = -(b+a)\}$ has zero measure.

(ii) If $ab = 0$, then (3.3) implies

$$\operatorname{supp} \bar{\sigma}(t) = \{0\}, \quad \text{a.e. } t \in \{\bar{\varphi} = -(b+a)\}.$$

That is, for almost all $t \in \{\bar{\varphi} = -(b+a)\}$, $\bar{\sigma}(t)$ is a Dirac measure.

Generally, it follows that for almost all $t \in [0,1]$, $\bar{\sigma}(t)$ is a Dirac measure. This means that $(\bar{y}(\cdot), \bar{\sigma}(\cdot)) \in \mathcal{P}$ and Problem (P) admits at least one optimal pair.

V. Now, we consider cases of $a < 0 < b$. Define

$$\Big(Y(\cdot), v(\cdot), \alpha, \gamma\Big) = \begin{cases} \frac{1}{b}(y(\cdot), u(\cdot), -a, 1), & \text{if } -a \geq b, \\ \frac{1}{(-a)}(-y(\cdot), -u(\cdot), b, 1), & \text{if } -a < b. \end{cases}$$

We have $\alpha \geq 1$, $\gamma > 0$,

$$\frac{dY(t)}{dt} = v(t), \quad t \in [0,1], \ v(t) \in [-\alpha, 1]$$

and

$$J\big(y(\cdot), u(\cdot)\big) = \frac{1}{\gamma^2} \int_0^1 \Big[AY^2(t) + BY(t)v(t) - v^2(t)\Big] dt.$$

Therefore, we may assume that $a = -\alpha \leq -1$ and $b = 1$ without loss of generality. After doing the above works, we restate Proposition 2.2 in the following form:

Proposition 3.1 *Suppose that $A > 0$, $C = -1$, $a = -\alpha \leq -1$, $b = 1$, $t_0 = 0$, and $T = 1$. Let $(\bar{y}(\cdot), \bar{\sigma}(\cdot)) \in \mathcal{RP}$ be an optimal pair of Problem* (R) *and $\bar{\varphi}(\cdot)$ be the solution of*

$$\begin{cases} \frac{d}{dt}\bar{\varphi}(t) = 2A\bar{y}(t), & \text{in } [0,1], \\ \bar{\varphi}(0) = -B\bar{y}(0), & \bar{\varphi}(1) = -B\bar{y}(1). \end{cases} \tag{3.4}$$

Then

$$\operatorname{supp} \bar{\sigma}(t) \subseteq \begin{cases} \{-\alpha\}, & \text{if } \bar{\varphi}(t) < \alpha - 1, \\ \{1\}, & \text{if } \bar{\varphi}(t) > \alpha - 1, \\ \{-\alpha, 1\}, & \text{if } \bar{\varphi}(t) = \alpha - 1, \end{cases} \quad \text{a.e. } t \in [0,1]. \tag{3.5}$$

4 Elementary Analysis on Optimal Relaxed Pair

By Proposition 3.1, we have (3.2) and

$$\operatorname{supp}\bar{\sigma}(t) \subseteq \{-\alpha, 1\}.$$

Thus, one can easily see that

$$\bar{\varphi}(t) = \alpha - 1, \quad \text{on } [\ell_1, \ell_2] \Longrightarrow \begin{cases} \bar{y}(t) = 0, \\ \bar{\sigma}(t) = \frac{\delta_{-\alpha} + \alpha\delta_1}{1+\alpha}, \end{cases} \quad \text{a.e. } t \in [\ell_1, \ell_2]. \tag{4.1}$$

On the other hand, by Proposition 3.1, we can get an interesting property of $\bar{\varphi}(\cdot)$.

Lemma 4.1 *Under the assumption of Proposition 3.1, if for some* $0 \le \ell_1 < \ell_2 \le 1$,

$$\bar{\varphi}(\ell_1) = \bar{\varphi}(\ell_2) = \alpha - 1,$$

then

$$\bar{\varphi}(t) = \alpha - 1, \quad \forall t \in [\ell_1, \ell_2].$$

Proof To see this, we need to prove that both $\{\bar{\varphi} < \alpha - 1\} \cap [\ell_1, \ell_2]$ and $\{\bar{\varphi} > \alpha - 1\} \cap [\ell_1, \ell_2]$ are empty. Otherwise suppose that $E \equiv \{\bar{\varphi} < \alpha - 1\} \cap [\ell_1, \ell_2]$ is not empty. Then E is a nonempty open set since $\bar{\varphi}(\cdot)$ is continuous. We have

$$\bar{\varphi}|_{\partial E} = \alpha - 1. \tag{4.2}$$

By (3.5),

$$\operatorname{supp}\bar{\sigma}(t) = \{-\alpha\}, \quad \text{a.e. } t \in E. \tag{4.3}$$

Therefore by (3.2), we get that

$$\frac{d^2\bar{\varphi}(t)}{dt^2} = 2A\alpha < 0, \quad \text{a.e. } t \in E. \tag{4.4}$$

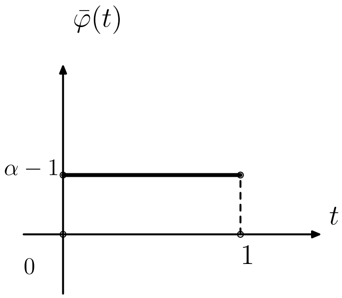
Combining (4.2) and (4.4), we have

$$\bar{\varphi}(t) \ge \alpha - 1, \quad \text{a.e. } t \in E. \tag{4.5}$$

This is a contradiction.

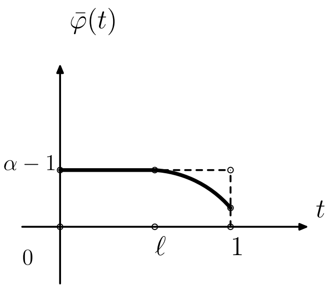
Similarly, we can prove that $\{\bar{\varphi} > \alpha - 1\} \cap [\ell_1, \ell_2]$ is empty. □

Based on the above interesting and important property, we will use Proposition 3.1 to calculate all possible solutions to Problem (R) by discussing 13 cases of $\bar{\varphi}(\cdot)$.



Case 1

$\bar{\varphi}(t) = \alpha - 1,\ t \in [0, 1]$

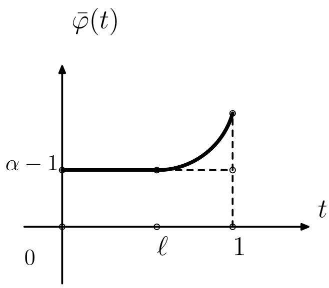


Case 2

$\bar{\varphi}(t) = \alpha - 1,\ t \in [0, \ell]$

$\bar{\varphi}(t) < \alpha - 1,\ t \in (\ell, 1]$

$\ell \in (0, 1)$

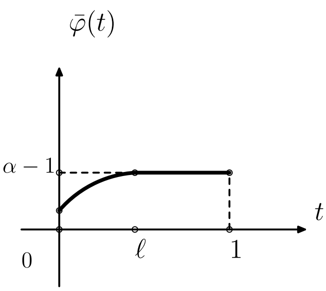


Case 3

$\bar{\varphi}(t) = \alpha - 1,\ t \in [0, \ell]$

$\bar{\varphi}(t) > \alpha - 1,\ t \in (\ell, 1]$

$\ell \in (0, 1)$

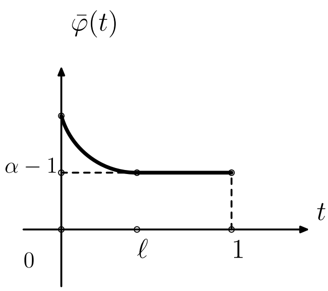


Case 4

$\bar{\varphi}(t) < \alpha - 1,\ t \in [0, \ell)$

$\bar{\varphi}(t) = \alpha - 1,\ t \in (\ell, 1]$

$\ell \in (0, 1)$

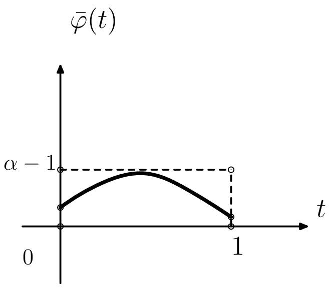


Case 5

$\bar{\varphi}(t) > \alpha - 1,\ t \in [0, \ell)$

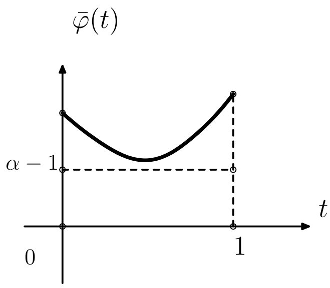
$\bar{\varphi}(t) = \alpha - 1,\ t \in [\ell, 1]$

$\ell \in (0, 1)$



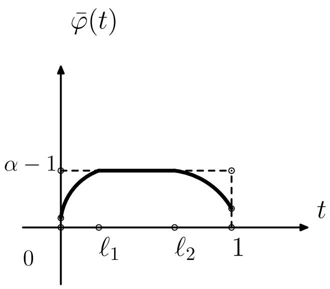
Case 6

$\bar{\varphi}(t) < \alpha - 1$, a.e. $t \in [0, 1]$



Case 7

$\bar{\varphi}(t) > \alpha - 1$, a.e. $t \in [0, 1]$

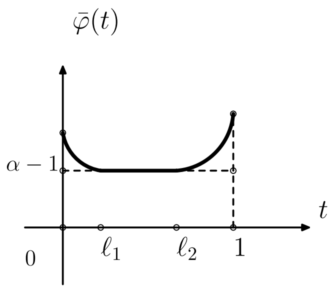


Case 8

$\bar{\varphi}(t) < \alpha - 1$, $t \notin [\ell_1, \ell_2]$

$\bar{\varphi}(t) = \alpha - 1$, $t \in [\ell_1, \ell_2]$

$0 < \ell_1 < \ell_2 < 1$

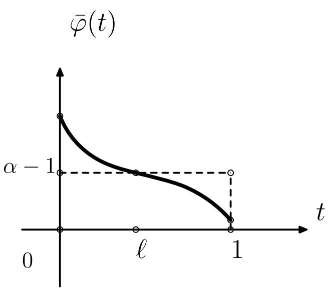


Case 9

$\bar{\varphi}(t) > \alpha - 1$, $t \notin [\ell_1, \ell_2]$

$\bar{\varphi}(t) = \alpha - 1$, $t \in [\ell_1, \ell_2]$

$0 < \ell_1 < \ell_2 < 1$

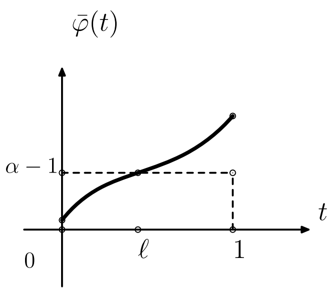


Case 10

$\bar{\varphi}(t) > \alpha - 1$, $t \in [0, \ell)$

$\bar{\varphi}(t) < \alpha - 1$, $t \in (\ell, 1]$

$0 < \ell < 1$

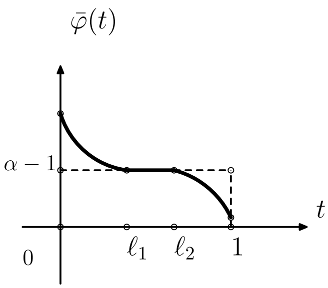


Case 11

$\bar{\varphi}(t) < \alpha - 1$, $t \in [0, \ell)$

$\bar{\varphi}(t) > \alpha - 1$, $t \in (\ell, 1]$

$0 < \ell < 1$



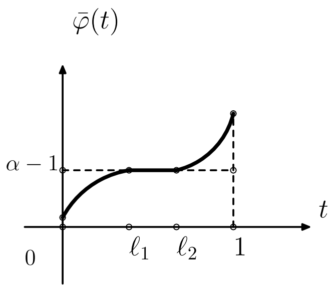
Case 12

$\bar{\varphi}(t) > \alpha - 1,\ t \in [0, \ell_1)$

$\bar{\varphi}(t) = \alpha - 1,\ t \in [\ell_1, \ell_2]$

$\bar{\varphi}(t) < \alpha - 1,\ t \in (\ell_2, 1]$

$0 < \ell_1 < \ell_2 < 1$



Case 13

$\bar{\varphi}(t) < \alpha - 1,\ t \in [0, \ell_1)$

$\bar{\varphi}(t) = \alpha - 1,\ t \in [\ell_1, \ell_2]$

$\bar{\varphi}(t) > \alpha - 1,\ t \in (\ell_2, 1]$

$0 < \ell_1 < \ell_2 < 1$

5 Calculating Critical Relaxed Pairs

For convenience, we call $(\bar{y}(\cdot), \bar{\sigma}(\cdot)) \in \mathcal{RP}$ a critical relaxed pair (and call $\bar{y}(\cdot)$ a critical state) of type k if the corresponding triple $(\bar{y}(\cdot), \bar{\sigma}(\cdot), \bar{\varphi}(\cdot))$ satisfies (3.4)–(3.5) and $\bar{\varphi}(\cdot)$ is in Case k. We will find all critical relaxed pairs by analyzing the 13 cases of $\bar{\varphi}(\cdot)$. Noting that we have supposed that

$$A > 0, \qquad C = -1, \qquad a = -\alpha \le -1, \qquad b = 1, \qquad t_0 = 0, \qquad T = 1 \tag{5.1}$$

and (4.1) holds, one can easily get that[1]

I. A critical relaxed pair of type 1 exists if and only if $\alpha = 1$. The corresponding state is [2]

$$\bar{y}(t) = 0, \quad \forall t \in [0, 1] \tag{5.2}$$

with

$$I_1 = I(\bar{y}(\cdot), \bar{\sigma}(\cdot)) = -1.$$

II. A critical relaxed pair of type 2/type 3 exists if and only if $\alpha = 1$, $-A < B < 0$. The corresponding state is

$$\bar{y}(t) = \pm(\ell - t)\chi_{[\ell,1]}(t), \quad \forall t \in [0, 1] \tag{5.3}$$

with $\ell = 1 + \frac{B}{A}$ and

$$I_2 = I_3 = I(\bar{y}(\cdot), \bar{\sigma}(\cdot)) = \frac{B^3}{6A^2} - 1.$$

[1] The following discussion will cover cases of $B = 0$ though they are essentially solved in [15].

[2] We will not write the expression of $\bar{\sigma}(\cdot)$ since it can be get immediately from (3.5) and (4.1).

III. A critical relaxed pair of type 4/type 5 exists if and only if $\alpha = 1$, $0 < B < A$. The corresponding critical relaxed state is

$$\bar{y}(t) = \pm(t - \ell)\chi_{[0,\ell]}(t), \quad \forall t \in [0,1] \tag{5.4}$$

with $\ell = \frac{B}{A}$ and

$$I_4 = I_5 = I(\bar{y}(\cdot), \bar{\sigma}(\cdot)) = -\frac{B^3}{6A^2} - 1.$$

IV. A critical relaxed pair of type 6 exists if and only if[3]

$$B^2 \geq A^2 - 4A\left(1 - \frac{1}{\alpha}\right). \tag{5.5}$$

The corresponding critical relaxed state is

$$\bar{y}(t) = \alpha\left(\frac{A+B}{2A} - t\right), \quad \forall t \in [0,1] \tag{5.6}$$

with

$$I_6 = I(\bar{y}(\cdot), \bar{\sigma}(\cdot)) = \left(\frac{A}{12} - \frac{B^2}{4A} - 1\right)\alpha^2.$$

To see this, we note that

$$\max_{t\in[0,1]} \bar{\varphi}(t) = \begin{cases} \frac{A^2-B^2}{4A}\alpha, & \text{if } |B| \leq A, \\ \left(-\frac{B^2}{2A} + \frac{|B|}{2}\right)\alpha, & \text{if } |B| \geq A. \end{cases}$$

Thus, we always have $\bar{\varphi}(t) < \alpha - 1$ a.e. when $|B| \geq A$. On the other hand, when $|B| < A$, we can see that $\bar{\varphi}(t) < \alpha - 1$ a.e. if and only if

$$A^2 > B^2 \geq A^2 - 4A\left(1 - \frac{1}{\alpha}\right).$$

Finally, we get that $\bar{\varphi}(t) < \alpha - 1$ a.e. if and only if (5.5) holds.

V. A critical relaxed pair of type 7 exists if and only if[4]

$$|B| \geq \frac{A + \sqrt{A^2 + 8A(\alpha - 1)}}{2}. \tag{5.7}$$

The corresponding critical relaxed state is

$$\bar{y}(t) = -\frac{A+B}{2A} + t, \quad \forall t \in [0,1] \tag{5.8}$$

[3] Equation (5.5) is equivalent to $\bar{\varphi} \leq \alpha - 1$ a.e.[0, 1].

[4] Equation (5.7) is equivalent to $\bar{\varphi} > \alpha - 1$ a.e.[0, 1].

with

$$I_7 = I(\bar{y}(\cdot), \bar{\sigma}(\cdot)) = \frac{A}{12} - \frac{B^2}{4A} - 1.$$

Thus, noting that I_1 and I_6 can be always achieved, we can see that, when $\alpha > 1$, the above $\bar{y}(\cdot)$ should not be an optimal relaxed state since

$$I_7 > \min(I_1, I_6).$$

In fact, in this case, it holds that

$$\bar{\varphi}(t) = At^2 - (A+B)t + \frac{B(A+B)}{2A}$$

and therefore

$$\min_{t\in[0,1]} \bar{\varphi}(t) = \begin{cases} \frac{B^2-A^2}{4A}, & \text{if } |B| \le A, \\ \left(\frac{B^2}{2A} - \frac{|B|}{2}\right), & \text{if } |B| \ge A, \end{cases}$$

Thus, $\bar{\varphi}(t) > \alpha - 1$ a.e. $[0, 1]$ if and only if (5.7) holds.

On the other hand, let

$$y_1(t) = 0, \quad \sigma_1(t) = \frac{\delta_1 + \delta_{-1}}{2}, \quad \forall t \in [0,1],$$

$$y_2(t) = \alpha\left(\frac{A+B}{2A} - t\right), \quad \sigma_2(t) = \delta_{-\alpha}, \quad \forall t \in [0,1].$$

Then

$$I(y_1(\cdot), \sigma_1(\cdot)) = -1,$$

$$I(y_2(\cdot), \sigma_2(\cdot)) = \left(\frac{A}{12} - \frac{B^2}{4A} - 1\right)\alpha^2.$$

That is, I_1 and I_6 can be always achieved.

When $\alpha > 1$, if

$$\frac{A}{12} - \frac{B^2}{4A} - 1 \ge 0,$$

then

$$I_7 > I(y_1(\cdot), \sigma_1(\cdot));$$

if

$$\frac{A}{12} - \frac{B^2}{4A} - 1 < 0,$$

then

$$I_7 > I(y_2(\cdot), \sigma_2(\cdot)).$$

Therefore, when $\alpha > 1$, a critical relaxed state of type 7 should not be an optimal relaxed state.

VI. A critical relaxed pair of type 8 exists if and only if $\alpha > 1$ and

$$B^2 < A^2 - 4A\left(1 - \frac{1}{\alpha}\right). \tag{5.9}$$

The corresponding critical relaxed state is

$$\bar{y}(t) = \begin{cases} -\alpha(t - \ell_1), & t \in [0, \ell_1], \\ 0, & t \in (\ell_1, \ell_2), \\ -\alpha(t - \ell_2), & t \in [\ell_2, 1], \end{cases} \tag{5.10}$$

with

$$\ell_1 = \frac{B + \sqrt{B^2 + 4A\left(1 - \frac{1}{\alpha}\right)}}{2A}, \qquad \ell_2 = \frac{B - \sqrt{B^2 + 4A\left(1 - \frac{1}{\alpha}\right)}}{2A} + 1 \tag{5.11}$$

and

$$I_8 = I(\bar{y}(\cdot), \bar{\sigma}(\cdot)) = -\frac{\sqrt{B^2 + 4A\left(1 - \frac{1}{\alpha}\right)}}{6A^2}\left[B^2 + 4A\left(1 - \frac{1}{\alpha}\right)\right]\alpha^2 - \alpha.$$

In fact, we have

$$\bar{\varphi}(t) = \begin{cases} \alpha - 1 - A\alpha(t - \ell_1)^2, & t \in [0, \ell_1], \\ \alpha - 1, & t \in (\ell_1, \ell_2), \\ \alpha - 1 - A\alpha(t - \ell_2)^2, & t \in [\ell_2, 1]. \end{cases}$$

Thus, by (3.4),

$$\ell_1 = \frac{B \pm \sqrt{B^2 + 4A\left(1 - \frac{1}{\alpha}\right)}}{2A},$$

$$\ell_2 - 1 = \frac{B \pm \sqrt{B^2 + 4A\left(1 - \frac{1}{\alpha}\right)}}{2A}.$$

Consequently, $0 < \ell_1 < \ell_2 < 1$ if and only if $\alpha > 1$, (5.9) and (5.11) hold.

VII. There is no critical relaxed pair of type 9.

Otherwise, let $(\bar{y}(\cdot), \bar{\sigma}(\cdot))$ be a critical relaxed pair of type 9. We have

$$\bar{y}(t) = \begin{cases} t - \ell_1, & t \in [0, \ell_1], \\ 0, & t \in (\ell_1, \ell_2), \\ t - \ell_2, & t \in [\ell_1, \ell_2] \end{cases}$$

and

$$\bar{\varphi}(t) = \begin{cases} \alpha - 1 + A(t-\ell_1)^2, & t \in [0, \ell_1], \\ \alpha - 1, & t \in (\ell_1, \ell_2), \\ \alpha - 1 + A(t-\ell_2)^2, & t \in [\ell_1, \ell_2]. \end{cases}$$

Thus, by (3.4),

$$\ell_1 = \frac{B \pm \sqrt{B^2 - 4A\big(1 - \frac{1}{\alpha}\big)}}{2A},$$

$$\ell_2 - 1 = \frac{B \pm \sqrt{B^2 - 4A\big(1 - \frac{1}{\alpha}\big)}}{2A}.$$

However, we can verify that $0 < \ell_1 < \ell_2 < 1$ can not hold.

VIII. If $\bar{y}(\cdot)$ is a critical state of types 10–13, then it should not be an optimal relaxed state.

Let $\bar{y}(\cdot)$ be a critical state of type 10. Then

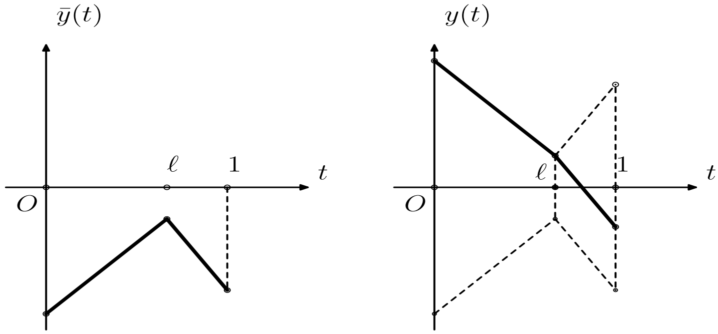
$$\bar{y}(t) = \begin{cases} \bar{y}(\ell) + t - \ell, & t \in [0, \ell], \\ \bar{y}(\ell) - \alpha(t - \ell), & t \in [\ell, 1] \end{cases}$$

and

$$\bar{y}(\ell) = \frac{1}{2A}\frac{d\bar{\varphi}}{dt}(\ell) = \frac{1}{2A}\lim_{t\to\ell}\frac{\bar{\varphi}(t) - \bar{\varphi}(\ell)}{t - \ell} \le 0.$$

Thus $\bar{y}(0) < 0$, $\bar{y}(1) < 0$ and consequently,

$$B = -\frac{\bar{\varphi}(0)}{\bar{y}(0)} > 0, \qquad \alpha - 1 > \bar{\varphi}(1) = -B\bar{y}(1) > 0. \tag{5.12}$$



Define

$$y(t) = \begin{cases} -\bar{y}(t), & t \in [0, \ell], \\ -\bar{y}(\ell) - \alpha(t - \ell), & t \in [\ell, 1], \end{cases}$$

and

$$\sigma(t)=\begin{cases}\delta_{-1}, & t\in[0,\ell],\\ \delta_{-\alpha}, & t\in[\ell,1],\end{cases}\quad \text{a.e. } [0,1].$$

Then $(y(\cdot),\sigma(\cdot))\in\mathcal{RP}$ and

$$|y(t)|\le|\bar{y}(t)|,\quad \forall t\in[\ell,1].$$

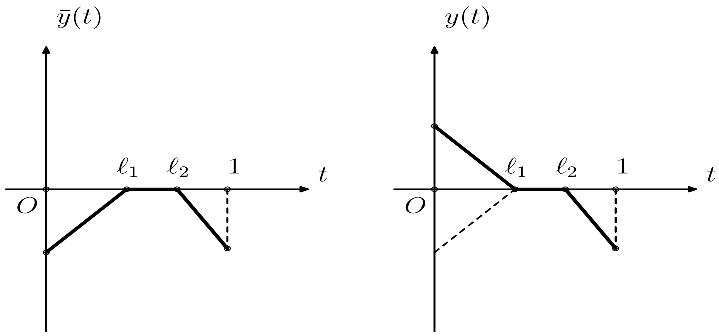
Thus,

$$\begin{aligned}&I(\bar{y}(\cdot),\bar{\sigma}(\cdot))\\ &\quad=I(y(\cdot),\sigma(\cdot))+\int_\ell^1\Big[A\Big(\bar{y}^2(t)-y^2(t)\Big)-2B\alpha\bar{y}(\ell)\Big]dt\\ &\quad\ge I(y(\cdot),\sigma(\cdot)).\end{aligned}\tag{5.13}$$

By (5.12), $-1\notin\{1,-\alpha\}$. Therefore, it follows from (3.5) that $(y(\cdot),\sigma(\cdot))\in\mathcal{RP}$ should not be an optimal relaxed pair. Consequently, $\bar{y}(\cdot)$ should not be an optimal relaxed state.

Similarly, if $(\bar{y}(\cdot),\bar{\sigma}(\cdot))$ is a critical relaxed pair of type 12, then we have $B>0$, $\alpha>1$ too. Moreover

$$\bar{y}(t)=\begin{cases}t-\ell_1, & t\in[0,\ell_1],\\ 0, & t\in[\ell_1,\ell_2],\\ -\alpha(t-\ell_2), & t\in[\ell_2,1].\end{cases}$$



Let

$$y(t)=\begin{cases}\ell_1-t, & t\in[0,\ell_1],\\ 0, & t\in[\ell_1,\ell_2],\\ -\alpha(t-\ell_2), & t\in[\ell_2,1]\end{cases}$$

and

$$\sigma(t)=\begin{cases}\delta_{-1}, & t\in[0,\ell_1],\\ \frac{\delta_{-\alpha}+\alpha\delta_1}{1+\alpha}, & t\in[\ell_1,\ell_2],\\ \delta_{-\alpha}, & t\in[\ell_2,1].\end{cases}$$

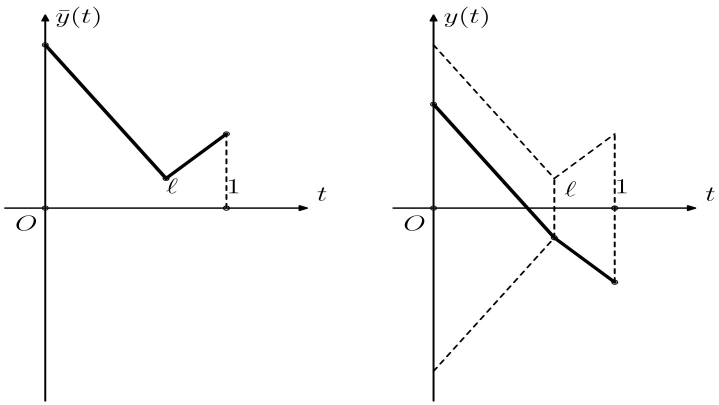
Then

$$I(\bar{y}(\cdot), \bar{\sigma}(\cdot)) = I(y(\cdot), \sigma(\cdot))$$

and therefore $\bar{y}(\cdot)$ should not be an optimal relaxed state since $(y(\cdot), \sigma(\cdot)) \in \mathcal{RP}$ should not be an optimal relaxed pair.

If $(\bar{y}(\cdot), \bar{\sigma}(\cdot))$ is a critical relaxed pair of type 11, then it can be seen that $\alpha > 1$, $\bar{y}(\ell) \geq 0$, $B < 0$ and

$$\bar{y}(t) = \begin{cases} \bar{y}(\ell) - \alpha(t - \ell), & t \in [0, \ell], \\ \bar{y}(\ell) + t - \ell, & t \in [\ell, 1]. \end{cases}$$



We set

$$y(t) = \begin{cases} -\bar{y}(\ell) - \alpha(t - \ell), & t \in [0, \ell], \\ -\bar{y}(t), & t \in [\ell, 1] \end{cases}$$

and

$$\sigma(t) = \begin{cases} \delta_{-\alpha}, & t \in [0, \ell], \\ \delta_{-1}, & t \in [\ell, 1]. \end{cases}$$

Then $(y(\cdot), \sigma(\cdot)) \in \mathcal{RP}$ and

$$|y(t)| \leq |\bar{y}(t)|, \quad \forall t \in [0, \ell].$$

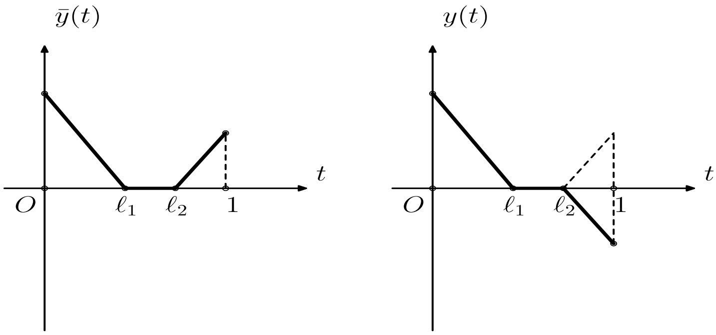
It holds

$$\begin{aligned} &I(\bar{y}(\cdot), \bar{\sigma}(\cdot)) \\ &\quad = I(y(\cdot), \sigma(\cdot)) + \int_0^\ell \Big[A\Big(\bar{y}^2(t) - y^2(t)\Big) - 2B\alpha\bar{y}(\ell)\Big] dt \\ &\quad \geq I(y(\cdot), \sigma(\cdot)). \end{aligned}$$

Therefore, $\bar{y}(\cdot)$ should not be an optimal relaxed state.

If $(\bar{y}(\cdot), \bar{\sigma}(\cdot))$ is a critical relaxed pair of type 13, then $\alpha > 1$, $B < 0$ and

$$\bar{y}(t) = \begin{cases} -\alpha(t - \ell_1), & t \in [0, \ell_1], \\ 0, & t \in [\ell_1, \ell_2], \\ t - \ell_2, & t \in [\ell_2, 1]. \end{cases}$$



Define

$$y(t) = \begin{cases} -\alpha(t - \ell_1), & t \in [0, \ell_1], \\ 0, & t \in [\ell_1, \ell_2], \\ \ell_2 - t, & t \in [\ell_2, 1] \end{cases}$$

and

$$\sigma(t) = \begin{cases} \delta_{-\alpha}, & t \in [0, \ell_1], \\ \frac{\delta_{-\alpha} + \alpha\delta_1}{1+\alpha}, & t \in [\ell_1, \ell_2], \\ \delta_{-1}, & t \in [\ell_2, 1]. \end{cases}$$

Then $(y(\cdot), \sigma(\cdot)) \in \mathcal{RP}$ and

$$I(\bar{y}(\cdot), \bar{\sigma}(\cdot)) = I(y(\cdot), \sigma(\cdot)).$$

Therefore $\bar{y}(\cdot)$ should not be an optimal relaxed state.

6 Main Results

Now, we can state our main results. By Sect. 5, we can list all candidate values of $\overline{I} \equiv \inf_{(y(\cdot),u(\cdot))\in\mathcal{P}} I(y(\cdot), u(\cdot))$ in Table 2:

We state the above results as two theorems.

Theorem 6.1 *Assume that* $A > 0$, $C = -1$, $[a, b] = [-1, 1]$ *and* $[t_0, T] = [0, 1]$.

(i) *If* $B = 0$, *then Problem* (P) *admits no optimal pair. There exists a unique optimal relaxed pair. That is,*

$$\begin{pmatrix} \bar{y}(t) \\ \bar{\sigma}(t) \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{\delta_1 + \delta_{-1}}{2} \end{pmatrix}, \quad t \in [0, 1].$$

Table 2 Candidate values of $\overline{I}$

	Cases	Candidate values of $\overline{I}$	Remark
$\alpha = 1$	$B = 0$	I_1	
	$-A < B < 0$	I_1, I_2, I_3	$I_2 = I_3$
	$0 < B < A$	I_1, I_4, I_5	$I_4 = I_5$
	$\lvert B\rvert \geq A$	I_1, I_6, I_7	$I_6 = I_7$
$\alpha > 1$	$B^2 \geq A^2 - 4A(1 - \frac{1}{\alpha})$	I_6	
	$B^2 < A^2 - 4A(1 - \frac{1}{\alpha})$	I_8	

Moreover,

$$\inf_{(y(\cdot),u(\cdot))\in\mathcal{P}} I(y(\cdot), u(\cdot)) = -1.$$

(ii) *If* $0 < |B| < A$, *then Problem* (P) *admits no optimal pair. There are two optimal relaxed pairs. That is, if* $-A < B < 0$

$$\begin{pmatrix} \bar{y}_i(t) \\ \bar{\sigma}_i(t) \end{pmatrix} = \begin{cases} \begin{pmatrix} 0 \\ \frac{\delta_1+\delta_{-1}}{2} \end{pmatrix}, & \text{if } t \in [0, 1 + \frac{B}{A}], \\ \begin{pmatrix} (-1)^i(t - 1 - \frac{B}{A}) \\ \delta_{(-1)^i} \end{pmatrix}, & \text{if } t \in (1 + \frac{B}{A}, 1], \end{cases} \qquad i = 1, 2,$$

and if $A > B > 0$,

$$\begin{pmatrix} \bar{y}_i(t) \\ \bar{\sigma}_i(t) \end{pmatrix} = \begin{cases} \begin{pmatrix} (-1)^i(t - \frac{B}{A}) \\ \delta_{(-1)^i} \end{pmatrix}, & \text{if } t \in [0, \frac{B}{A}], \\ \begin{pmatrix} 0 \\ \frac{\delta_1+\delta_{-1}}{2} \end{pmatrix}, & \text{if } t \in (\frac{B}{A}, 1]. \end{cases} \qquad i = 1, 2.$$

Moreover,

$$\inf_{(y(\cdot),u(\cdot))\in\mathcal{P}} I(y(\cdot), u(\cdot)) = -\frac{|B|^3}{6A^2} - 1.$$

(iii) *If* $|B| \geq A$, *then Problem* (P) *admits two optimal pairs. That is*

$$\begin{pmatrix} \bar{y}_i(t) \\ \bar{u}_i(t) \end{pmatrix} = (-1)^i \begin{pmatrix} t - \frac{B}{2A} - \frac{1}{2} \\ 1 \end{pmatrix}, \quad t \in [0, 1], \ i = 1, 2.$$

Moreover,

$$\inf_{(y(\cdot),u(\cdot))\in\mathcal{P}} I(y(\cdot), u(\cdot)) = \frac{A}{12} - \frac{B^2}{4A} - 1.$$

Theorem 6.2 *Assume that* $A > 0$, $C = -1$, $[a, b] = [-\alpha, 1]$, $\alpha > 1$ *and* $[t_0, T] = [0, 1]$.

(i) *If*

$$B^2 < A^2 - 4A\left(1 - \frac{1}{\alpha}\right),$$

then Problem (*P*) *admits no optimal pair. There is a unique optimal relaxed pair* $(\bar{y}(\cdot), \bar{\sigma}(\cdot))$ *with*

$$\bar{y}(t) = \begin{cases} -\alpha(t - \ell_1), & t \in [0, \ell_1], \\ 0, & t \in (\ell_1, \ell_2), \\ -\alpha(t - \ell_2), & t \in [\ell_2, 1], \end{cases}$$

$$\bar{\sigma}(t) = \begin{cases} \delta_{-\alpha}, & t \notin (\ell_1, \ell_2), \\ \frac{\delta_{-\alpha} + \alpha\delta_1}{1+\alpha}, & t \in (\ell_1, \ell_2), \end{cases}$$

where

$$\ell_1 = \frac{B + \sqrt{B^2 + 4A(1 - \frac{1}{\alpha})}}{2A}, \qquad \ell_2 = \frac{B - \sqrt{B^2 + 4A(1 - \frac{1}{\alpha})}}{2A} + 1.$$

Moreover,

$$\inf_{(y(\cdot), u(\cdot)) \in \mathcal{P}} I(y(\cdot), u(\cdot)) = -\frac{\sqrt{B^2 + 4A(1 - \frac{1}{\alpha})}}{6A^2}\left[B^2 + 4A\left(1 - \frac{1}{\alpha}\right)\right]\alpha^2 - \alpha.$$

(ii) *If*

$$B^2 \geq A^2 - 4A\left(1 - \frac{1}{\alpha}\right),$$

then Problem (*P*) *admits a unique optimal pair* $(\bar{y}(\cdot), \bar{u}(\cdot))$ *with*

$$\begin{pmatrix} \bar{y}(t) \\ \bar{u}(t) \end{pmatrix} = \begin{pmatrix} -\alpha(t - \frac{B}{2A} - \frac{1}{2}) \\ -\alpha \end{pmatrix}, \quad t \in [0, 1].$$

Moreover,

$$\inf_{(y(\cdot), u(\cdot)) \in \mathcal{P}} I(y(\cdot), u(\cdot)) = \left(\frac{A}{12} - \frac{B^2}{4A} - 1\right)\alpha^2.$$

We can see from Theorems 6.1, 6.2 that, if

$$A > 0, \qquad C = -1, \qquad [a, b] = [-\alpha, 1], \qquad [t_0, T] = [0, 1], \tag{6.1}$$

then when $\alpha \geq 1$, Problem (P) admits at least one optimal control if and only if

$$B^2 \geq A^2 - 4A\left(1 - \frac{1}{\alpha}\right). \tag{6.2}$$

On the other hand, combining Theorem 6.2 with Part V of Sect. 3, it can be shown that under the assumption (6.1), if $\alpha \in (0, 1)$, then Problem (P) admits at least one optimal control if and only if

$$B^2 \geq A^2 - 4A(1-\alpha).$$

More precisely, it holds

Theorem 6.3 *Assume that* $A > 0$, $C = -1$, $[a, b] = [-\alpha, 1]$, $\alpha \in (0, 1)$ *and* $[t_0, T] = [0, 1]$.

(i) *If*

$$B^2 < A^2 - 4A(1-\alpha),$$

then Problem (*P*) *admits no optimal pair. There is a unique optimal relaxed pair* $(\bar{y}(\cdot), \bar{\sigma}(\cdot))$ *with*

$$\bar{y}(t) = \begin{cases} t - \ell_1, & t \in [0, \ell_1], \\ 0, & t \in (\ell_1, \ell_2), \\ t - \ell_2, & t \in [\ell_2, 1], \end{cases}$$

$$\bar{\sigma}(t) = \begin{cases} \delta_1, & t \notin (\ell_1, \ell_2), \\ \frac{\alpha\delta_1 + \delta_{-\alpha}}{1+\alpha}, & t \in (\ell_1, \ell_2), \end{cases}$$

where

$$\ell_1 = \frac{B + \sqrt{B^2 + 4A(1-\alpha)}}{2A}, \qquad \ell_2 = \frac{B - \sqrt{B^2 + 4A(1-\alpha)}}{2A} + 1.$$

Moreover,

$$\begin{aligned} &\inf_{(y(\cdot),u(\cdot))\in\mathcal{P}} I(y(\cdot), u(\cdot)) \\ &\quad = -\frac{\sqrt{B^2 + 4A(1-\alpha)}}{6A^2}\Big[B^2 + 4A(1-\alpha)\Big]\alpha^2 - \alpha. \end{aligned}$$

(ii) *If*

$$B^2 \geq A^2 - 4A(1-\alpha),$$

then Problem (*P*) *admits a unique optimal pair* $(\bar{y}(\cdot), \bar{u}(\cdot))$ *with*

$$\begin{pmatrix} \bar{y}(t) \\ \bar{u}(t) \end{pmatrix} = \begin{pmatrix} t - \frac{B}{2A} - \frac{1}{2} \\ 1 \end{pmatrix}, \qquad t \in [0, 1].$$

Moreover,

$$\inf_{(y(\cdot),u(\cdot))\in\mathcal{P}} I(y(\cdot), u(\cdot)) = \frac{A}{12} - \frac{B^2}{4A} - 1.$$

Thus, if $\alpha > 0$ and (6.1) holds, then Problem (P) admits at least one optimal control if and only if

$$B^2 \geq A^2 - 4A\left(1 - \min\left(\alpha, \frac{1}{\alpha}\right)\right). \tag{6.3}$$

In particular, when $A = 1$, $B = 0$, (6.3) becomes

$$\alpha \leq \frac{3}{4}, \quad \text{or} \quad \alpha \geq \frac{4}{3}, \tag{6.4}$$

which coincides with the results in [15].

In general, using Theorems 6.1, 6.2 and the discussion we posed in Sect. 3, we know whether an optimal control to Problem (P) exists or not. For the convenience of readers, in Sect. 1, we list the results in Table 1.

Remark 6.1 By constructing a special pair which is impossible to be an optimal relaxed pair, we proved that critical relaxed pairs of types 10–13 should not be optimal relaxed pairs. This method could not be used directly if $U = [-\alpha, 1]$ is replaced by $U = \{-\alpha, 1\}$. However, when $C < 0$, Proposition 3.1 shows that optimal relaxed pairs do not change when $U = [-\alpha, 1]$ is replaced by $\{-\alpha, 1\}$. Therefore, Theorems 6.1 and 6.2 remain true if $U = [-\alpha, 1]$ is replaced by $U = \{-\alpha, 1\}$.

Remark 6.2 For the cases of $U = \{a, b\}$ and $C \geq 0$, the problem of whether an optimal control exists or not is not trivial since $U = \{a, b\}$ is not convex. We can use these paper's methods to study such a problem.

Remark 6.3 If $y(t_0)$ and $y(T)$ are fixed, similar problems can be considered. At that time, the cost functional I is independent of B and the corresponding $\bar{\varphi}(t_0)$ and $\bar{\varphi}(T)$ will be free. Nevertheless, we can solve the problem. Why we choose cases of $y(t_0)$ and $y(T)$ being free is that we think such cases are more interesting than that of $y(t_0)$, $y(t)$ being fixed.

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