Asymptotic periodicity of the Volterra equation with infinite delay

ARTICLE *in* NONLINEAR ANALYSIS · JANUARY 2008

Impact Factor: 1.61 · DOI: 10.1016/j.na.2006.10.050

Jinliang Wang
Qingdao University of Science and Tec…

Yanbin Tang
Huazhong University of Science and T…

Available online at www.sciencedirect.com
ScienceDirect
Nonlinear Analysis 68 (2008) 315–328
Nonlinear Analysis
www.elsevier.com/locate/na

Asymptotic periodicity of the Volterra equation with infinite delay

Jinliang Wang[a,*], Li Zhou[b], Yanbin Tang[b]

[a] Ocean Current and Wave Laboratory, Institute of Oceanology, Chinese Academy of Science, Qingdao 266071, PR China
[b] Department of Mathematics, Huazhong University of Science and Technology, Wuhan, Hubei 430074, PR China

Received 1 June 2003; accepted 14 October 2006

Abstract

For some species, hereditary factors have great effects on their population evolution, which can be described by the well-known Volterra model. A model developed is investigated in this article, considering the seasonal variation of the environment, where the diffusive effect of the population is also considered. The main approaches employed here are the upper–lower solution method and the monotone iteration technique. The results show that whether the species dies out or not depends on the relations among the birth rate, the death rate, the competition rate, the diffusivity and the hereditary effects. The evolution of the population may show asymptotic periodicity, provided a certain condition is satisfied for the above factors.

MSC: 35B10; 35B40; 35K55

Keywords: Volterra; Asymptotic periodicity; Delay; Existence; Uniqueness

1. Introduction

A simple model for describing the evolution of a single species population is given by Volterra in [11] as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = x(t)\left[a - bx(t) - \int_0^\infty K(\tau)x(t-\tau)\mathrm{d}\tau\right], \quad t \in R^+,$$
$$x(t) = \phi(t) \geq 0, \quad t \in R_0^-,$$

where $R^+ = (0, \infty)$, $R_0^- = (-\infty, 0]$, and $x(t)$ is the population density, a and b are positive constants. The integral part is a hereditary term concerning the effect of the past history on the present growth rate.

In [1,8,9] the researchers studied the Volterra type population model with variable coefficients in the case the growth rate is affected by the variation of the environment where the diffusion was also taken into consideration. For other arguments regarding the Volterra equations one can also refer to [2,7,12]. In [9] the authors use comparison methods and get some prior bounds on the solution, where the bounds are positive constants related to the homogeneous Neumann boundary condition. In this paper we will make a further study of the asymptotic behavior of the solution, for the case where the coefficients vary periodically in time t; we can prove that the solution has asymptotic periodicity

* Corresponding author.
E-mail addresses: wangjinliang10@sina.com.cn (J.L. Wang), zhouli@public.wh.hb.cn (L. Zhou), tangyb@public.wh.hb.cn (Y.B. Tang).

0362-546X/$ - see front matter
doi:10.1016/j.na.2006.10.050

in the long run. Our hypotheses as regards the coefficients are based on the birth and death rates, rates of diffusion, rates of interaction, and environment carrying capacities possibly varying on a seasonal scale. The model is given as follows:

$$Lu = u\left[a - bu - c\int_0^\infty u(t-\tau,x)\mathrm{d}\mu(\tau)\right], \quad (t,x) \in R^+ \times \Omega, \tag{1.1}$$

$$B[u](t,x) = 0, \quad (t,x) \in R^+ \times \partial\Omega, \tag{1.2}$$

$$u(t,x) = \phi(t,x), \quad (t,x) \in R_0^- \times \overline{\Omega}. \tag{1.3}$$

We give fundamental hypotheses more general than those in [9] as follows:

(H_1) Ω is a bounded domain in R^n with boundary $\partial\Omega \in C^{2+\alpha}$ $(0 < \alpha < 1)$. The differential operator L is defined as $L = \partial/\partial t + A$, and

$$Af(x) = -\sum_{i,j=1}^{n} \alpha_{ij}(t,x)\frac{\partial^2 f(x)}{\partial x_i \partial x_j} + \sum_{j=1}^{n} \beta_j(t,x)\frac{\partial f(x)}{\partial x_j},$$

where all the coefficients α_{ij}, β_j are Hölder continuous in t, x and T-periodic in t; here T is a positive constant. A is uniformly strong elliptic, i.e. there is a constant $\delta > 0$ such that, for all $(\xi_1, \xi_2, \ldots, \xi_n) \in R^n$,

$$\sum_{i,j=1}^{n} \alpha_{ij}(t,x)\xi_i\xi_j \geq \delta\sum_{j=1}^{n}\xi_j^2, \quad (t,x) \in [0,T] \times \Omega.$$

The boundary condition is given by $B[u] = u$ or $B[u] = \partial u/\partial n + \gamma(x)u$, where $\gamma \in C^{1+\alpha}(\partial\Omega)$ is a nonnegative bounded function, $\partial/\partial n$ denotes the outward normal derivative on $\partial\Omega$.

(H_2) The coefficients $a(t,x)$, $b(t,x)$, $c(t,x)$ are T-periodic in t and Hölder continuous on $[0,T] \times \overline{\Omega}$ with $b(t,x) > 0, c(t,x) \geq 0$ on $[0,T] \times \overline{\Omega}$. We also denote as a_1, b_1, c_1 and a_2, b_2, c_2 the minimum and maximum values of $a(t,x)$, $b(t,x)$, $c(t,x)$ on $[0,T] \times \overline{\Omega}$ with $c_2 > 0$, respectively.

(H_3) $\phi \in C(R_0^- \times \overline{\Omega})$ is a nonnegative bounded function. $\mu(\cdot)$ is a bounded variation function with $\mu(0) = 0$. Let $M(t)$ denote the total variation of $\mu(\cdot)$ on $[0,t]$, and assume $M_0 = \lim_{t\to\infty} M(t) < \infty$ and $\mu_0 = \lim_{t\to\infty}\mu(t) < \infty$.

Define $M^{\pm}(t) = (M(t) \pm \mu(t))/2$. It follows from ($H_3$) that $M(t)$ and $M^{\pm}(t)$ are nonnegative and nondecreasing in R^+. Set $M_0^{\pm} = \lim_{t\to\infty} M^{\pm}(t) < \infty$; then

$$M_0^+ + M_0^- = M_0, \qquad M_0^+ - M_0^- = \mu_0. \tag{1.4}$$

(H_4) Assume $a_2 > 0$ and $b_1 > c_2 M_0^-$.

This paper is arranged as follows. In the next section we study the quasi-solutions of the boundary value problem (1.1) and (1.2). Meanwhile, the asymptotic behavior of the solution for the initial–boundary value problem (1.1)–(1.3) is also considered. Section 3 concerns the proof of the existence and uniqueness of the periodic solution for the boundary value problem (1.1) and (1.2). And the last section concerns the numerical simulations.

2. Quasi-solutions and asymptotic behavior

Lemma 2.1. *If there exist a pair of functions $\overline{u}$ and $\underline{u}$ (called coupled upper and lower solutions) such that $\overline{u} \geq \underline{u}$ on $R \times \overline{\Omega}$, and they satisfy the following inequalities:*

$$\begin{aligned}
&L\overline{u} \geq \overline{u}\left[a - b\overline{u} + c\int_0^\infty \overline{u}(t-\tau,x)\mathrm{d}M^-(\tau) - c\int_0^\infty \underline{u}(t-\tau,x)\mathrm{d}M^+(\tau)\right],\\
&L\underline{u} \leq \underline{u}\left[a - b\underline{u} + c\int_0^\infty \underline{u}(t-\tau,x)\mathrm{d}M^-(\tau) - c\int_0^\infty \overline{u}(t-\tau,x)\mathrm{d}M^+(\tau)\right],\\
&B[\overline{u}](t,x) \geq 0 \geq B[\underline{u}](t,x), \quad (t,x) \in R^+ \times \partial\Omega,\\
&\overline{u}(t,x) \geq \phi(t,x) \geq \underline{u}(t,x), \quad (t,x) \in R_0^- \times \overline{\Omega},
\end{aligned} \tag{2.1}$$

where the first and the second inequalities are defined on $R^+ \times \Omega$, then the initial–boundary value problem (1.1)–(1.3) has a unique solution u with $\overline{u} \geq u \geq \underline{u}$ on $R \times \overline{\Omega}$.

It follows from $\mu(\tau) = M^+(\tau) - M^-(\tau)$ that

$$\int_0^\infty u(t-\tau, x)\mathrm{d}\mu(\tau) = \int_0^\infty u(t-\tau, x)\mathrm{d}M^+(\tau) - \int_0^\infty u(t-\tau, x)\mathrm{d}M^-(\tau).$$

In view of the monotonicity of $M^+(\cdot)$ and $M^-(\cdot)$, we have

$$\begin{cases} 0 \le \int_0^\infty u_1(t-\tau, x)\mathrm{d}M^+(\tau) \le \int_0^\infty u_2(t-\tau, x)\mathrm{d}M^+(\tau), \\ 0 \le \int_0^\infty u_1(t-\tau, x)\mathrm{d}M^-(\tau) \le \int_0^\infty u_2(t-\tau, x)\mathrm{d}M^-(\tau), \end{cases} \tag{2.2}$$

for $0 \le u_1 \le u_2$. Hence, by reference to the monotonicity method in [5,6], the above lemma is easy to prove.

To study the asymptotic behavior, we introduce some results which were given by [3] for the following boundary value problem:

$$\begin{aligned} &Lu(t,x) = u(t,x)[e(t,x) - b(t,x)u(t,x)], \quad (t,x) \in R^+ \times \Omega, \\ &B[u] = 0, \quad (t,x) \in R^+ \times \partial\Omega, \end{aligned} \tag{2.3}$$

where $e(t,x)$ and $b(t,x)$ are T-periodic in t with $b(t,x) > 0$, and the other conditions are the same as for (1.1) and (1.2).

Proposition 2.1 (*Theorem in [3]*). *The eigenvalue problem*

$$\begin{aligned} &L\varphi(t,x) - e(t,x)\varphi(t,x) = \sigma(e)\varphi(t,x), \quad (t,x) \in R^+ \times \Omega, \\ &B[\varphi](t,x) = 0, \quad (t,x) \in R^+ \times \partial\Omega, \\ &\varphi \text{ is } T\text{-periodic in } t, \end{aligned} \tag{2.4}$$

has a principal eigenvalue $\sigma(e)$ with positive eigenfunction.

(1) *If $\sigma(e) \ge 0$, then the trivial solution 0 is globally asymptotically stable in (2.3) with respect to every nonnegative initial condition.*

(2) *If $\sigma(e) < 0$, then the problem (2.3) admits a positive T-periodic solution $\theta(t,x)$ which is globally asymptotically stable with respect to every nonnegative, nontrivial initial function.*

It is easy to check that 0 is a lower solution of (1.1)–(1.3). We search for a positive constant upper solution by solving

$$P\left[a - bP + c\int_0^\infty P\mathrm{d}M^-(\tau) - c\int_0^\infty 0\mathrm{d}M^+(\tau)\right] \le 0,$$

which yields $P \ge a/(b - cM_0^-)$. Noticing that $a_1 \le a \le a_2, b_1 \le b \le b_2, c_1 \le c \le c_2$ and the relations in hypothesis (H_4), we choose $P = a_2/(b_1 - c_2M_0^-)$.

Then for every initial function ϕ with value in $[0, P]$, it is easy to check that P and 0 are a pair of coupled upper and lower solutions of (1.1)–(1.3). So it follows from Lemma 2.1 that the problem (1.1)–(1.3) has a unique solution on $R \times \overline{\Omega}$.

Theorem 2.1. *Under the hypotheses* (H_1)–(H_4), *if $\sigma(a(t,x) + PM_0^- c(t,x)) \ge 0$, then the trivial solution 0 is globally asymptotically stable in (1.1)–(1.3) with respect to every nonnegative initial function $\phi(t,x)$ with $0 \le \phi(t,x) \le P$.*

Proof. Let $U(t,x)$ be the solution of the following parabolic problem:

$$\begin{aligned} &LU = U[a + PM_0^- c - bU], \quad (t,x) \in R^+ \times \Omega, \\ &B[U](t,x) = 0, \quad (t,x) \in R^+ \times \partial\Omega, \\ &U(0,x) = \phi(0,x), \quad x \in \overline{\Omega}. \end{aligned} \tag{2.5}$$

As $\phi(0,x) \ge 0$ on $\overline{\Omega}$, it is easy to see that $U(t,x) \ge 0$ on $R^+ \times \overline{\Omega}$. Define the function $\tilde{U}(t,x)$ as $\tilde{U}(t,x) = \phi(t,x)$ on $R_0^- \times \overline{\Omega}$ and $\tilde{U}(t,x) = U(t,x)$ on $R^+ \times \overline{\Omega}$; then $\tilde{U}$ and 0 are a pair of upper and lower solutions of (1.1)–(1.3) on

$R \times \overline{\Omega}$ according to (2.1). Therefore, by Lemma 2.1, there exists a unique solution u for (1.1)–(1.3) with $0 \le u \le \tilde{U}$ on $R \times \overline{\Omega}$. Let $e(t,x) = a + PM_0^- c$; then $\sigma(e) = \sigma(a + PM_0^- c) \ge 0$, and it follows from Proposition 2.1 that

$$\lim_{t\to\infty} \|u(t,\cdot)\|_{C(\overline{\Omega})} \le \lim_{t\to\infty} \|\tilde{U}(t,\cdot)\|_{C(\overline{\Omega})} = \lim_{t\to\infty} \|U(t,\cdot)\|_{C(\overline{\Omega})} = 0.$$

The proof is finished. □

For the case $\sigma(a + PM_0^- c) < 0$, Proposition 2.1 implies the existence of a periodic solution θ_0 of the boundary value problem (2.3) for $e = a + PM_0^- c$.

We consider the system below:

$$\begin{aligned} L\theta_1 &= \theta_1 \left[a - b\theta_1 + c\int_0^\infty \theta_1(t-\tau,x)\mathrm{d}M^-(\tau) - c\int_0^\infty \theta_2(t-\tau,x)\mathrm{d}M^+(\tau)\right],\\ L\theta_2 &= \theta_2 \left[a - b\theta_2 + c\int_0^\infty \theta_2(t-\tau,x)\mathrm{d}M^-(\tau) - c\int_0^\infty \theta_1(t-\tau,x)\mathrm{d}M^+(\tau)\right],\\ B[\theta_1](t,x) &= B[\theta_2](t,x) = 0, \quad (t,x) \in R \times \partial\Omega, \end{aligned} \tag{2.6}$$

where the first and the second equations are defined on $R^+ \times \Omega$. Define

$$\theta^*(t,x) \equiv c(t,x)\int_0^\infty \theta_0(t-\tau,x)\mathrm{d}M^+(\tau).$$

It follows from $\theta_0 > 0$ and $c \ge 0$ on $R^+ \times \Omega$ that $\theta^* \ge 0$. In fact, $\theta^*(t,x)$ is also T-periodic in t;

$$\begin{aligned} \theta^*(t+T,x) &= c(t+T,x)\int_0^\infty \theta_0(t+T-\tau,x)\mathrm{d}M^+(\tau)\\ &= c(t,x)\int_0^\infty \theta_0(t-\tau,x)\mathrm{d}M^+(\tau) = \theta^*(t,x). \end{aligned}$$

Now let $\Theta(t,x)$ be the positive T-periodic solution of the following problem:

$$\begin{aligned} &Lv(t,x) = v(t,x)[a(t,x) - \theta^*(t,x) - b(t,x)v(t,x)], \quad (t,x) \in R^+ \times \Omega,\\ &B[v](t,x) = 0, \quad (t,x) \in R \times \partial\Omega,\\ &v(t,x) \text{ is } T\text{-periodic in } t. \end{aligned} \tag{2.7}$$

It is easy to check that θ_0 and Θ are upper and lower solutions of problem (1.1) and (1.2) according to the relations in (2.1). At the same time, (θ_0, θ_0), (Θ, Θ) can be also seen as a pair of upper and lower solutions of system (2.6).

Theorem 2.2. *Under the hypotheses* (H_1)–(H_4), *if* $\sigma(a + PM_0^- c) < 0$ *and* $\sigma(a - \theta^*) < 0$, *then the boundary value problem* (1.1) *and* (1.2) *has a pair of ordered positive* T*-periodic upper and lower quasi-solutions* $\overline{\theta}$, $\underline{\theta}$ *which satisfy* (2.6) *with* $\Theta \le \underline{\theta} \le \overline{\theta} \le \theta_0$ *on* $R \times \overline{\Omega}$. *And for every nonnegative nontrivial initial function* ϕ *with* $0 \le \phi \le P$, *the time-dependent solution* u *of* (1.1)–(1.3) *satisfies*

$$\liminf_{t\to\infty}[u(t,\cdot) - \underline{\theta}(t,\cdot)] \ge 0 \ge \limsup_{t\to\infty}[u(t,\cdot) - \overline{\theta}(t,\cdot)] \quad \text{in } C(\overline{\Omega}). \tag{2.8}$$

Proof. For the eigenvalue problem (2.4) with $e = a + PM_0^- c$, noticing that $\sigma(a + PM_0^- c) < 0$ and $0 \le u \le \tilde{U}$, it follows from Proposition 2.1 that

$$\limsup_{t\to\infty}[u(t,\cdot) - \theta_0(t,\cdot)] \le \lim_{t\to\infty}[\tilde{U}(t,\cdot) - \theta_0(t,\cdot)] = \lim_{t\to\infty}[U(t,\cdot) - \theta_0(t,\cdot)] = 0, \tag{2.9}$$

where U and $\tilde{U}$ are given in the proof of Theorem 2.1. The relations in (2.9) imply that for every $\varepsilon > 0$, there exists $t_1 > 0$; in the case $t > t_1$,

$$u \le U \le \theta_0 + \varepsilon/(2c_2M_0^+).$$

Noting that $\lim_{t\to\infty} M^+(t) = M_0^+$, for the same ε, there exists $t_2 > 0$, in the case $t > t_2$,

$$M_0^+ - M^+(t) \le \varepsilon/(2c_2 P).$$

Let $T_\varepsilon = \max\{t_1, t_2\}$; then the above two inequalities are satisfied for $t \ge 2T_\varepsilon$, and

$$\begin{aligned} c\int_0^\infty u(t-\tau,x)\mathrm{d}M^+(\tau) &= c\left\{\int_0^{T_\varepsilon} u(t-\tau,x)\mathrm{d}M^+(\tau) + \int_{T_\varepsilon}^\infty u(t-\tau,x)\mathrm{d}M^+(\tau)\right\} \\ &\le c\left\{\int_0^{T_\varepsilon}\left[\theta_0(t-\tau,x) + \frac{\varepsilon}{2c_2 M_0^+}\right]\mathrm{d}M^+(\tau) + P\left[M_0^+ - M^+(T_\varepsilon)\right]\right\} \\ &\le c\int_0^\infty \theta_0(t-\tau,x)\mathrm{d}M^+(\tau) + \frac{c\varepsilon}{2c_2 M_0^+}\cdot M_0^+ + \frac{c\varepsilon}{2c_2} \\ &\le \theta^* + \varepsilon. \end{aligned} \tag{2.10}$$

Hence for $(t,x) \in (2T_\varepsilon, \infty)\times\Omega$, we have

$$\begin{aligned} Lu &= u\left[a - bu + c\int_0^\infty u(t-\tau,x)\mathrm{d}M^-(\tau) - c\int_0^\infty u(t-\tau,x)\mathrm{d}M^+(\tau)\right] \\ &\ge u\left[a - bu - c\int_0^\infty u(t-\tau,x)\mathrm{d}M^+(\tau)\right] \\ &\ge u(a - bu - \varepsilon - \theta^*). \end{aligned}$$

The comparison arguments imply $u(t,x) \ge V(t,x)$ on $[2T_\varepsilon,\infty)\times\overline{\Omega}$, where V is the solution of the following parabolic initial–boundary value problem:

$$\begin{aligned} &LV = V\left[(a-\varepsilon-\theta^*) - bV\right], \quad (t,x)\in(2T_\varepsilon,\infty)\times\Omega, \\ &B[V](t,x) = 0, \quad (t,x)\in(2T_\varepsilon,\infty)\times\partial\Omega, \\ &V(2T_\varepsilon,x) = u(2T_\varepsilon,x), \quad x\in\Omega. \end{aligned}$$

In relation to problem (2.7), the arbitrariness of ε and Proposition 2.1 imply that

$$\liminf_{t\to\infty}[u(t,\cdot) - \Theta(t,\cdot)] \ge \lim_{t\to\infty}[V(t,\cdot) - \Theta(t,\cdot)] = 0 \quad \text{in } C(\overline{\Omega}). \tag{2.11}$$

Let $\overline{\theta}^{(0)} = \theta_0$, $\underline{\theta}^{(0)} = \Theta$; then it is known from (2.9) and (2.11) that

$$\liminf_{t\to\infty}[u(t,\cdot) - \underline{\theta}^{(0)}(t,\cdot)] \ge 0; \qquad \limsup_{t\to\infty}[u(t,\cdot) - \overline{\theta}^{(0)}(t,\cdot)] \le 0 \quad \text{in } C(\overline{\Omega}). \tag{2.12}$$

Relation (2.12) also indicates that $\overline{\theta}^{(0)} \ge \underline{\theta}^{(0)}$ on $[0,T]\times\overline{\Omega}$. Let $\{\overline{\theta}^{(k)}\}$ and $\{\underline{\theta}^{(k)}\}$ $(k = 0,1,2,\ldots)$ be two sequences of T-periodic functions to be defined in the following, and $\overline{\theta}_+^{(k)}$, $\underline{\theta}_+^{(k)}$, $\overline{\theta}_-^{(k)}$, $\underline{\theta}_-^{(k)}$ $(k = 0,1,2,\ldots)$ be the positive T-periodic functions defined as

$$\begin{aligned} \overline{\theta}_+^{(k)}(t,x) &= c(t,x)\int_0^\infty \overline{\theta}^{(k)}(t-\tau,x)\mathrm{d}M^+(\tau); \\ \underline{\theta}_+^{(k)}(t,x) &= c(t,x)\int_0^\infty \underline{\theta}^{(k)}(t-\tau,x)\mathrm{d}M^+(\tau); \\ \overline{\theta}_-^{(k)}(t,x) &= c(t,x)\int_0^\infty \overline{\theta}^{(k)}(t-\tau,x)\mathrm{d}M^-(\tau); \\ \underline{\theta}_-^{(k)}(t,x) &= c(t,x)\int_0^\infty \underline{\theta}^{(k)}(t-\tau,x)\mathrm{d}M^-(\tau). \end{aligned}$$

We construct periodic sequences as follows:

$$\begin{aligned} L\overline{\theta}^{(k)} &= \overline{\theta}^{(k)}\left(a - b\overline{\theta}^{(k)} + \overline{\theta}_-^{(k-1)} - \underline{\theta}_+^{(k-1)}\right), \quad (t,x) \in R^+ \times \Omega, \\ L\underline{\theta}^{(k)} &= \underline{\theta}^{(k)}\left(a - b\underline{\theta}^{(k)} + \underline{\theta}_-^{(k-1)} - \overline{\theta}_+^{(k-1)}\right), \quad (t,x) \in R^+ \times \Omega, \\ B[\overline{\theta}^{(k)}](t,x) &= B[\underline{\theta}^{(k)}](t,x) = 0, \quad (t,x) \in R \times \partial\Omega,\ k \in N. \end{aligned} \tag{2.13}$$

Lemma 2.2. *The sequences $\{\overline{\theta}^{(k)}\}$ and $\{\underline{\theta}^{(k)}\}$ $(k = 0, 1, 2, \ldots)$ given by (2.13) are well defined, and they possess the monotone property*

$$\theta_0 = \overline{\theta}^{(0)} \geq \overline{\theta}^{(k)} \geq \overline{\theta}^{(k+1)} \geq \underline{\theta}^{(k+1)} \geq \underline{\theta}^{(k)} \geq \underline{\theta}^{(0)} = \Theta, \quad (t,x) \in [0,T] \times \overline{\Omega}. \tag{2.14}$$

Moreover, for each integer k,

$$\liminf_{t\to\infty}[u(t,\cdot) - \underline{\theta}^{(k)}(t,\cdot)] \geq 0; \qquad \limsup_{t\to\infty}[u(t,\cdot) - \overline{\theta}^{(k)}(t,\cdot)] \leq 0 \quad \text{in } C(\overline{\Omega}). \tag{2.15}$$

Proof. The regularity arguments in [6,12] imply that the sequences are well defined. In the following, we prove (2.14) and (2.15). First, we prove that relations (2.14) and (2.15) are satisfied for $k = 1$.

Step 1. To prove $\overline{\theta}^{(0)} \geq \overline{\theta}^{(1)}$. Notice that

$$\begin{aligned} \underline{\theta}_+^{(0)}(t,x) &= c(t,x)\int_0^\infty \Theta(t-\tau,x)\mathrm{d}M^+(\tau) \geq 0, \\ \overline{\theta}_-^{(0)}(t,x) &= c(t,x)\int_0^\infty \theta_0(t-\tau,x)\mathrm{d}M^-(\tau) \leq PM_0^- c(t,x), \end{aligned}$$

on $R^+ \times \Omega$; it follows from the iteration schemes (2.13) that

$$L\overline{\theta}^{(1)} = \overline{\theta}^{(1)}\left(a - b\overline{\theta}^{(1)} + \overline{\theta}_-^{(0)} - \underline{\theta}_+^{(0)}\right) \leq \overline{\theta}^{(1)}\left(a - b\overline{\theta}^{(1)} + PM_0^- c\right).$$

At the same time, for $(t,x) \in R \times \partial\Omega$ we have $B[\overline{\theta}^{(1)}] = 0$. So $\overline{\theta}^{(1)}$ is a lower solution of (2.3) with $e = a + PM_0^- c$ and $\overline{\theta}^{(1)} \leq \overline{\theta}^{(0)}$.

Step 2. To prove $\underline{\theta}^{(1)} \geq \underline{\theta}^{(0)}$. From (2.13), we have

$$L\underline{\theta}^{(1)} = \underline{\theta}^{(1)}\left(a - b\underline{\theta}^{(1)} + \underline{\theta}_-^{(0)} - \overline{\theta}_+^{(0)}\right) \geq \underline{\theta}^{(1)}\left(a - b\underline{\theta}^{(1)} - \overline{\theta}_+^{(0)}\right),$$

so $\underline{\theta}^{(1)}$ is an upper solution of problem (2.7) and $\underline{\theta}^{(1)} \geq \underline{\theta}^{(0)} = \Theta$.

Step 3. To prove $\underline{\theta}^{(1)} \leq \overline{\theta}^{(1)}$ and

$$\liminf_{t\to\infty}[u(t,\cdot) - \underline{\theta}^{(1)}(t,\cdot)] \geq 0 \geq \limsup_{t\to\infty}[u(t,\cdot) - \overline{\theta}^{(1)}(t,\cdot)] \quad \text{in } C(\overline{\Omega}).$$

Since $\theta_0(t,x) \geq \Theta(t,x)$ on $R \times \Omega$ we obtain

$$\begin{aligned} \left(\overline{\theta}_-^{(0)} - \underline{\theta}_+^{(0)}\right) - \left(\underline{\theta}_-^{(0)} - \overline{\theta}_+^{(0)}\right) &= \left[\overline{\theta}_-^{(0)} - \underline{\theta}_-^{(0)}\right] + \left[\overline{\theta}_+^{(0)} - \underline{\theta}_+^{(0)}\right] \\ &= c\left\{\int_0^\infty [\theta_0(t-\tau,x) - \Theta(t-\tau,x)]\,\mathrm{d}M^-(\tau) \right. \\ &\quad \left. + \int_0^\infty [\theta_0(t-\tau,x) - \Theta(t-\tau,x)]\,\mathrm{d}M^+(\tau)\right\} \geq 0. \end{aligned}$$

Hence

$$Lu = u\left(a - bu + \overline{\theta}_-^{(0)} - \underline{\theta}_+^{(0)}\right) \geq u\left(a - bu + \underline{\theta}_-^{(0)} - \overline{\theta}_+^{(0)}\right).$$

By using the comparison result, it follows from (2.13) that $\overline{\theta}^{(1)} \geq \underline{\theta}^{(1)}$.

Replacing $M^+(\tau)$ by $M^-(\tau)$ in (2.10), we see that, for each $\varepsilon > 0$, there exists $t_3 > 0$; in the case $t > t_3$,

$$c(t,x)\int_0^\infty u(t-\tau,x)\mathrm{d}M^-(\tau) \le c(t,x)\int_0^\infty \theta_0(t-\tau,x)\mathrm{d}M^-(\tau)+\varepsilon = \overline{\theta}_-^{(0)}(t,x)+\varepsilon.$$

It follows from (2.11) that for the same ε, there exists $t_4 > 0$; in the case $t > t_4$,

$$u(t,\cdot) \ge V(t,\cdot) \ge \underline{\theta}^{(0)}(t,\cdot) - \varepsilon/(2c_2M_0^+).$$

Considering that $\underline{\theta}^{(0)}(t,x)$ is T-periodic in t, and continuous on $[0,T]\times\overline{\Omega}$, we can set

$$\underline{\theta}_M = \max_{[0,T]\times\overline{\Omega}} \underline{\theta}^{(0)}(t,x).$$

Noticing that $\lim_{t\to\infty} M^+(t) = M_0^+$, for the above ε, there should exist $t_5 > 0$; and in the case $t > t_5$,

$$M_0^+ - M^+(t) \le \varepsilon/(2c_2\underline{\theta}_M).$$

Let $\delta = \max\{t_3, t_4, t_5\}$. Then the above three inequalities are satisfied for $t \ge 2\delta$. Now considering that

$$\int_\delta^\infty u(t-\tau,x)\mathrm{d}M^+(\tau) \ge 0,$$

we can get the following relations:

$$\begin{aligned}
c\int_0^\infty u(t-\tau,x)\mathrm{d}M^+(\tau) &\ge c\left\{\int_0^\delta \left[\underline{\theta}^{(0)}(t-\tau,x) - \frac{\varepsilon}{2c_2M_0^+}\right]\mathrm{d}M^+(\tau) + \int_\delta^\infty u(t-\tau,x)\mathrm{d}M^+(\tau)\right\}\\
&\ge c\left\{\int_0^\infty \underline{\theta}^{(0)}(t-\tau,x)\mathrm{d}M^+(\tau) - \int_\delta^\infty \underline{\theta}^{(0)}(t-\tau,x)\mathrm{d}M^+(\tau) - \frac{\varepsilon}{2c_2M_0^+}\cdot M_0^+\right\}\\
&\ge \underline{\theta}_+^{(0)} - c\left[\underline{\theta}_M\left(M_0^+ - M(t-\delta)\right) + \frac{\varepsilon}{2c_2}\right]\\
&\ge \underline{\theta}_+^{(0)} - c\left[\underline{\theta}_M \cdot \frac{\varepsilon}{2c_2\underline{\theta}_M} + \frac{\varepsilon}{2c_2}\right]\\
&\ge \underline{\theta}_+^{(0)} - \varepsilon.
\end{aligned}$$

It follows that

$$\begin{aligned}
Lu &= u\left[a - bu + c\int_0^\infty u(t-\tau,x)\mathrm{d}M^-(\tau) - c\int_0^\infty u(t-\tau,x)\mathrm{d}M^+(\tau)\right]\\
&\le u(t,x)\left[\left(a + \overline{\theta}_-^{(0)} - \underline{\theta}_+^{(0)} + 2\varepsilon\right) - bu\right].
\end{aligned}$$

By the comparison argument, we have $u(t,x) \le \overline{U}(t,x)$ on $[2\delta,\infty)\times\overline{\Omega}$, where $\overline{U}(t,x)$ is a solution of the following problem:

$$\begin{aligned}
&L\overline{U} = \overline{U}\left[\left(a + \overline{\theta}_-^{(0)} - \underline{\theta}_+^{(0)} + 2\varepsilon\right) - b\overline{U}\right], \quad (t,x)\in(2\delta,\infty)\times\Omega,\\
&B[\overline{U}](t,x) = 0, \quad (t,x)\in(2\delta,\infty)\times\partial\Omega,\\
&\overline{U}(2\delta,x) = u(2\delta,x), \quad x\in\overline{\Omega}.
\end{aligned} \tag{2.16}$$

It follows from $a + \overline{\theta}_-^{(0)} - \underline{\theta}_+^{(0)} \ge a - \underline{\theta}_+^{(0)} \ge a - \theta^*$ that

$$\sigma(a + \overline{\theta}_-^{(0)} - \underline{\theta}_+^{(0)}) \le \sigma(a - \underline{\theta}_+^{(0)}) \le \sigma(a-\theta^*) < 0.$$

The arbitrariness of ε implies that

$$\limsup_{t\to\infty}[u(t,\cdot) - \overline{\theta}^{(1)}(t,\cdot)] \le \lim_{t\to\infty}[\overline{U}(t,\cdot) - \overline{\theta}^{(1)}(t,\cdot)] = 0 \quad \text{in } C(\overline{\Omega}), \tag{2.17}$$

where $\overline{\theta}^{(1)}$ is the positive T-periodic solution of the following problem:

$$\begin{aligned}
&Lu = u\left(a + \overline{\theta}_{-}^{(0)}(t,x) - \underline{\theta}_{+}^{(0)} - bu\right) \quad \text{in } R^{+} \times \Omega,\\
&B[u](t,x) = 0 \quad \text{on } R \times \partial\Omega.
\end{aligned}$$

Through the same process, we can get

$$\liminf_{t\to\infty}[u(t,\cdot) - \underline{\theta}^{(1)}(t,\cdot)] \geq 0. \tag{2.18}$$

So from the above three steps, we know that (2.14) and (2.15) hold for $k = 1$.

We continue our discussion by deduction as follows. If the relations in (2.14) hold for integer $k-1$, then according to relation (2.2),

$$\begin{aligned}
&\theta^{*} = \overline{\theta}_{+}^{(0)} \geq \overline{\theta}_{+}^{(k-1)} \geq \overline{\theta}_{+}^{(k)} \geq \underline{\theta}_{+}^{(k)} \geq \underline{\theta}_{+}^{(k-1)} \geq \underline{\theta}_{+}^{(0)},\\
&\overline{\theta}_{-}^{(0)} \geq \overline{\theta}_{-}^{(k-1)} \geq \overline{\theta}_{-}^{(k)} \geq \underline{\theta}_{-}^{(k)} \geq \underline{\theta}_{-}^{(k-1)} \geq \underline{\theta}_{-}^{(0)},
\end{aligned}$$

and

$$\begin{aligned}
&a + \underline{\theta}_{-}^{(k)} - \overline{\theta}_{+}^{(k)} \geq a + \underline{\theta}_{-}^{(k-1)} - \overline{\theta}_{+}^{(k-1)} \geq a + \underline{\theta}_{-}^{(k-1)} - \theta^{*} \geq a - \theta^{*},\\
&a + \overline{\theta}_{-}^{(k)} - \underline{\theta}_{+}^{(k)} \geq a - \underline{\theta}_{+}^{(k)} \geq a - \overline{\theta}_{+}^{(0)} = a - \theta^{*}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\sigma(a + \underline{\theta}_{-}^{(k)} - \overline{\theta}_{+}^{(k)}) \leq \sigma(a - \theta^{*}) < 0,\\
&\sigma(a + \overline{\theta}_{-}^{(k)} - \underline{\theta}_{+}^{(k)}) \leq \sigma(a - \theta^{*}) < 0,
\end{aligned}$$

which fulfil all the conditions for Proposition 2.1, and the iteration and induction process can continue through. Using the same method as previously, the induction process implies (2.14) and (2.15). For the iterations methods one can also refer to [4,5,15]. The proof of this lemma is finished. □

Lemma 2.2 implies the existence of the limits of $\{\overline{\theta}^{(k)}\}$ and $\{\underline{\theta}^{(k)}\}$; say

$$\lim_{k\to\infty} \overline{\theta}^{(k)}(t,x) = \overline{\theta}(t,x), \qquad \lim_{k\to\infty} \underline{\theta}^{(k)} = \underline{\theta}(t,x), \quad (t,x) \in [0,T] \times \overline{\Omega}, \tag{2.19}$$

where $\overline{\theta}, \underline{\theta}$ are positive T-periodic functions with $\overline{\theta} \geq \underline{\theta}$ on $[0,T] \times \overline{\Omega}$.

The regularity argument in [3,6] shows that $\overline{\theta}, \underline{\theta}$ are Hölder continuous on $R^{+} \times \overline{\Omega}$. Let $k \to \infty$ in (2.13); we know that $\overline{\theta}$ and $\underline{\theta}$ also satisfy the equations in (2.6), i.e.

$$\begin{aligned}
&L\overline{\theta} = \overline{\theta}\left[a - b\overline{\theta} + c\int_{0}^{\infty} \overline{\theta}(t-\tau,x)\mathrm{d}M^{-}(\tau) - c\int_{0}^{\infty} \underline{\theta}(t-\tau,x)\mathrm{d}M^{+}(\tau)\right],\\
&L\underline{\theta} = \underline{\theta}\left[a - b\underline{\theta} + c\int_{0}^{\infty} \underline{\theta}(t-\tau,x)\mathrm{d}M^{-}(\tau) - c\int_{0}^{\infty} \overline{\theta}(t-\tau,x)\mathrm{d}M^{+}(\tau)\right],\\
&B[\overline{\theta}](t,x) = B[\underline{\theta}](t,x) = 0, \quad (t,x) \in R \times \partial\Omega,
\end{aligned} \tag{2.20}$$

where the first and the second equations are defined on $R^{+} \times \Omega$. Furthermore, it is easy to check that $\overline{\theta}$ and $\underline{\theta}$ are a pair of coupled upper and lower solutions of problem (1.1) and (1.2). It follows from (2.15) that

$$\liminf_{t\to\infty}[u(t,\cdot) - \underline{\theta}(t,\cdot)] \geq 0, \qquad \limsup_{t\to\infty}[u(t,\cdot) - \overline{\theta}(t,\cdot)] \leq 0 \quad \text{in } C(\overline{\Omega}),$$

and this finishes the proof of Theorem 2.2. □

3. Existence and uniqueness

To study the existence of periodic solutions of the boundary value problem (1.1) and (1.2), it suffices to show $\overline{\theta} \equiv \underline{\theta}$. For convenience of argument, here we only consider the case $L = \partial/\partial t - \triangle$, where $\triangle$ denotes the Laplacian. Define

$$\begin{aligned}
&\overline{\theta}_t = \frac{\partial \overline{\theta}}{\partial t}, \qquad \underline{\theta}_t = \frac{\partial \underline{\theta}}{\partial t},\\
&\theta_{0-} = \int_0^\infty \theta_0(t-\tau, x)\mathrm{d}M^-(\tau), \qquad \Theta_+ = \int_0^\infty \Theta(t-\tau, x)\mathrm{d}M^+(\tau),\\
&\overline{\theta}_+ = \int_0^\infty \overline{\theta}(t-\tau, x)\mathrm{d}M^+(\tau), \qquad \underline{\theta}_+ = \int_0^\infty \underline{\theta}(t-\tau, x)\mathrm{d}M^+(\tau),\\
&\overline{\theta}_- = \int_0^\infty \overline{\theta}(t-\tau, x)\mathrm{d}M^-(\tau), \qquad \underline{\theta}_- = \int_0^\infty \underline{\theta}(t-\tau, x)\mathrm{d}M^-(\tau),\\
&(\overline{\theta} - \underline{\theta})_M = \int_0^\infty [\overline{\theta}(t-\tau, x) - \underline{\theta}(t-\tau, x)]\mathrm{d}M(\tau).
\end{aligned} \tag{3.1}$$

From (2.20) we can get the following relations:

$$\begin{aligned}
(\overline{\theta}_t - \underline{\theta}_t) - \triangle(\overline{\theta} - \underline{\theta}) &= a(\overline{\theta} - \underline{\theta}) - b(\overline{\theta}^2 - \underline{\theta}^2) + c(\overline{\theta}\,\overline{\theta}_- - \underline{\theta}\,\underline{\theta}_-) - c(\overline{\theta}\underline{\theta}_+ - \underline{\theta}\overline{\theta}_+)\\
&= [a - b(\overline{\theta} + \underline{\theta}) + c(\overline{\theta}_- - \underline{\theta}_+)](\overline{\theta} - \underline{\theta}) + c\underline{\theta}(\overline{\theta} - \underline{\theta})_M.
\end{aligned} \tag{3.2}$$

In the following, we use the method of [13,14] to search for the conditions for $\overline{\theta} \equiv \underline{\theta}$ according to the boundary conditions.

Part A. Dirichlet condition

For the Dirichlet condition $\overline{\theta} = \underline{\theta} = 0$ on $\partial\Omega$, consider that $\overline{\theta} - \underline{\theta} \geq 0$, multiply (3.2) by $(\overline{\theta} - \underline{\theta})$, and integrate in x on Ω; then the left-hand I and the right-hand II can be written as follows:

$$\begin{aligned}
I &= \int_\Omega (\overline{\theta} - \underline{\theta})(\overline{\theta}_t - \underline{\theta}_t)dx - \int_\Omega (\overline{\theta} - \underline{\theta})\triangle(\overline{\theta} - \underline{\theta})\mathrm{d}x\\
&= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (\overline{\theta} - \underline{\theta})^2\mathrm{d}x + \int_\Omega |\nabla(\overline{\theta} - \underline{\theta})|^2\mathrm{d}x,
\end{aligned} \tag{3.3}$$

$$II = \int_\Omega \{[a - b(\overline{\theta} + \underline{\theta}) + c(\overline{\theta}_- - \underline{\theta}_+)](\overline{\theta} - \underline{\theta})^2 + c\underline{\theta}(\overline{\theta} - \underline{\theta})(\overline{\theta} - \underline{\theta})_M\}\mathrm{d}x. \tag{3.4}$$

Let $\|\cdot\|_{L^2(\Omega)}$ (denote $\|\cdot\|$ for simplicity) be the L^2 norm on Ω, then

$$I \geq \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\overline{\theta} - \underline{\theta}\|^2 + \lambda_1\|\overline{\theta} - \underline{\theta}\|^2, \tag{3.5}$$

where we have used the Poincaré inequality (see [10]),

$$\int_\Omega |\nabla(\overline{\theta} - \underline{\theta})|^2\mathrm{d}x \geq \lambda_1 \int_\Omega (\overline{\theta} - \underline{\theta})^2\mathrm{d}x,$$

and λ_1 is the principal eigenvalue of $-\triangle$ on Ω with homogeneous Dirichlet boundary condition. Define

$$G = \sup_{[0,T]\times\Omega} [a - 2b\Theta + c(\theta_{0-} - \Theta_+)], \qquad H = \sup_{[0,T]\times\Omega} (c\theta_0). \tag{3.6}$$

With $\Theta \leq \underline{\theta} \leq \overline{\theta} \leq \theta_0$ in mind, we have from (3.4)

$$II \leq \int_\Omega [a - 2b\Theta + c(\theta_{0-} - \Theta_+)](\overline{\theta} - \underline{\theta})^2\mathrm{d}x + \int_\Omega c\theta_0(\overline{\theta} - \underline{\theta})(\overline{\theta} - \underline{\theta})_M\mathrm{d}x$$

$$
\begin{aligned}
&\leq G\int_{\Omega}(\overline{\theta}-\underline{\theta})^2\mathrm{d}x + H\int_{\Omega}(\overline{\theta}-\underline{\theta})(\overline{\theta}-\underline{\theta})_M\mathrm{d}x\\
&= G\|\overline{\theta}-\underline{\theta}\|^2 + H\int_{\Omega}[\overline{\theta}(t,x)-\underline{\theta}(t,x)]\int_0^{\infty}[\overline{\theta}(t-\tau,x)-\underline{\theta}(t-\tau,x)]\mathrm{d}M(\tau)\mathrm{d}x\\
&\leq G\|\overline{\theta}-\underline{\theta}\|^2 + H\int_0^{\infty}\|\overline{\theta}(t,\cdot)-\underline{\theta}(t,\cdot)\|\cdot\|\overline{\theta}(t-\tau,\cdot)-\underline{\theta}(t-\tau,\cdot)\|\mathrm{d}M(\tau)\\
&\leq G\|\overline{\theta}-\underline{\theta}\|^2 + \frac{H}{2}\int_0^{\infty}\{\|\overline{\theta}(t,\cdot)-\underline{\theta}(t,\cdot)\|^2+\|\overline{\theta}(t-\tau,\cdot)-\underline{\theta}(t-\tau,\cdot)\|^2\}\mathrm{d}M(\tau)\\
&= \left(G+\frac{HM_0}{2}\right)\|\overline{\theta}-\underline{\theta}\|^2 + \frac{H}{2}\int_0^{\infty}\|\overline{\theta}(t-\tau,\cdot)-\underline{\theta}(t-\tau,\cdot)\|^2\mathrm{d}M(\tau).
\end{aligned}
\tag{3.7}
$$

It follows from (3.5) and (3.7) that

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\overline{\theta}-\underline{\theta}\|^2 \leq \left(G+\frac{HM_0}{2}-\lambda_1\right)\|\overline{\theta}-\underline{\theta}\|^2 + \frac{H}{2}\int_0^{\infty}\|\overline{\theta}(t-\tau,\cdot)-\underline{\theta}(t-\tau,\cdot)\|^2\mathrm{d}M(\tau). \tag{3.8}
$$

Integrate (3.8) in t over $[0, T]$, and consider that $\|\overline{\theta}-\underline{\theta}\|^2$ is T-periodic in t:

$$
\begin{aligned}
0 &= \int_0^T\left[\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\overline{\theta}-\underline{\theta}\|^2\right]\mathrm{d}t\\
&\leq \left(G+\frac{HM_0}{2}-\lambda_1\right)\int_0^T\|\overline{\theta}-\underline{\theta}\|^2\mathrm{d}t + \frac{H}{2}\int_0^{\infty}\int_0^T\|\overline{\theta}(t-\tau,x)-\underline{\theta}(t-\tau,x)\|^2\mathrm{d}t\mathrm{d}M(\tau)\\
&= \left(G+\frac{HM_0}{2}-\lambda_1\right)\int_0^T\|\overline{\theta}-\underline{\theta}\|^2\mathrm{d}t + \frac{H}{2}\int_0^{\infty}\int_0^T\|\overline{\theta}(t,x)-\underline{\theta}(t,x)\|^2\mathrm{d}t\mathrm{d}M(\tau)\\
&= (G+HM_0-\lambda_1)\int_0^T\|\overline{\theta}-\underline{\theta}\|^2\mathrm{d}t.
\end{aligned}
\tag{3.9}
$$

If $G + HM_0 - \lambda_1 < 0$, then it follows from (3.9) that

$$
\int_0^T\|\overline{\theta}-\underline{\theta}\|^2\mathrm{d}t \equiv 0,
$$

which implies that $\overline{\theta} \equiv \underline{\theta}$ on $R^+ \times \overline{\Omega}$. So the boundary value problem (1.1) and (1.2) has a T-periodic solution.

Now if θ^1 is any other solution with $\Theta \leq \theta^1 \leq \theta_0$, then θ^1 and $\underline{\theta}$ are a pair of upper and lower solutions of problem (1.1) and (1.2) which also satisfy (2.20). The same reasoning as previously for $\overline{\theta}$ and $\underline{\theta}$ yields $\theta^1 \equiv \underline{\theta}$ on $R^+ \times \overline{\Omega}$. So the periodic solution of problem (1.1) and (1.2) is also unique.

To ensure the existence of $\overline{\theta}$ and $\underline{\theta}$, one needs $\sigma(a + PM_0^- c) < 0$ and $\sigma(a - \theta^*) < 0$, and it suffices to have $a - \theta^* > \lambda_1$. So from the above argument we can get the sufficient conditions for $\overline{\theta} \equiv \underline{\theta}$ as follows:

$$
\text{(i) } a-\theta^* > \lambda_1, \qquad \text{(ii) } G + HM_0 < \lambda_1, \tag{3.10}
$$

where G, H are given by (3.6) and

$$
\theta^*(t,x) = c(t,x)\int_0^{\infty}\theta_0(t-\tau,x)\mathrm{d}M^+(\tau).
$$

Theorem 3.1. *Under the hypotheses* (H_1)–(H_4)*, for the Dirichlet boundary condition, if the conditions in* (3.10) *are satisfied, then the boundary value problem* (1.1) *and* (1.2) *has a unique smooth T-periodic solution $\theta(t,x)$. Moreover, for every nonnegative nontrivial initial function $\phi(t,x)$ with $0 \leq \phi(t,x) \leq P$, the solution $u(t,x)$ of the initial–boundary value problem* (1.1)–(1.3) *has the asymptotic behavior*

$$
\lim_{t\to\infty}[u(t,\cdot)-\theta(t,\cdot)] = 0 \quad \text{in } C(\overline{\Omega}).
$$

Part B. Neumann condition

In the case $\gamma(x) \equiv 0$ on $\partial\Omega$, there is a Neumann boundary condition

$$\frac{\partial\overline{\theta}}{\partial n} = \frac{\partial\underline{\theta}}{\partial n} = 0.$$

According to (2.1), for suitable initial function $\phi(t,x)$, we can get a pair of coupled positive constant upper and lower solutions k_2, k_1 from the following system:

$$\begin{aligned} &k_2(a_2 - b_1k_2 + c_2M_0^- k_2 - c_1M_0^+ k_1) = 0,\\ &k_1(a_1 - b_2k_1 + c_1M_0^- k_1 - c_2M_0^+ k_2) = 0. \end{aligned} \tag{3.11}$$

If $(b_1 - c_2M_0^-)(b_2 - c_1M_0^-) > c_1c_2M_0^{+2}$ and $a_1(b_1 - c_2M_0^-) > a_2c_2M_0^+$, then we can solve (3.11) and get

$$\begin{aligned} k_1 &= \frac{a_1(b_1 - c_2M_0^-) - a_2c_2M_0^+}{(b_1 - c_2M_0^-)(b_2 - c_1M_0^-) - c_1c_2M_0^{+2}},\\ k_2 &= \frac{a_2(b_2 - c_1M_0^-) - a_1c_1M_0^+}{(b_1 - c_2M_0^-)(b_2 - c_1M_0^-) - c_1c_2M_0^{+2}}. \end{aligned} \tag{3.12}$$

It is easy to check that $0 < k_1 \le \Theta \le \underline{\theta} \le \overline{\theta} \le \theta_0 \le k_2 \le P$ and (3.6) yields

$$\begin{aligned} G &= \sup_{[0,T]\times\Omega} [a - 2b\Theta + c(\theta_{0-} - \Theta_+)]\\ &\le a_2 - 2b_1k_1 + c_2M_0^- k_2 - c_1M_0^+ k_1\\ &= a_2 - (2b_1 + c_1M_0^+)k_1 + c_2M_0^- k_2,\\ H &= \sup_{[0,T]\times\Omega} (c\theta_0) \le c_2k_2. \end{aligned} \tag{3.13}$$

The same method as in *Part* A reveals that

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\overline{\theta} - \underline{\theta}\|^2 + \|\nabla(\overline{\theta} - \underline{\theta})\|^2\\ &\quad \le \left(G + \frac{HM_0}{2}\right)\|\overline{\theta} - \underline{\theta}\|^2 + \frac{H}{2}\int_0^\infty \|\overline{\theta}(t-\tau,\cdot) - \underline{\theta}(t-\tau,\cdot)\|^2 \mathrm{d}M(\tau). \end{aligned} \tag{3.14}$$

Integrating (3.14) in t on $[0,T]$, and considering that $\|\overline{\theta} - \underline{\theta}\|^2$ is T-periodic in t,

$$\begin{aligned} &0 + \int_0^T \|\nabla(\overline{\theta} - \underline{\theta})\|^2 \mathrm{d}t\\ &\quad \le \left(G + \frac{HM_0}{2}\right)\int_0^T \|\overline{\theta} - \underline{\theta}\|^2\mathrm{d}t + \frac{H}{2}\int_0^\infty\int_0^T \|\overline{\theta}(t-\tau,\cdot) - \underline{\theta}(t-\tau,\cdot)\|^2\mathrm{d}t\mathrm{d}M(\tau)\\ &\quad = (G + HM_0)\int_0^T \|\overline{\theta} - \underline{\theta}\|^2\mathrm{d}t\\ &\quad \le [a_2 - (2b_1 + c_1M_0^+)k_1 + c_2M_0^- k_2 + c_2k_2M_0]\int_0^T \|\overline{\theta} - \underline{\theta}\|^2\mathrm{d}t. \end{aligned} \tag{3.15}$$

So if $a_2 - (2b_1 + c_1M_0^+)k_1 + (M_0^- + M_0)c_2k_2 < 0$, then it follows from (3.15) that $\int_0^T \|\overline{\theta} - \underline{\theta}\|^2\mathrm{d}t \equiv 0$, which implies that $\overline{\theta} \equiv \underline{\theta}$. As the principal eigenvalue of $-\triangle$ on Ω with homogeneous Neumann boundary condition is $\lambda_1 = 0$, to ensure the existence of $\overline{\theta}$ and $\underline{\theta}$, one needs $\sigma(a(t,x) + PM_0^- c(t,x)) < 0$ and $\sigma(a - \theta^*) < 0$ and it suffices to have $a_1 - c_2M_0^+ k_2 > 0$. On the other hand, it follows from $a_1(b_1 - c_2M_0^-) > a_2c_2M_0^+$ that $a_1 - c_2M_0^+ k_2 > 0$. So the sufficient conditions for $\overline{\theta} \equiv \underline{\theta}$ can be summed up as

(i) $(b_1 - c_2M_0^-)(b_2 - c_1M_0^-) > c_1c_2M_0^{+2}, \qquad a_1(b_1 - c_2M_0^-) > a_2c_2M_0^+,$

$$\text{(ii) } a_2 - (2b_1 + c_1 M_0^+)k_1 + (M_0^- + M_0)c_2 k_2 < 0. \tag{3.16}$$

where k_1, k_2 are given by (3.12).

Theorem 3.2. *Under the hypotheses* (H_1)–(H_4)*, for the Neumann boundary condition, if the conditions in* (3.16) *are satisfied, then the boundary value problem* (1.1) *and* (1.2) *has a unique smooth* T*-periodic solution* θ*. Moreover, for every nonnegative nontrivial initial function* $\phi(t,x)$ *with* $k_1 \le \phi(t,x) \le k_2$*, the solution* $u(t,x)$ *of the initial–boundary value problem* (1.1)–(1.3) *has the asymptotic behavior*

$$\lim_{t\to\infty}[u(t,\cdot) - \theta(t,\cdot)] = 0 \quad \text{in } C(\overline{\Omega}).$$

Part C. Robin condition

For the Robin boundary condition

$$\frac{\partial\overline{\theta}}{\partial t} + \gamma\,\overline{\theta} = 0, \qquad \frac{\partial\underline{\theta}}{\partial t} + \gamma\,\underline{\theta} = 0 \quad \text{on } R^+ \times \partial\Omega,$$

we obtain the following relations by using the method of *Part* A:

$$\begin{aligned} I &= \int_\Omega (\overline{\theta} - \underline{\theta})(\overline{\theta}_t - \underline{\theta}_t)\mathrm{d}x - \int_\Omega (\overline{\theta} - \underline{\theta})\triangle(\overline{\theta} - \underline{\theta})\mathrm{d}x \\ &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (\overline{\theta} - \underline{\theta})^2\mathrm{d}x + \int_{\partial\Omega}\gamma(s)(\overline{\theta} - \underline{\theta})^2\mathrm{d}s + \int_\Omega |\nabla(\overline{\theta} - \underline{\theta})|^2\mathrm{d}x \\ &\ge \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\overline{\theta} - \underline{\theta}\|^2 + \|\nabla(\overline{\theta} - \underline{\theta})\|^2. \end{aligned} \tag{3.17}$$

Further calculation as in *Part* B reveals that (3.14) also holds for the Robin boundary condition. So the sufficient condition for $\overline{\theta} \equiv \underline{\theta}$ should be $G + HM_0 < 0$, that is,

$$\sup_{[0,T]\times\Omega}[a - 2b\Theta + c(\theta_{0-} - \Theta_+)] + \sup_{[0,T]\times\Omega}(c\theta_0)M_0 < 0, \tag{3.18}$$

where Θ, θ_0 are related to the Robin boundary condition.

Theorem 3.3. *Under the hypotheses* (H_1)–(H_4)*, for the Robin boundary condition, if* $\sigma(a - \theta^*) < 0$ *and the inequality in* (3.18) *are satisfied, then the boundary value problem* (1.1) *and* (1.2) *has a unique smooth* T*-periodic solution* θ*. Moreover, for every nonnegative nontrivial initial function* $\phi(t,x)$ *with* $0 \le \phi(t,x) \le P$*, the solution* $u(t,x)$ *of the initial–boundary value problem* (1.1)–(1.3) *has the asymptotic behavior*

$$\lim_{t\to\infty}[u(t,\cdot) - \theta(t,\cdot)] = 0 \quad \text{in } C(\overline{\Omega}).$$

4. Numerical simulations

In this section, we give some numerical simulations of the initial–boundary value problem (1.1)–(1.3) in the one-dimensional spatial domain $\Omega = (0,1)$.

Example 1. We consider the problem below:

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = u\left[a - bu - c\int_0^\infty u(t-\tau,x)\mathrm{d}\mu(\tau)\right], \quad (t,x) \in (0,+\infty)\times(0,1), \tag{4.1}$$

$$u(t,0) = u(t,1) = 0, \quad t \in (0,+\infty), \tag{4.2}$$

$$u(t,x) = \phi(t,x), \quad (t,x) \in (-\infty,0]\times[0,1]. \tag{4.3}$$

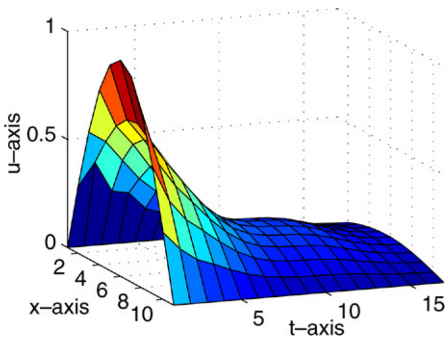

Fig. 1. The global asymptotic stability of the trivial solution for problem (4.1)–(4.3).

It is easy to see that the principal eigenvalue of $-\partial^2 u/\partial x^2$ on $\Omega = (0, 1)$ with homogeneous Dirichlet boundary conditions is $\lambda_1 = \pi^2$. We choose

$$a(t,x) = 7.5 + 2\sin(2\pi t); \qquad b(t,x) = 5 + \sin(2\pi t); \qquad c(t,x) = 1;$$
$$\mu(\tau) = \tau/(\tau+1); \qquad \phi(t,x) = \sin(\pi x).$$

It is easy to see that $M^+(\tau) = \tau/(\tau+1)$, $M^-(\tau) = 0$ and $T = 1$. So $P = a_2/(b_1 - c_2 M_0^-) = 9.5/(4-0) = 2.375$ and

$$a + cPM_0^- \le a_2 + c_2 PM_0^- = 9.5 + 1 \times 2.375 \times 0 = 9.5 < \pi^2.$$

Hence $\sigma(a + cPM_0^-) > 0$ and it follows from Theorem 2.1 that the solution of (4.1)–(4.3) satisfies $\lim_{t\to\infty} u(t,x) = 0$ for all $x \in [0, 1]$ (see Fig. 1).

Example 2. For convenience of simulation, we consider a particular case of problem (1.1)–(1.3) in which the time delay does not vary any longer:

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = u[a - bu - cu(t-0.5, x)], \quad (t,x) \in (0,+\infty) \times (0,1), \tag{4.4}$$
$$\frac{\partial u(t,0)}{\partial x} = \frac{\partial u(t,1)}{\partial x} = 0, \quad t \in (0,+\infty), \tag{4.5}$$
$$u(t,x) = \phi(t,x), \quad (t,x) \in [-0.5, 0] \times [0,1]. \tag{4.6}$$

It is easy to see that $M_0^+ = 1$ and $M_0^- = 0$ at this time. We choose

$$a(t,x) = 14 + 2\sin(2\pi t); \qquad b(t,x) = 30;$$
$$c(t,x) = 1 + \cos(2\pi t); \qquad \phi(t,x) = [\sin(\pi x) + 1.2]/3.$$

It follows that $a_1 = 12$, $a_2 = 16$, $b_1 = b_2 = 30$, $c_1 = 0$ and $c_2 = 2$. Hence

$$(b_1 - c_2 M_0^-)(b_2 - c_1 M_0^-) - c_1 c_2 M_0^{+2} = (30-0)(30-0) - 0 = 900 > 0,$$
$$a_1(b_1 - c_2 M_0^-) - a_2 c_2 M_0^+ = 12 \times (30-0) - 16 \times 2 = 328 > 0.$$

So condition (i) in (3.16) is satisfied. On the other hand, it follows from (3.12) that

$$\begin{aligned} k_1 &= \frac{a_1(b_1 - c_2 M_0^-) - a_2 c_2 M_0^+}{(b_1 - c_2 M_0^-)(b_2 - c_1 M_0^-) - c_1 c_2 M_0^{+2}} \\ &= \frac{12 \times (30-0) - 16 \times 2}{(30-0) \times (30-0) - 0} = \frac{328}{900}; \\ k_2 &= \frac{a_2(b_2 - c_1 M_0^-) - a_1 c_1 M_0^+}{(b_1 - c_2 M_0^-)(b_2 - c_1 M_0^-) - c_1 c_2 M_0^{+2}} \\ &= \frac{16 \times (30-0) - 0}{(30-0) \times (30-0) - 0} = \frac{16}{30}. \end{aligned}$$

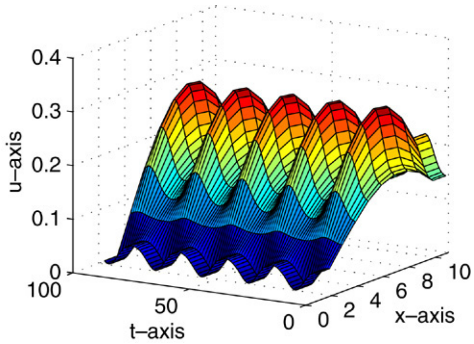

Fig. 2. The asymptotic periodicity of the problem (4.4)–(4.6).

The condition (ii) in (3.16) is satisfied; indeed,

$$\begin{aligned} a_2 - (2b_1 + c_1 M_0^+)k_1 + (M_0^- + M_0)c_2k_2 &= 16 - (2\times 30 + 0)\times \frac{328}{900} + (0+1)\times 2\times \frac{16}{30} \\ &= 16 - 24 + \frac{96}{30} = -4.8 < 0. \end{aligned}$$

According to Theorem 3.2, the boundary value problem (4.4) and (4.5) has a unique 1-periodic solution θ. Moreover, the solution $u(t,x)$ of the initial–boundary value problem (4.4)–(4.6) has the asymptotic periodicity $\lim_{t\to\infty}[u(t,x) - \theta(t,x)] = 0$ for all $x \in [0,1]$ (see Fig. 2).

References

[1] K. Gopalsamy, X.Z. He, Dynamics of an almost periodic logistic integro-differential equation, Bulletin of the Institute of Mathematics. Academia Sinica 20 (3) (1992) 267–284.
[2] H.R. Henriquez, Regularity of solutions of abstract retarded functional differential equations with unbounded delay, Nonlinear Analysis 28 (1997) 513–531.
[3] P. Hess, Periodic–Parabolic Boundary Value Problems and Positivity, in: Pitman Research Notes in Mathematics Series, vol. 247, Pitman, New York, 1991.
[4] X. Lu, Periodic solution and oscillation in a competition model with diffusion and distributed delay effects, Nonlinear Analysis 27 (6) (1996) 699–709.
[5] C.V. Pao, Coupled nonlinear parabolic systems with time delays, Journal of Mathematical Analysis and Applications 196 (1995) 237–265.
[6] C.V. Pao, Nonlinear parabolic and Elliptic Equations, Plenum Press, New York, 1992.
[7] S.G. Ruan, J.H. Wu, Reaction–diffusion equations with infinite delay, The Canadian Applied Mathematics Quarterly 2 (1994) 485–550.
[8] R. Redlinger, Existence theorems for semi-linear parabolic systems with functionals, Nonlinear Analysis 8 (6) (1984) 667–682.
[9] B. Shi, Y. Chen, A prior bounds and stability of solutions for a Volterra reaction–diffusion equation with infinite delay, Nonlinear Analysis 44 (2001) 97–121.
[10] J. Smoller, Shock Waves and Reaction–Diffusion Equations, Springer, 1983.
[11] V. Volterra, Lecons sur la theorie mathematique de la lutte pour la vie, Gauthier-Villars, Paris, 1931.
[12] J.H. Wu, Theory and Applications of Partial Functional Differential Equations, Springer, New York, 1996.
[13] J.L. Wang, L. Zhou, Existence and uniqueness of periodic solution of delayed Logistic equation and its asymptotic behavior, Journal of Partial Differential Equations 16 (4) (2003) 1–13.
[14] J.L. Wang, L. Zhou, Y.B. Tang, Asymptotic periodicity of a food-limited diffusive population model with time-delay, Journal of Mathematical Analysis and Applications 313 (2) (2006) 381–399.
[15] L. Zhou, Y.P. Fu, Existence and stability of periodic quasi-solutions in nonlinear parabolic systems with discrete delays, Journal of Mathematical Analysis and Applications 250 (2000) 139–161.