Accurate Period Approximation for Any Simple Pendulum Amplitude *

XUE De-Sheng(薛德胜), ZHOU Zhao(周钊), GAO Mei-Zhen(高美珍)**

Key Lab for Magnetism and Magnetic Materials of the Ministry of Education, Lanzhou University, Lanzhou 730000

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Accurate approximate analytical formulae of the pendulum period composed of a few elementary functions for any amplitude are constructed. Based on an approximation of the elliptic integral, two new logarithmic formulae for large amplitude close to 180° are obtained. Considering the trigonometric function modulation results from the dependence of relative error on the amplitude, we realize accurate approximation period expressions for any amplitude between 0 and 180°. A relative error less than 0.02% is achieved for any amplitude. This kind of modulation is also effective for other large-amplitude logarithmic approximation expressions.

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The simple pendulum, one of the most popular paradigms of linear and nonlinear problems, is widely used for analyzing harmonic motion or nonlinear oscillatory systems in textbooks[1] and undergraduate courses. As a periodic process,[2] however, it is difficult to obtain an analytical solution[2,3] in terms of elementary functions for the pendulum dynamic equation, which is a nonlinear differential equation.[4] In this sense, a linear approximation for small amplitude oscillation of the simple pendulum is often taken, where a trigonometric function solution is usually described by a simple harmonic motion with a period as a function of the length of the pendulum L and the local acceleration of gravity g only.[5,6] With increasing amplitudes, the nonlinear nature of the pendulum motion becomes apparent, and numerical solutions for both the motion equation and the period are often used, which makes the pendulum motion a pedagogically rich theme.[1,7]

Considering the importance of the period in the description of the motion of the simple pendulum and the simplicity of the period description for students and researchers, it is necessary to find analytical solutions in the form of elementary functions for the simple pendulum period because of applications in introductory physics labs,[8,9] classical mechanics, electromagnetism courses,[1,10] and even in physics research (acoustics, electronics, superconductivity).[1,11,12] The approximate formulae of the period[9,13−26] found by different authors can be classified into three groups. Most of the approximation formulae are constructed by comparing to the series expansion of the exact period.[9,18−24] These yield good estimates for the small-amplitude period. The maximum amplitude describing the period within 0.1% relative error is 133°.[24] Based on the asymptotic approximation around amplitude π, Cromer[25] and Amore *et al.*[26] constructed a logarithmic formula which is in good agreement for large amplitude. Its relative error of small amplitude is much smaller than that of the large-amplitude results from the small-amplitude approximate period. Therefore, it is also described as a period approximation expression for any amplitude. In order to obtain a more accurate expression of period for any amplitude, Lima combines the small- and large-amplitude formulae.[2] Recently, Zhang modified Lima's combined formula by analyzing the type of relative error, and an accurate approximation period with relative error less than 0.08% was derived, however it is complicated.[27]

Certainly, a good approximate analytical expression of the period composed of the elementary functions should be accurate and simple for an amplitude θ_0 range as large as possible. In this Letter, we try to find such an expression of period. Comparison with other existing formulae for any amplitude will also be presented.

It is known that the exact pendulum period $T_{\rm ex}$ can be written as[2]

$$\frac{T_{\rm ex}}{T_0}=\frac{2}{\pi}\int_0^{\pi/2}\frac{d\phi}{\sqrt{1-k^2\sin^2\phi}},$$

$$K(k)\equiv\int_0^{\pi/2}\frac{d\phi}{\sqrt{1-k^2\sin^2\phi}},\tag{1}$$

where T_0 is the harmonic period of the pendulum, $K(k)$ is the complete elliptic integral of the first kind, $k=\sin(\theta_0/2)$ is the modulus of the elliptic integral, and $k\sin\varphi=\sin(\theta/2)$ with θ being the angular displacement of the pendulum. Most of the approximate expressions are worked out by means of a term-by-term comparison of the power-series expansion for

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**Corresponding author. Email: gaomz@lzu.edu.cn

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the approximate period with the corresponding series for the exact expansion of $2K(k)/\pi$. From small-amplitude series of expansion, the more the equal terms of series taken into account, the more accurate the amplitude described.[24] From large-amplitude series expansion around $k = 1$, a simple logarithmic formula, accurate for an amplitude around π, is constructed by taking into account the first term of the series by Cromer[25] and Amore *et al.*[26]

Alternatively, one can choose some approximate forms of the integrated function; the approximate $K(k)$ can be integrated with only a few elementary functions. By modifying approximate $K(k)$ further, a good estimate can be yielded for the period. In this sense, simple and accurate approximate formulae can be found. We reform the complete elliptic integral of the first kind as

$$K(k) = \int_0^1 \frac{dt}{\sqrt{1-k^2t^2}\sqrt{1-t^2}}, \tag{2}$$

where $t = \sin\varphi$. In order to complete the elliptic integral by approximating, we rewrite Eq. (2) as

$$K(k)=\int_0^1 \frac{dt}{(1-k^2t^2)\sqrt{1-(1-k^2)t^2/(1-k^2t^2)}}, \tag{3a}$$

$$K(k)=\int_0^1 \frac{dt}{\sqrt{[1-(1+k^2)t^2/2]^2-(1-k^2)^2t^4/4}}, \tag{3b}$$

where the last term in the radical sign is close to zero both at $\theta_0 = 0$ and $\theta_0 = \pi$. If we omit this part, the integral can be completed. To ensure that the approximate $K(k)$ is accurate at $k \to 1$, the approximate expressions (3) can be written as

$$K(k) \approx \frac{1}{2}\Big(\ln 4 + \frac{1}{k}\ln\frac{1+k}{1-k}\Big), \tag{4a}$$

$$K(k) \approx \frac{1}{2}\Big(\ln 2 + \sqrt{\frac{2}{1+k^2}}\ln\frac{\sqrt{2}+\sqrt{1+k^2}}{\sqrt{2}-\sqrt{1+k^2}}\Big). \tag{4b}$$

With these approximate integrals, from Eq. (1) the approximate periods of the simple pendulum simply read

$$\frac{T_1}{T_0} \approx \frac{1}{\pi}\Big(\ln 4 + \frac{1}{k}\ln\frac{1+k}{1-k}\Big), \tag{5a}$$

$$\frac{T_2}{T_0} \approx \frac{1}{\pi}\Big[\ln 2 + \sqrt{\frac{2}{1+k^2}}\ln\frac{\sqrt{2}+\sqrt{1+k^2}}{\sqrt{2}-\sqrt{1+k^2}}\Big], \tag{5b}$$

where T_1 and T_2 are two pendulum periods from the two different expressions. The amplitude dependence of the pendulum period is shown in Fig. 1. It is found that both T_1 and T_2 always overestimate the period, but approximate T_{ex} asymptotically when $\theta_0 \to \pi$. Note that the relative errors of T_1 and T_2 increase with decreasing amplitude, reaching a maximum less than 8% and 1.5% for $\theta_0 \to 0$, respectively. Within 0.1% relative error, the ranges of accurately described amplitude are 171–180° and 148–180° for T_1 and T_2, respectively. Meanwhile, the range of the Cromer expression[25] is 172–180°.

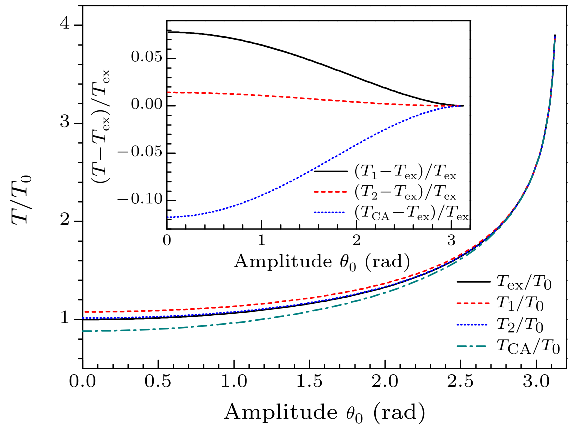

Fig. 1. Amplitude θ_0 dependence of the pendulum period T. The solid line (black) is the exact period T_{ex}, the dashed line (red) is T_1, the dotted line (blue) is T_2, and the dot-dashed line (dark cyan) is for the T_{CA} of the Cromer formula. Inset: the relative errors for T_1, T_2 and T_{CA}.

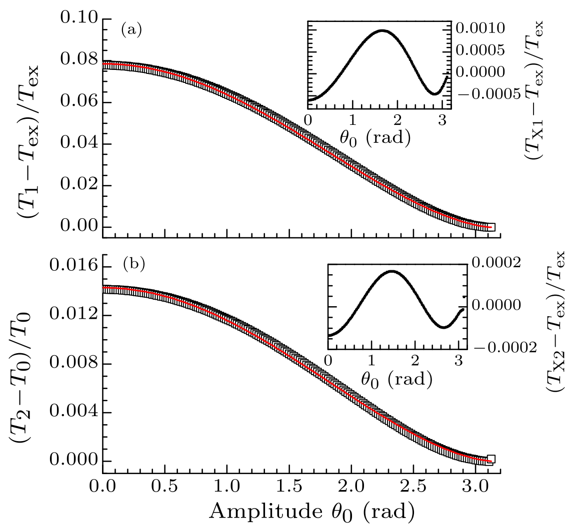

Fig. 2. Amplitude θ_0 dependence of the relative error of the approximate period (a) T_1 and (b) T_2. The open block (black) is the calculating relative error of the approximate period, and the solid line (red) is the fitting curve of the cosine function. Inset: the relative errors of T_{X1} and T_{X2}.

Let us see the relative errors of T_1 and T_2 shown in Fig. 2 in detail. It is found that they can be described approximately as

$$\frac{T_1 - T_{ex}}{T_{ex}} \approx \frac{\pi}{40}\Big(\cos\frac{\theta_0}{2}\Big)^{1.6}, \tag{6a}$$

$$\frac{T_2 - T_0}{T_0} \approx \frac{1}{70}\Big(\cos\frac{\theta_0}{2}\Big)^{1.6}. \tag{6b}$$

From Eq. (6), the accurate approximate period can be described as

$$\frac{T_{\rm X1}}{T_0} = \frac{1}{\pi[1+\pi k'^{1.6}/40]}\Big(\ln 4 + \frac{1}{k}\ln\frac{1+k}{1-k}\Big), \quad (7a)$$

$$\frac{T_{\rm X2}}{T_0} = \frac{1}{\pi}\Big(\ln 2 + \sqrt{\frac{2}{1+k^2}}\ln\frac{\sqrt{2}+\sqrt{1+k^2}}{\sqrt{2}-\sqrt{1+k^2}}\Big) - \frac{k'^{1.6}}{70}, \quad (7b)$$

where $k' = \cos(\theta_0/2)$. The relative errors of T_{X1} and T_{X2} are also plotted in the inset of Fig. 2. The maximum relative errors of T_{X1} and T_{X2} are less than 0.11% and 0.02% for any amplitude, respectively.

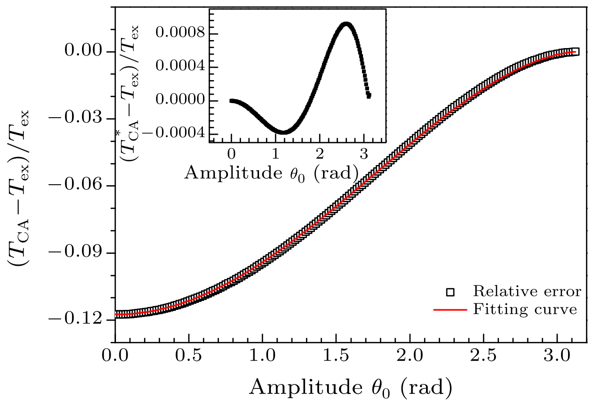

Fig. 3. Amplitude θ_0 dependence of the relative error of the approximate period $T_{\rm CA}$. The open squares represent the calculating relative error of the approximate period, and the solid line is the fitting curve of the cosine function. Inset: the relative error of $T^*_{\rm CA}$.

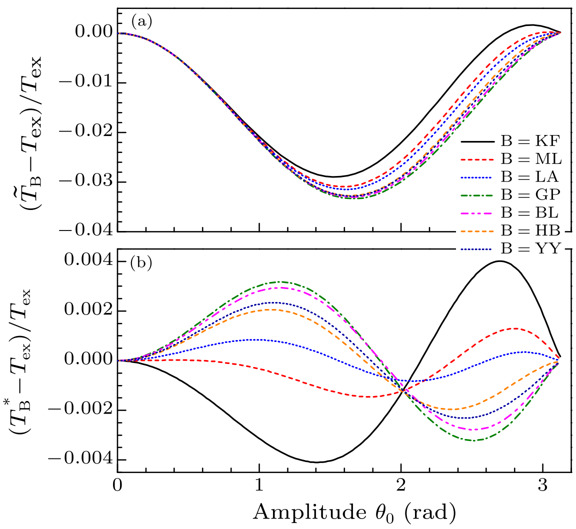

Fig. 4. Amplitude θ_0 dependence of the relative error of the approximate period (a) $\tilde{T}_{\rm B}$ and (b) $T^*_{\rm B}$, in combination with the Kidd and Fogg expression (black solid line), the Molina expression (red dashed line), the Lima and Arun expression (blue dotted line), the Ganley and Parwani expression (dark cyan dot-dashed line), the Beléndez expression (magenta dot-dot-dashed line), the Hite and Beléndez expression (dark yellow short dashed line), and the Yu and Yuan expression (navy short dotted line).

In order to find the effectiveness of the modulated idea, we apply the above process to the first real large-amplitude expression for any amplitude constructed by Cromer[25] and Amore *et al.*,[26]

$$\frac{T_{\rm CA}}{T_0} = \frac{2}{\pi}\ln\Big(\frac{4}{\cos(\theta_0/2)}\Big). \quad (8)$$

If we plot the deviation of $T_{\rm CA}$ from the exact period as $(T_{\rm CA}-T_{\rm ex})/T_{\rm ex} \sim \theta_0$ or $(T_{\rm CA}-T_0)/T_0 \sim \theta_0$ curves, both of them can be fitted by the trigonometric function approximately. However, the $(T_{\rm CA}-T_{\rm ex})/T_{\rm ex} \sim \theta_0$ curve can be fitted much better, as shown in Fig. 3, with the formula

$$\frac{T_{\rm CA}-T_{\rm ex}}{T_{\rm ex}} = -\frac{1}{2.71\pi}\Big(\cos\frac{\theta_0}{2}\Big)^{1.7}. \quad (9)$$

Considering the modulation of relative error, another accurate period expression for any amplitude is

$$\frac{T^*_{\rm CA}}{T_0} = \frac{2}{\pi - \cos^{1.7}(\theta_0/2)/2.71}\ln\Big(\frac{4}{\cos(\theta_0/2)}\Big). \quad (10)$$

It is found that the relative error of $T^*_{\rm CA}$ is less than 0.1% for any amplitude, as plotted in the inset of Fig. 3.

By noting that the trigonometric relation $k^2 + k'^2 = 1$ is valid for all amplitudes, with $k(k')$ ending to 0(1) when $\theta_0 \to 0$ and $k(k')$ ending to 1(0) when $\theta_0 \to \pi$, Lima[2] deduced that a weighted average with statistical weights depending on θ_0 through k andk' could be taken for approximating the behavior of the exact period, as given by

$$TA_{\rm B} = k'^2 T_{\rm B} + k^2 T_{\rm CA}. \quad (11)$$

From the relative errors of different $TA_{\rm B}$, as shown in Fig. 4(a), except for several degrees at the small-amplitude and large-amplitude sides, most of them have the maximal relative error in the range of the middle amplitude, among which the least maximum is about 3%. If we fit the relative errors with trigonometric functions, as suggested by Zhang and Yuan,[27]

$$\frac{TA_{\rm B}-T_{\rm ex}}{T_{\rm ex}} = -\frac{\pi}{a}\sin^2\theta_0, \quad (12)$$

where a is a fitting parameter, and the modulated approximate period can be written as

$$T^*_{\rm B} = \frac{TA_{\rm B}}{1-(\pi/a)\sin^2\theta_0}. \quad (13)$$

As shown in Fig. 4(b) the maximum relative errors of $T^*_{\rm B}$ are 0.4%, 0.15%, 0.08%, 0.33%, 0.3%, 0.21%, and 0.24% by optimizing a = 125, 106, 100, 90, 91, 94 and 93 for the combinations of the formula $T_{\rm CA}$ (Cromer[25] and Amore *et al.*[26]) with the Kidd and Fogg (KF) expression,[17] the Molina (ML) expression,[18] the Lima and Arun (LA) expression,[19] the Ganley and Parwani (GP) expression,[19,20] the Beléndez (BL) expression,[21] the Hite and Beléndez (HB) expression,[22,23] and the Yu and Yuan (YY) expression,[24] respectively. Clearly, the modulation method is still effective here.

In this work, we have used a cut-off approximation for the elliptic integral to derive new approximate logarithmic formulae of the approximate period from the large-amplitude side around π. Considering the modulation results from the relative error of the logarithmic approximate period, two new accurate approximate formulae for any amplitude have been obtained. The relative error of the accurate approximate period is less than 0.02% for amplitudes between 0 and π rad. The improvement in the accuracy by the simple modulation is significant in comparison with the previous approximate formulae. With a similar modulation, the relative error of the approximate formula of Cromer[25] and Amore *et al.*[26] decreases from 12% to 0.1%. The modulation results from the relative error are effective for an approximate period for any amplitude, and especially effective for the large-amplitude approximate period.

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