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MULTIPLE SOLUTIONS FOR THE SCHRÖDINGER EQUATION WITH MAGNETIC FIELD*

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Abstract The authors consider the semilinear Schrödinger equation

$$-\Delta_A u + V_\lambda(x)u = Q(x)|u|^{\gamma-2}u \quad \text{in} \quad \mathbb{R}^N,$$

where $1 < \gamma < 2^*$ and $\gamma \neq 2$, $V_\lambda = V^+ - \lambda V^-$. Exploiting the relation between the Nehari manifold and fibrering maps, the existence of nontrivial solutions for the problem is discussed.

Key words Nehari manifold, fibrering maps, Schrödinger equation

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1 Introduction

In this article, we study the existence of nontrivial solutions of the semilinear Schrödinger equation

$$-\Delta_A u + V_\lambda(x)u = Q(x)|u|^{\gamma-2}u, \quad x \in \mathbb{R}^N, \tag{1.1}$$

where $-\Delta_A = (-i\nabla + A)^2$, $u : \mathbb{R}^N \to \mathbb{C}$, $N \geq 3$, $1 < \gamma < 2^*$ and $\gamma \neq 2$. The coefficient V_λ is the scalar (or electric) potential and $A = (A_1, \cdots, A_N) : \mathbb{R}^N \to \mathbb{R}^N$ the vector (or magnetic) potential. We assume in this paper that $A \in L^2_{\text{loc}}(\mathbb{R}^N)$, $V_\lambda(x)$ and $Q(x)$ are continuous functions changing signs on $\mathbb{R}^N$. $V_\lambda(x) = V^+(x) - \lambda V^-(x)$, where $V^+(x) = \max(V(x), 0)$, $V^-(x) = \max(-V(x), 0)$ and $V^-(x) \in L^{\frac{N}{2}}(\mathbb{R}^N)$. It is assumed that $\lim\limits_{|x|\to\infty} Q(x) = Q(\infty) < 0$. Further assumptions on $V_\lambda(x)$ and $Q(x)$ will be formulated later.

In the case $A = 0$, the problem was extensively studied. In particular, in a bounded domain Ω, it was established in [4] the existence and multiplicity of non-negative solutions of

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(1.1) for $\gamma > 2$. Later, the case $1 < \gamma < 2$ was considered in [2]. In the whole space $\mathbb{R}^N$, if $V \in L^{\frac{N}{2}}(\mathbb{R}^N)$, the eigenvalue problem

$$-\Delta u = \lambda V(x)u \qquad \text{in } \mathbb{R}^N$$

has a sequence of eigenvalues, $0 < \lambda_1(V) \leq \lambda_2(V) \leq \cdots \leq \lambda_n(V) \leq \cdots$, of finite multiplicity and going to infinity. Under this condition, it was proved in [8], that, for $V_\lambda = -\lambda V$,

(i) problem (1.1) has a positive solution for every $0 < \lambda < \lambda_1(V)$,

(ii) if $\int_{\mathbb{R}^N} Q\phi_1^\gamma \mathrm{d}x < 0$, where ϕ_1 is the eigenfunction corresponding to $\lambda_1(V)$, then there exists a constant $\delta > 0$ such that problem (1.1) admits at least two positive solutions for every $\lambda_1(V) < \lambda < \lambda_1(V) + \delta$. These solutions were obtained by the mountain-pass lemma and local minimization. Similar results were obtained for the $p-$Laplacian in $\mathbb{R}^N$ in [6] and [10].

Recently, much interest in the case $A \neq 0$ has arisen and various existence results were obtained, see for instance, [1], [5], [7], [11] and references therein. Inspired by [2] and [4], we consider the existence of nontrivial solutions for (1.1) with $A \neq 0$. We classify the Nehari manifold, and find solutions of (1.1) as minimizers of the associated functional on two distinct components of the Nehari manifold.

It is known from [7] that the eigenvalue problem

$$-\Delta_A u + V^+(x)u = \mu V^-(x)u \qquad \text{in } \mathbb{R}^N \tag{1.2}$$

has a sequence of eigenvalues $0 < \mu_1 < \mu_2 \leq \mu_3 \leq \ldots \leq \mu_n \to \infty$ if $V^- \neq 0$ and $V^- \in L^{\frac{N}{2}}(\mathbb{R}^N)$. Let us denote the corresponding orthonormal system of eigenfunctions by $\varphi_1(x), \varphi_2(x), \cdots$. The sequence is complete in the Hilbert space $H^1_{A,V^+}(\mathbb{R}^N)$, where $H^1_{A,V^+}(\mathbb{R}^N)$ is the closure of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\| = \Big(\int_{\mathbb{R}^N} \left(|\nabla_A u|^2 + V^+(x)|u|^2 \right) \mathrm{d}x \Big)^{\frac{1}{2}},$$

and $\nabla_A u = (\nabla + iA)u$, $V^+(x) = \max(V(x), 0)$. The first eigenvalue μ_1 is defined by the Rayleigh quotient

$$\mu_1 = \inf_{u \in H^1_{A,V^+}(\mathbb{R}^N)} \frac{\int_{\mathbb{R}^N} (|\nabla_A u|^2 + V^+(x)|u|^2)\mathrm{d}x}{\int_{\mathbb{R}^N} V^-(x)|u|^2 \mathrm{d}x}. \tag{1.3}$$

Our main result is as follows.

Theorem 1.1 If $2 < \gamma < 2^*$, then

(i) problem (1.1) has a solution for $0 < \lambda < \mu_1$,

(ii) if $\int_{\mathbb{R}^N} Q(x)|\varphi_1|^\gamma \mathrm{d}x < 0$ and $\lambda = \mu_1$, then problem (1.1) has a solution.

(iii) if $\int_{\mathbb{R}^N} Q(x)|\varphi_1|^\gamma \mathrm{d}x < 0$, then there exists a constant $\delta > 0$ such that problem (1.1) admits at least two solutions for $\mu_1 < \lambda < \mu_1 + \delta$.

For the case of $1 < \gamma < 2$, we have

Theorem 1.2 (i) Problem (1.1) has a solution for $0 < \lambda < \mu_1$,

(ii) If $\int_{\mathbb{R}^N} Q(x)|\varphi_1|^\gamma \mathrm{d}x < 0$, then there exists a constant $\delta > 0$ such that problem (1.1) admits at least two solutions for $\mu_1 < \lambda < \mu_1 + \delta$.

We point out that φ_1 may not belong to $L^\gamma(\mathbb{R}^N)$. The condition $\int_{\mathbb{R}^N} Q(x)|\varphi_1|^\gamma \mathrm{d}x < 0$ is an extra assumption on Q .

In Section 2 we discuss the relation between the Nehari manifold and the fibrering maps. Theorems 1.1 and 1.2 are proved in Section 3 and Section 4.

2 Preliminaries

Suppose $u \in H^1_{A,V^+}(\mathbb{R}^N)$, by the diamagnetic inequality ([12], Theorem 7.21), $|u| \in D^{1,2}(\mathbb{R}^N)$, where $D^{1,2}(\mathbb{R}^N)$ is the usual Sobolev space of real valued functionals defined by

$$D^{1,2}(\mathbb{R}^N) = \Big\{u;\ u \in L^{2^*}(\mathbb{R}^N),\ \nabla u \in L^2(\mathbb{R}^N)\ \Big\}.$$

Therefore, $u \in L^{2^*}(\mathbb{R}^N)$, where $2^* = \frac{2N}{N-2}$. Functions in $H^1_{A,V^+}(\mathbb{R}^N)$ may not belong to $L^\gamma(\mathbb{R}^N)$ with $1 < \gamma < 2$ or $2 < \gamma < 2^*$. So we look for solutions of problem (1.1) in the space $E = H^1_{A,V^+}(\mathbb{R}^N) \cap L^\gamma(\mathbb{R}^N)$ equipped with norm

$$\|u\|_E = \Big(\int_{\mathbb{R}^N} (|\nabla_A u|^2 + V^+(x)|u|^2)\mathrm{d}x + (\int_{\mathbb{R}^N} |u|^\gamma \mathrm{d}x)^{\frac{2}{\gamma}}\Big)^{\frac{1}{2}}.$$

Alternatively, E can be defined as the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the above norm.

It is apparent that the functional

$$J_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla_A u|^2 + V_\lambda(x)|u|^2\right)\mathrm{d}x - \frac{1}{\gamma}\int_{\mathbb{R}^N} Q(x)|u|^\gamma \mathrm{d}x$$

is a C^1-functional in E. Critical points of J_λ in E are solutions of problem (1.1), which belong to the so-called Nehari manifold

$$S = \Big\{u \in E : \int_{\mathbb{R}^N} (|\nabla_A u|^2 + V_\lambda(x)|u|^2)\mathrm{d}x = \int_{\mathbb{R}^N} Q(x)|u|^\gamma \mathrm{d}x\Big\}.$$

On S, we have that

$$J_\lambda(u) = \Big(\frac{1}{2} - \frac{1}{\gamma}\Big)\int_{\mathbb{R}^N} (|\nabla_A u|^2 + V_\lambda(x)|u|^2)\mathrm{d}x = \Big(\frac{1}{2} - \frac{1}{\gamma}\Big)\int_{\mathbb{R}^N} Q(x)|u|^\gamma \mathrm{d}x. \tag{2.1}$$

The Nehari manifold S is closely linked to the behavior of functions $\Phi_u : t \to J_\lambda(tu)$ $(t \geq 0)$. Such maps are known as fibrering maps introduced in [9], and were also discussed in [2], [4], [6]. If $u \in E$, we have

$$\Phi_u(t) = \frac{t^2}{2}\int_{\mathbb{R}^N} (|\nabla_A u|^2 + V_\lambda(x)|u|^2)\mathrm{d}x - \frac{t^\gamma}{\gamma}\int_{\mathbb{R}^N} Q(x)|u|^\gamma \mathrm{d}x; \tag{2.2}$$

$$\Phi'_u(t) = t\int_{\mathbb{R}^N} (|\nabla_A u|^2 + V_\lambda(x)|u|^2)\mathrm{d}x - t^{\gamma-1}\int_{\mathbb{R}^N} Q(x)|u|^\gamma \mathrm{d}x; \tag{2.3}$$

$$\Phi''_u(t) = \int_{\mathbb{R}^N} (|\nabla_A u|^2 + V_\lambda(x)|u|^2)\mathrm{d}x - (\gamma - 1)t^{\gamma-2}\int_{\mathbb{R}^N} Q(x)|u|^\gamma \mathrm{d}x. \tag{2.4}$$

Obviously, $u \in S$ if and only if $\Phi'_u(1) = 0$. More generally, $\Phi'_u(t) = 0$ if and only if $tu \in S$, i.e., elements in S correspond to stationary points of fibrering maps. It follows from (**??**) and (**??**) that if $\Phi'_u(t) = 0$, then $\Phi''_u(t) = (2-\gamma)t^{\gamma-2}\int_{\mathbb{R}^N} Q(x)|u|^\gamma \mathrm{d}x$. So we may divide S into three subsets S^+, S^- and S^0 as follows:

$$S^+ = \Big\{u \in S : (2-\gamma)\int_{\mathbb{R}^N} Q(x)|u|^\gamma \mathrm{d}x > 0\Big\},$$

$$S^- = \Big\{u \in S : (2-\gamma)\int_{\mathbb{R}^N} Q(x)|u|^\gamma \mathrm{d}x < 0\Big\},$$

$$S^0 = \Big\{u \in S : (2-\gamma)\int_{\mathbb{R}^N} Q(x)|u|^\gamma \mathrm{d}x = 0\Big\}.$$

S^+, S^- and S^0 correspond to local minima, local maxima and inflection points of the fibrering maps $\Phi_u(t)$, respectively. Consequently,

Lemma 2.1 Let $u \in S$. Then

(i) $\Phi_u'(1) = 0$;

(ii) $u \in S^+, S^-, S^0$ if $\Phi_u''(1) > 0$, $\Phi_u''(1) < 0$, $\Phi_u''(1) = 0$, respectively.

On the other hand, for $u \in E$,

(i) if $\int_{\mathbb{R}^N}(|\nabla_A u|^2 + V_\lambda(x)|u|^2)\mathrm{d}x$ and $\int_{\mathbb{R}^N} Q(x)|u|^\gamma \mathrm{d}x$ have the same sign, Φ_u has a unique turning point at

$$t(u) = \left(\frac{\int_{\mathbb{R}^N}(|\nabla_A u|^2 + V_\lambda(x)|u|^2)\mathrm{d}x}{\int_{\mathbb{R}^N} Q(x)|u|^\gamma \mathrm{d}x}\right)^{\frac{1}{\gamma-2}}.$$

If $2 < \gamma < 2^*$, $t(u)$ is a local minimum (maximum) of $\Phi_u(t)$ and $t(u)u \in S^+$ (S^-) if and only if $\int_{\mathbb{R}^N} Q(x)|u|^\gamma \mathrm{d}x < 0 (> 0)$. The case $1 < \gamma < 2$ can be discussed analogously.

(ii) if $\int_{\mathbb{R}^N}(|\nabla_A u|^2 + V_\lambda(x)|u|^2)\mathrm{d}x$ and $\int_{\mathbb{R}^N} Q(x)|u|^\gamma \mathrm{d}x$ have different signs, then Φ_u has no turning points and so no multiples of u lying in S.

We define

$$L^+(\lambda) = \Big\{u \in E : \|u\| = 1, \quad \int_{\mathbb{R}^N}(|\nabla_A u|^2 + V_\lambda(x)|u|^2)\mathrm{d}x > 0\Big\}.$$

$L^-(\lambda)$, $L^0(\lambda)$ are defined by replacing $>$ in L^+ by $<$ and $=$ respectively. We also define

$$B^+ = \Big\{u \in E : \|u\| = 1, \quad \int_{\mathbb{R}^N} Q(x)|u|^\gamma \mathrm{d}x > 0\Big\},$$

and B^-, B^0 are defined by replacing $>$ in B^+ by $<$ and $=$, respectively.

Thus, if $2 < \gamma < 2^*$ $(1 < \gamma < 2)$ and $u \in L^+(\lambda) \cap B^+$, we have $\Phi_u(t) > 0 (< 0)$ for $t > 0$ small and $\Phi_u(t) \to -\infty(+\infty)$ as $t \to \infty$, $\Phi_u(t)$ has a unique maximum (minimum) point at $t(u)$ with $t(u)u \in S^-(S^+)$. Similarly, if $u \in L^-(\lambda) \cap B^-$, $\Phi_u(t) < 0 (> 0)$ for t small, $\Phi_u(t) \to +\infty(-\infty)$ as $t \to \infty$ and $\Phi_u(t)$ has a unique minimum (maximum) point at $t(u)$ with $t(u)u \in S^+(S^-)$. Finally, if $u \in L^+(\lambda) \cap B^-$ $(L^-(\lambda) \cap B^+)$, Φ_u is strictly increasing (decreasing) for all $t > 0$. Consequently, if $u \in E \backslash \{0\}$ and $2 < \gamma < 2^*$ $(1 < \gamma < 2)$, we have

(i) $t \to \Phi_u(t)$ has a local minimum (local maximum) at $t = t(u)$ and $t(u)u \in S^+(S^-)$ if and only if $\frac{u}{\|u\|} \in L^-(\lambda) \bigcap B^-$;

(ii) $t \to \Phi_u(t)$ has a local maximum (local minimum) at $t = t(u)$ and $t(u)u \in S^-(S^+)$ if and only if $\frac{u}{\|u\|} \in L^+(\lambda) \bigcap B^+$;

(iii) if $\frac{u}{\|u\|} \in L^-(\lambda) \bigcap B^+$ or $L^+(\lambda) \bigcap B^-$, no multiple of u lies in S.

We shall prove the existence of solutions of (1.1) by looking for minimizers of J_λ on S. Although S is a subset of E, minimizers of J_λ on S are actually critical points of J_λ on E. Indeed, as proved in [4], Theorem 2.3, we have

Lemma 2.2 Suppose that u is a local minimum for J_λ on S. If $u \notin S^0$, then u is a critical point of J_λ.

3 Superlinear Case

Suppose in this section that $2 < \gamma < 2^*$. Since the range of the parameter λ affects the existence of solutions of problem (1.1), we distinguish the following cases to be discussed:

(i) $0 < \lambda < \mu_1$;

(ii) $\lambda > \mu_1$;

(iii) $\lambda = \mu_1$.

In the case (i) $0 < \lambda < \mu_1$, by (1.3), we see that there exists a $\delta(\lambda) > 0$ such that

$$\int_{\mathbb{R}^N} (|\nabla_A u|^2 + V_\lambda(x)|u|^2)\mathrm{d}x \geq \delta(\lambda) \int_{\mathbb{R}^N} (|\nabla_A u|^2 + V^+(x)|u|^2)\mathrm{d}x > 0, \tag{3.1}$$

for every $u \in H^1_{A,V^+}(\mathbb{R}^N)\backslash\{0\}$. Thus, $L^-(\lambda)$, $L^0(\lambda)$ and S^+ are empty, and $S^0 = \{0\}$.

Lemma 31 Let $\{u_n\} \subset S^-$ be a minimizing sequence of $A = \inf_{u \in S^-} J_\lambda(u)$. Suppose $\{u_n\}$ is bounded in E, then $\{u_n\}$ has a subsequence strongly convergent in E.

Proof We may assume that $u_n \rightharpoonup u$ in E as $n \to \infty$. We first show that $u_n \to u$ in $L^\gamma(\mathbb{R}^N)$. By Brézis-Lieb Lemma,

$$\begin{aligned}
&\int_{\mathbb{R}^N} Q(x)|u_n|^\gamma \mathrm{d}x \\
=& \int_{\mathbb{R}^N} Q(x)|u|^\gamma \mathrm{d}x + \int_{\mathbb{R}^N} Q(x)|u_n - u|^\gamma \mathrm{d}x + o(1) \\
=& \int_{\mathbb{R}^N} Q(x)|u|^\gamma \mathrm{d}x + \int_{\{|x|\leq R\}} Q(x)|u_n - u|^\gamma \mathrm{d}x + \int_{\{|x|\geq R\}} Q(x)|u_n - u|^\gamma \mathrm{d}x + o(1) \\
=& \int_{\mathbb{R}^N} Q(x)|u|^\gamma \mathrm{d}x + \int_{\{|x|\geq R\}} Q(x)|u_n - u|^\gamma \mathrm{d}x + o(1),
\end{aligned} \tag{3.2}$$

where $Q(x) \leq 0$ if $|x| \geq R$. Suppose $u_n \not\to u$ in $L^\gamma(\mathbb{R}^N)$, by the assumption $0 < \lambda < \mu_1$ and (3.2), we would have

$$\begin{aligned}
0 &< \int_{\{|x|\geq R\}} (|\nabla_A u|^2 + V_\lambda(x)|u|^2)\mathrm{d}x \\
&\leq \liminf_{n\to\infty} \int_{\mathbb{R}^N} (|\nabla_A u_n|^2 + V_\lambda(x)|u_n|^2)\mathrm{d}x \\
&\leq \int_{\{|x|\geq R\}} Q(x)|u|^\gamma \mathrm{d}x + \int_{\{|x|\geq R\}} Q(x)|u_n - u|^\gamma \mathrm{d}x + o(1) \\
&< \int_{\mathbb{R}^N} Q(x)|u|^\gamma \mathrm{d}x.
\end{aligned} \tag{3.3}$$

So there is an $s (0 < s < 1)$ such that

$$\int_{\mathbb{R}^N} (|\nabla_A (su)|^2 + V_\lambda(x)|su|^2)\mathrm{d}x = \int_{\mathbb{R}^N} Q(x)|su|^\gamma \mathrm{d}x.$$

It implies from (**??**) that $su \in S^-$. On the other hand,

$$\begin{aligned}
\int_{\mathbb{R}^N} (|\nabla_A u|^2 + V_\lambda(x)|u|^2)\mathrm{d}x &\leq \liminf_{n\to\infty} \int_{\mathbb{R}^N} (|\nabla_A u_n|^2 + V_\lambda(x)|u_n|^2)\mathrm{d}x = \frac{A}{\frac{1}{2} - \frac{1}{\gamma}} \\
&\leq \int_{\mathbb{R}^N} (|\nabla_A (su)|^2 + V_\lambda(x)|su|^2)\mathrm{d}x,
\end{aligned}$$

this yields $s \geq 1$ which is a contradiction. Hence $u_n \to u$ in $L^\gamma(\mathbb{R}^N)$ as $n \to \infty$.

Next, we show that $u_n \to u$ in $H^1_{A,V^+}(\mathbb{R}^N)$ up to a subsequence. On the contrary, we would have

$$\begin{aligned}\int_{\mathbb{R}^N}(|\nabla_A u|^2 + V_\lambda(x)|u|^2 - Q(x)|u|^\gamma)\mathrm{d}x &< \liminf_{n\to\infty}\int_{\mathbb{R}^N}(|\nabla_A u_n|^2 + V_\lambda(x)|u_n|^2 - Q(x)|u_n|^\gamma)\mathrm{d}x\\ &= 0,\end{aligned}$$

which yields

$$\Phi_u'(1) = \int_{\mathbb{R}^N}(|\nabla_A u|^2 + V_\lambda(x)|u|^2 - Q(x)|u|^\gamma)\mathrm{d}x < 0.$$

So there exists $0 < \alpha < 1$ such that $\Phi_u'(\alpha) = 0$, that is, $\alpha u \in S^-$. Since each $\Phi_u(t)$ attains its maximum at $t = 1$ if $0 \leq t \leq 1$ and $u \in S^-$, we see that

$$J_\lambda(\alpha u) < \lim_{n\to\infty} J_\lambda(\alpha u_n) \leq \lim_{n\to\infty} J_\lambda(u_n) = A,$$

which is impossible. Therefore, $u_n \to u$ in $H^1_{A,V^+}(\mathbb{R}^N)$ and hence $u_n \to u$ in E as $n \to \infty$.

Proposition 3.1 We have

(i) $\inf\limits_{u\in S^-} J_\lambda(u) > 0$;

(ii) there exists $u \in S^-$ such that $J_\lambda(u) = \inf\limits_{v\in S^-} J_\lambda(v)$.

Proof (i) Obviously, $J_\lambda(u) \geq 0$ if $u \in S^-$. We claim that $\inf\limits_{u\in S^-} J_\lambda(u) > 0$. Indeed, for $u \in S^-$, $v = \frac{u}{\|u\|} \in L^+(\lambda) \cap B^+$ and $u = t(v)v$ with

$$t(v) = \left(\frac{\int_{\mathbb{R}^N}(|\nabla_A v|^2 + V_\lambda(x)|v|^2)\mathrm{d}x}{\int_{\mathbb{R}^N} Q(x)|v|^\gamma \mathrm{d}x}\right)^{\frac{1}{\gamma-2}}.$$

u satisfies

$$\begin{aligned}J_\lambda(u) = J_\lambda(t(v)v) &= \Big(\frac{1}{2} - \frac{1}{\gamma}\Big)t^2(v)\int_{\mathbb{R}^N}(|\nabla_A v|^2 + V_\lambda(x)|v|^2)\mathrm{d}x\\ &= \Big(\frac{1}{2} - \frac{1}{\gamma}\Big)\frac{\big(\int_{\mathbb{R}^N}(|\nabla_A v|^2 + V_\lambda(x)|v|^2)\mathrm{d}x\big)^{\frac{\gamma}{\gamma-2}}}{\big(\int_{\mathbb{R}^N} Q(x)|v|^\gamma \mathrm{d}x\big)^{\frac{2}{\gamma-2}}}\\ &\geq \Big(\frac{1}{2} - \frac{1}{\gamma}\Big)\frac{(\delta(\lambda))^{\frac{\gamma}{\gamma-2}}}{\big(\int_{\mathbb{R}^N} Q(x)|v|^\gamma \mathrm{d}x\big)^{\frac{2}{\gamma-2}}},\end{aligned}$$

by (3.1). To estimate the integral appeared in the denominator, we choose $R > 0$ such that $Q(x) < 0$ for $|x| \geq R$. Applying the Hölder and Sobolev inequalities, we have

$$\begin{aligned}\int_{\mathbb{R}^N} Q(x)|v|^\gamma \mathrm{d}x \leq \int_{\{|x|\leq R\}} Q(x)|v|^\gamma \mathrm{d}x &\leq c(R)\|Q\|_\infty\Big(\int_{\{|x|\leq R\}}|v|^{2^*}\mathrm{d}x\Big)^{\frac{\gamma}{2^*}}\\ &\leq c(R)\|Q\|_\infty\|v\|^\gamma = c(R)\|Q\|_\infty,\end{aligned}$$

where $c(R) > 0$ is a constant depending on R. It yields

$$\inf_{u\in S^-} J_\lambda(u) \geq \Big(\frac{1}{2} - \frac{1}{\gamma}\Big)\frac{(\delta(\lambda))^{\frac{\gamma}{\gamma-2}}}{(\|Q\|_\infty c(R))^{\frac{2}{\gamma-2}}} > 0.$$

(ii) Let $\{u_n\} \subset S^-$ be a minimizing sequence for $A = \inf\limits_{u\in S^-} J_\lambda(u)$. Since $0 < \lambda < \mu_1$, $\{u_n\}$ is bounded in $H^1_{A,V^+}(\mathbb{R}^N)$ and $\{\int_{\mathbb{R}^N} Q(x)|u_n|^\gamma \mathrm{d}x\}$ is also bounded. We now show that $\{u_n\}$ is bounded in $L^\gamma(\mathbb{R}^N)$. We choose $R > 0$ such that $Q(x) \le \frac{Q(\infty)}{2}$ for $|x| \ge R$, then

$$\begin{aligned}\int_{\mathbb{R}^N} Q(x)|u_n|^\gamma \mathrm{d}x &= \int_{\{|x|\le R\}} Q(x)|u_n|^\gamma \mathrm{d}x + \int_{\{|x|\ge R\}} Q(x)|u_n|^\gamma \mathrm{d}x \\ &\le \int_{\{|x|\le R\}} Q(x)|u_n|^\gamma \mathrm{d}x + \frac{Q(\infty)}{2}\int_{\{|x|\ge R\}} |u_n|^\gamma \mathrm{d}x.\end{aligned}$$

So

$$-\frac{Q(\infty)}{2}\int_{\{|x|\ge R\}} |u_n|^\gamma \mathrm{d}x \le -\int_{\mathbb{R}^N} Q(x)|u_n|^\gamma \mathrm{d}x + \int_{\{|x|\le R\}} Q(x)|u_n|^\gamma \mathrm{d}x. \tag{3.4}$$

It yields

$$\int_{\{|x|\ge R\}} |u_n|^\gamma \mathrm{d}x \le c(R). \tag{3.5}$$

Therefore, $\{u_n\}$ is bounded in $L^\gamma(\mathbb{R}^N)$ and then in E. By Lemma 3.1, we may assume that $u_n \to u$ in E as $n \to \infty$. If $u(x) = 0$ on $\mathbb{R}^N$, since

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} (|\nabla_A u_n|^2 + V_\lambda(x)|u_n|^2)\mathrm{d}x = \lim_{n\to\infty}\int_{\mathbb{R}^N} Q(x)|u_n|^\gamma \mathrm{d}x = \frac{A}{\frac{1}{2}-\frac{1}{\gamma}} > 0, \tag{3.6}$$

and $Q(x) < 0$ provided that $|x| \ge R$, we obtain from (3.2) that

$$0 < \lim_{n\to\infty}\int_{\mathbb{R}^N} Q(x)|u_n|^\gamma \mathrm{d}x \le \int_{\mathbb{R}^N} Q(x)|u|^\gamma \mathrm{d}x = 0,$$

a contradiction. Hence, $u \ne 0$. Furthermore, $J_\lambda(u) = \lim\limits_{n\to\infty} J_\lambda(u_n) = \inf\limits_{v\in S^-} J_\lambda(v)$, that is, u is a minimizer on S^-. This completes the proof.

In the case (ii) $\lambda > \mu_1$, we see that φ_1 satisfies

$$\int_{\mathbb{R}^N} (|\nabla_A \varphi_1|^2 + V_\lambda(x)|\varphi_1|^2)\mathrm{d}x = \int_{\mathbb{R}^N} (\mu_1 - \lambda)V^-(x)|\varphi_1|^2 \mathrm{d}x < 0.$$

This yields $\varphi_1 \in L^-(\lambda)$. If $\int_{\mathbb{R}^N} Q(x)|\varphi_1|^\gamma \mathrm{d}x < 0$, then $\varphi_1 \in L^-(\lambda) \cap B^-$ and S^+ is non-empty. In this case, S may consist of two distinct components, so it is possible to obtain two solutions by showing that J_λ has an appropriate minimizer on each component.

Lemma 3.2 Suppose $\int_{\mathbb{R}^N} Q(x)|\varphi_1|^\gamma \mathrm{d}x < 0$, then there exists $\delta > 0$ such that $\overline{L^-(\lambda)} \cap \overline{B^+} = \emptyset$ whenever $\mu_1 \le \lambda < \mu_1 + \delta$.

Proof Suppose that the result is false. Then there would exist sequences $\{\lambda_n\}$ and $\{u_n\}$ such that $\lambda_n \to \mu_1^+$, $\|u_n\| = 1$ and

$$\int_{\mathbb{R}^N} (|\nabla_A u_n|^2 + V_{\lambda_n}(x)|u_n|^2)\mathrm{d}x \le 0, \qquad \int_{\mathbb{R}^N} Q(x)|u_n|^\gamma \mathrm{d}x \ge 0.$$

Since $\{u_n\}$ is bounded in $H^1_{A,V^+}(\mathbb{R}^N)$ and $V^- \in L^{\frac{N}{2}}(\mathbb{R}^N)$, we may assume that $u_n \rightharpoonup u$ in $H^1_{A,V^+}(\mathbb{R}^N)$ and $\int_{\mathbb{R}^N} V^-(x)|u_n|^2 \mathrm{d}x \to \int_{\mathbb{R}^N} V^-(x)|u|^2 \mathrm{d}x$ as $n \to \infty$. We have $u_n \to u$ in $H^1_{A,V^+}(\mathbb{R}^N)$ as $n \to \infty$. Otherwise, we would have

$$\int_{\mathbb{R}^N} (|\nabla_A u|^2 + V_{\mu_1}(x)|u|^2)\mathrm{d}x < \liminf_{n\to\infty}\int_{\mathbb{R}^N} (|\nabla_A u_n|^2 + V_{\lambda_n}(x)|u_n|^2)\mathrm{d}x \le 0, \tag{3.7}$$

a contradiction. As a result

$$\int_{\mathbb{R}^N}(|\nabla_A u|^2+V^+(x)|u|^2-\mu_1 V^-(x)|u|^2)\mathrm{d}x=0.$$

It implies that there exists a constant k such that $u=k\varphi_1$. Since

$$\int_{\mathbb{R}^N}Q^-(x)|u_n|^\gamma\mathrm{d}x\le\int_{\mathbb{R}^N}Q^+(x)|u_n|^\gamma\mathrm{d}x,$$

and suppQ^+ is bounded, similar to the proof of Proposition 3.1 (ii), we may show that $\{u_n\}$ is bounded in $L^\gamma(\mathbb{R}^N)$, by Brézis-Lieb Lemma,

$$\int_{\mathbb{R}^N}Q(x)|u_n|^\gamma\mathrm{d}x=\int_{\mathbb{R}^N}Q(x)|k\varphi_1|^\gamma\mathrm{d}x+\int_{\mathbb{R}^N}Q(x)|u_n-k\varphi_1|^\gamma\mathrm{d}x+o(1).$$

This, together with $\int_{\mathbb{R}^N}Q(x)|u_n|^\gamma\mathrm{d}x\ge0$, implies that

$$\int_{\mathbb{R}^N}Q(x)|k\varphi_1|^\gamma\mathrm{d}x\ge0,$$

However, by the assumption that $\int_{\mathbb{R}^N}Q(x)|\varphi_1|^\gamma\mathrm{d}x<0$, we would have $k=0$. This is impossible as $\|u\|=\|k\varphi_1\|=1$.

Proposition 3.2 Suppose that $\overline{L^-(\lambda)}\cap\overline{B^+}=\emptyset$. Then

(i) $S^0=\{0\}$;

(ii) $0\notin\overline{S^-}$ and S^- is closed;

(iii) $\overline{S^-}\cap\overline{S^+}=\emptyset$;

(iv) S^+ is bounded.

Proof (i) Suppose that there is a $u\in S^0\backslash\{0\}$, then $\frac{u}{\|u\|}\in L^0(\lambda)\cap B^0\subset\overline{L^-(\lambda)}\cap\overline{B^+}=\emptyset$, which is impossible.

(ii) Arguing by contradiction, we assume that there exists $\{u_n\}\subset S^-$ such that $u_n\to0$ in E as $n\to\infty$. Hence

$$0<\int_{\mathbb{R}^N}(|\nabla_A u_n|^2+V_\lambda(x)|u_n|^2)\mathrm{d}x=\int_{\mathbb{R}^N}Q(x)|u_n|^\gamma\mathrm{d}x\to0$$

as $n\to\infty$. Let $v_n=\frac{u_n}{\|u_n\|}$. We may assume that $v_n\rightharpoonup v$ in $H^1_{A,V^+}(\mathbb{R}^N)$ and $\int_{\mathbb{R}^N}V^-(x)|v_n|^2\mathrm{d}x\to\int_{\mathbb{R}^N}V^-(x)|v|^2\mathrm{d}x$ as $n\to\infty$. Since the set $\{x:Q(x)>0\}$ is bounded, we see that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}Q^+(x)|v_n|^\gamma\|u_n\|^{\gamma-2}\mathrm{d}x=0$$

as $n\to\infty$. So

$$\begin{aligned}0&<\int_{\mathbb{R}^N}(|\nabla_A v_n|^2+V_\lambda(x)|v_n|^2)\mathrm{d}x=\int_{\mathbb{R}^N}Q(x)|v_n|^\gamma\|u_n\|^{\gamma-2}\mathrm{d}x\\&\le\int_{\mathbb{R}^N}Q^+(x)|v_n|^\gamma\|u_n\|^{\gamma-2}\mathrm{d}x.\end{aligned}$$

This yields

$$\lim_{n\to\infty}\lambda\int_{\mathbb{R}^N}V^-(x)|v_n|^2\mathrm{d}x=\lambda\int_{\mathbb{R}^N}V^-(x)|v|^2\mathrm{d}x=1,$$

and $v \neq 0$. We also have

$$\int_{\mathbb{R}^N} (|\nabla_A v|^2 + V_\lambda(x)|v|^2)\mathrm{d}x \leq \liminf_{n\to\infty} \int_{\mathbb{R}^N} (|\nabla_A v_n|^2 + V_\lambda(x)|v_n|^2)\mathrm{d}x = 0,$$

which implies $\frac{v}{\|v\|} \in \overline{L^-(\lambda)}$. On the other hand, we may deduce from $\int_{\mathbb{R}^N} Q(x)|v_n|^\gamma \mathrm{d}x > 0$ and the Brézis-Lieb Lemma that $\int_{\mathbb{R}^N} Q(x)|v|^\gamma \mathrm{d}x \geq 0$, so $\frac{v}{\|v\|} \in \overline{B^+}$. Consequently, $\frac{v}{\|v\|} \in \overline{L^-(\lambda)} \cap \overline{B^+}$, contradicting to the assumption. Hence, $0 \notin \overline{S^-}$.

Next, we prove that S^- is closed. By (i), we know that $\overline{S^-} \subset S^- \cup S^0 = S^- \cup \{0\}$. Since $0 \notin \overline{S^-}$, it follows that $\overline{S^-} = S^-$.

(iii) According to (i) and (ii) we have

$$\overline{S^-} \cap \overline{S^+} \subset \overline{S^-} \cap (S^+ \cup S^0) = S^- \cap (S^+ \cup \{0\}) = (S^- \cap S^+) \cup (S^- \cap \{0\}) = \emptyset.$$

(iv) Suppose by contradiction that S^+ is unbounded, then there would exist a sequence $\{u_n\} \subset S^+$ such that $\|u_n\|_E \to \infty$. Setting $v_n = \frac{u_n}{\|u_n\|_E}$, we have

$$\int_{\mathbb{R}^N} (|\nabla_A v_n|^2 + V_\lambda(x)|v_n|^2)\mathrm{d}x = \int_{\mathbb{R}^N} Q(x)|v_n|^\gamma \|u_n\|_E^{\gamma-2}\mathrm{d}x$$

giving

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} Q(x)|v_n|^\gamma \mathrm{d}x = 0,$$

as $n \to \infty$. Suppose $v_n \rightharpoonup v$ in E, we may deduce as before that $\int_{\mathbb{R}^N} Q(x)|v|^\gamma \mathrm{d}x \geq 0$.

We have $v_n \to v$ in $H^1_{A,V^+}(\mathbb{R}^N)$. Indeed, if it is not true, then

$$\int_{\mathbb{R}^N} (|\nabla_A v|^2 + V_\lambda(x)|v|^2)\mathrm{d}x < \lim_{n\to\infty} \int_{\mathbb{R}^N} (|\nabla_A v_n|^2 + V_\lambda(x)|v_n|^2)\mathrm{d}x \leq 0$$

implies $v \neq 0$ and $\frac{v}{\|v\|} \in \overline{L^-(\lambda)} \cap \overline{B^+}$, a contradiction. Thus, $\int_{\mathbb{R}^N} (|\nabla_A v|^2 + V_\lambda(x)|v|^2)\mathrm{d}x \leq 0$.

We distinguish two cases to be discussed, (a) $v_n \to v$ in $L^\gamma(\mathbb{R}^N)$; (b) $v_n \not\to v$ in $L^\gamma(\mathbb{R}^N)$. If (a) occurs, then $v_n \to v$ in E, $\|v\|_E = 1$ and $\frac{v}{\|v\|} \in \overline{L^-(\lambda)} \cap \overline{B^+}$, which is impossible. If (b) occurs, by Brézis-Lieb lemma, $\int_{\mathbb{R}^N} Q(x)|v|^\gamma \mathrm{d}x > 0$, again we have $\frac{v}{\|v\|} \in \overline{L^-(\lambda)} \cap \overline{B^+}$, a contradiction. Hence, S^+ is bounded.

Lemma 3.3 Suppose that $\overline{L^-(\lambda)} \cap \overline{B^+} = \emptyset$. Then

(i) every minimizing sequence for J_λ on S^- is bounded;

(ii) $\inf\limits_{u\in S^-} J_\lambda(u) > 0$;

(iii) there exists a minimizer of $J_\lambda(u)$ on S^-.

Proof (i) Let $\{u_n\} \subset S^-$ be a minimizing sequence for J_λ. Then

$$\int_{\mathbb{R}^N} (|\nabla_A u_n|^2 + V_\lambda(x)|u_n|^2)\mathrm{d}x = \int_{\mathbb{R}^N} Q(x)|u_n|^\gamma \mathrm{d}x \to c \geq 0. \tag{3.8}$$

Suppose by contradiction that $\|u_n\|_E \to \infty$ as $n \to \infty$. Let $v_n = \frac{u_n}{\|u_n\|_E}$, we have

$$\int_{\mathbb{R}^N} (|\nabla_A v_n|^2 + V_\lambda(x)|v_n|^2)\mathrm{d}x = \int_{\mathbb{R}^N} Q(x)|v_n|^\gamma \|u_n\|_E^{\gamma-2}\mathrm{d}x \to 0,$$

and

$$\int_{\mathbb{R}^N} Q(x)|v_n|^\gamma \mathrm{d}x \to 0,$$

as $n \to \infty$. It implies $\int_{\mathbb{R}^N} Q(x)|v|^\gamma \mathrm{d}x \geq 0$. Actually, we have $v_n \to v$ in $H^1_{A,V^+}(\mathbb{R}^N)$ as $n \to \infty$. In fact, otherwise we would have

$$\int_{\mathbb{R}^N} (|\nabla_A v|^2 + V_\lambda(x)|v|^2)\mathrm{d}x < \liminf_{n\to\infty} \int_{\mathbb{R}^N} (|\nabla_A v_n|^2 + V_\lambda(x)|v_n|^2)\mathrm{d}x = 0,$$

and so $v \neq 0$ and $\frac{v}{\|v\|} \in \overline{L^-(\lambda)} \cap \overline{B^+}$, a contradiction. Now, we may obtain a contradiction as the proof of (iv) of Proposition 3.2, we omit the detail.

(ii) It is apparent that $\inf\limits_{u\in S^-} J_\lambda(u) \geq 0$. We claim that $\inf\limits_{u\in S^-} J_\lambda(u) > 0$. In fact, if $\inf\limits_{u\in S^-} J_\lambda(u) = 0$, let $\{u_n\} \subset S^-$ be a minimizing sequence, then

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} (|\nabla_A u_n|^2 + V_\lambda(x)|u_n|^2)\mathrm{d}x = \lim_{n\to\infty} \int_{\mathbb{R}^N} Q(x)|u_n|^\gamma \mathrm{d}x = 0.$$

By (i), $\{u_n\}$ is bounded in E. Applying the arguments of the proof of (iv) of Proposition 3.2, we may show that $u_n \to u$ in $H^1_{A,V^+}(\mathbb{R}^N)$ as $n \to \infty$ and

$$\int_{\mathbb{R}^N} (|\nabla_A u|^2 + V_\lambda(x)|u|^2)\mathrm{d}x = 0.$$

If $u_n \not\to u$ in $L^\gamma(\mathbb{R}^N)$, we have $\int_{\mathbb{R}^N} Q(x)|u|^\gamma \mathrm{d}x > 0$, and then $\frac{u}{\|u\|} \in L^0(\lambda) \cap B^+ \subset \overline{L^-(\lambda)} \cap \overline{B^+}$, a contradiction. Therefore, $u_n \to u$ in E as $n \to \infty$. By Proposition 3.1 (ii), we know that $0 \notin \overline{S^-}$ and S^- is closed, so $u \neq 0$. It yields $\frac{u}{\|u\|} \in L^0(\lambda) \cap B^0 \subset \overline{L^-(\lambda)} \cap \overline{B^+}$, a contradiction.

(iii) Let $\{u_n\}$ be a minimizing sequence for J_λ on S^-. By (i), $\{u_n\}$ is bounded in E. Since

$$\Big(\frac{1}{2} - \frac{1}{\gamma}\Big) \lim_{n\to\infty} \int_{\mathbb{R}^N} Q(x)|u_n|^\gamma \mathrm{d}x = \inf_{u\in S^-} J_\lambda(u) > 0,$$

we have $\int_{\mathbb{R}^N} Q(x)|u|^\gamma \mathrm{d}x > 0$. The assumption $\overline{L^-(\lambda)} \cap \overline{B^+} = \emptyset$ implies $B^+ \subset L^+(\lambda)$. Consequently

$$\int_{\mathbb{R}^N} (|\nabla_A u|^2 + V_\lambda(x)|u|^2)\mathrm{d}x > 0.$$

So we may assume by Lemma 3.1 that $u_n \to u$ in E as $n \to \infty$. Hence, $u \in S$. We know from $\int_{\mathbb{R}^N} Q(x)|u|^\gamma \mathrm{d}x > 0$ that $u \in S^-$. It follows

$$J_\lambda(u) = \lim_{n\to\infty} J_\lambda(u_n) = \inf_{u\in S^-} J_\lambda(u),$$

i.e., u is a minimizer for $J_\lambda(u)$ on S^-.

We now proceed to the investigation of J_λ on S^+.

Lemma 3.4 If $L^-(\lambda) \neq \emptyset$ and $\overline{L^-(\lambda)} \cap \overline{B^+} = \emptyset$, then there exists $u \in S^+$ such that $J_\lambda(u) = \inf\limits_{v\in S^+} J_\lambda(v)$.

Proof By the assumptions, $L^-(\lambda) \subset B^-$, so $S^+ \neq \emptyset$. By Proposition 3.1 (iv), S^+ is bounded. Using Hölder inequality and Sobolev inequality, we find for $u \in S^+$ that

$$\begin{aligned} J_\lambda(u) &= \Big(\frac{1}{2}-\frac{1}{\gamma}\Big)\int_{\mathbb{R}^N}(|\nabla_A u|^2+V_\lambda(x)|u|^2)\mathrm{d}x \geq -\Big(\frac{1}{2}-\frac{1}{\gamma}\Big)\int_{\mathbb{R}^N}\lambda V^-(x)|u|^2\mathrm{d}x \\ &\geq -C\|V^-\|_{\frac{N}{2}}, \end{aligned}$$

so the problem $B := \inf\limits_{u\in S^+} J_\lambda(u)$ is well defined, it is obvious that $B<0$. Let $\{u_n\}\subset S^+$ be a minimizing sequence for J_λ. Then

$$J_\lambda(u_n) = \Big(\frac{1}{2}-\frac{1}{\gamma}\Big)\int_{\mathbb{R}^N}(|\nabla_A u_n|^2+V_\lambda(x)|u_n|^2)\mathrm{d}x = \Big(\frac{1}{2}-\frac{1}{\gamma}\Big)\int_{\mathbb{R}^N}Q(x)|u_n|^\gamma\mathrm{d}x \to B<0.$$

Since $\{u_n\}$ is bounded in E, assuming $u_n \rightharpoonup u$ in E, we obtain

$$\int_{\mathbb{R}^N}(|\nabla_A u|^2+V_\lambda(x)|u|^2)\mathrm{d}x \leq \lim_{n\to\infty}\int_{\mathbb{R}^N}(|\nabla_A u_n|^2+V_\lambda(x)|u_n|^2)\mathrm{d}x<0.$$

It yields $\frac{u}{\|u\|}\in L^-\subset B^-$ and there is a $t(u)$ such that $t(u)u\in S^+$ with

$$t(u)=\left(\frac{\int_{\mathbb{R}^N}(|\nabla_A u|^2+V_\lambda(x)|u|^2)\mathrm{d}x}{\int_{\mathbb{R}^N}Q(x)|u|^\gamma\mathrm{d}x}\right)^{\frac{1}{\gamma-2}}.$$

We now show that $u_n\to u$ in E as $n\to\infty$. First we establish the convergence of $\{u_n\}$ in $H^1_{A,V^+}(\mathbb{R}^N)$. In the contrary case, there would hold

$$\begin{aligned} \int_{\mathbb{R}^N}(|\nabla_A u|^2+V_\lambda(x)|u|^2)\mathrm{d}x &< \lim_{n\to\infty}\int_{\mathbb{R}^N}(|\nabla_A u_n|^2+V_\lambda(x)|u_n|^2)\mathrm{d}x \\ &= \lim_{n\to\infty}\int_{\mathbb{R}^N}Q(x)|u_n|^\gamma\mathrm{d}x \\ &\leq \int_{\mathbb{R}^N}Q(x)|u|^\gamma\mathrm{d}x<0, \end{aligned}$$

because $\frac{u}{\|u\|}\in B^-$. From this we derive that $t(u)>1$, it leads to a contradiction as

$$J_\lambda(t(u)u)<J_\lambda(u)\leq\lim_{n\to\infty}J_\lambda(u_n)=B.$$

Next, we show $u_n\to u$ in $L^\gamma(\mathbb{R}^N)$ as $n\to\infty$. If it is not true, by Brézis-Lieb Lemma,

$$\begin{aligned} \int_{\mathbb{R}^N}(|\nabla_A u|^2+V_\lambda(x)|u|^2)\mathrm{d}x &= \lim_{n\to\infty}\int_{\mathbb{R}^N}(|\nabla_A u_n|^2+V_\lambda(x)|u_n|^2)\mathrm{d}x \\ &= \lim_{n\to\infty}\int_{\mathbb{R}^N}Q(x)|u_n|^\gamma\mathrm{d}x \\ &< \int_{\mathbb{R}^N}Q(x)|u|^\gamma\mathrm{d}x, \end{aligned}$$

which implies $t(u)>1$ and leads to a contradiction. Consequently, $u_n\to u$ in E as $n\to\infty$, and then

$$\int_{\mathbb{R}^N}(|\nabla_A u|^2+V_\lambda(x)|u|^2)\mathrm{d}x=\int_{\mathbb{R}^N}Q(x)|u|^\gamma\mathrm{d}x<0,$$

i.e., $u \in S^+$ and

$$J_\lambda(u) = \lim_{n\to\infty} J_\lambda(u_n) = \inf_{v\in S^+} J_\lambda(v).$$

Thus, u is a minimizer for $J_\lambda(u)$ on S^+.

Suppose $\int_{\mathbb{R}^N} Q(x)|\varphi_1|^\gamma \mathrm{d}x < 0$, then $\varphi_1 \in L^-(\lambda)$ if $\lambda > \mu_1$, $L^-(\lambda) \neq \emptyset$. By Lemmas 3.2–3.4, there exists a $\delta > 0$ such that J_λ has a minimizer on S^- and S^+ respectively whenever $\mu_1 < \lambda < \mu_1 + \delta$. These minimizers are different from each other because $\overline{L^-(\lambda)} \cap \overline{B^+} = \emptyset$. By Lemma 2.2, we have

Proposition 3.3 If $\int_{\mathbb{R}^N} Q(x)|\varphi_1|^\gamma \mathrm{d}x < 0$, then there exists a $\delta > 0$ such that problem (1.1) has two distinct solutions for $\mu_1 < \lambda < \mu_1 + \delta$.

In the case (iii) $\lambda = \mu_1$, we prove that there is a mountain-pass solution of (1.1). We commence by establishing the $(\mathrm{PS})_c$ condition.

Proposition 3.4 Suppose that $\int_{\mathbb{R}^N} Q(x)|\varphi_1|^\gamma \mathrm{d}x < 0$. Then the functional J_λ satisfies the $(\mathrm{PS})_c$ condition for $c \in \mathbb{R}$.

Proof Let $\{u_n\}$ be a $(\mathrm{PS})_c$ sequence.We show that $\{u_n\}$ is bounded in E. If it is not the case, suppose $\|u_n\|_E \to \infty$ as $n \to \infty$. Let $v_n = \frac{u_n}{\|u_n\|_E}$, we may assume that $v_n \rightharpoonup v$ in E and $\int_{\mathbb{R}^N} V^-(x)|v_n|^2 \mathrm{d}x \to \int_{\mathbb{R}^N} V^-(x)|v|^2 \mathrm{d}x$ as $n \to \infty$. $\{v_n\}$ satisfies

$$\int_{\mathbb{R}^N} (|\nabla_A v_n|^2 + V_{\mu_1}(x)|v_n|^2)\mathrm{d}x = \int_{\mathbb{R}^N} Q(x)|v_n|^\gamma \|u_n\|^{\gamma-2}\mathrm{d}x + o(1),$$

and

$$\frac{\gamma}{2}\int_{\mathbb{R}^N} (|\nabla_A v_n|^2 + V_{\mu_1}(x)|v_n|^2)\mathrm{d}x = \int_{\mathbb{R}^N} Q(x)|v_n|^\gamma \|u_n\|^{\gamma-2}\mathrm{d}x + o(1).$$

This implies

$$\int_{\mathbb{R}^N} (|\nabla_A v_n|^2 + V_{\mu_1}(x)|v_n|^2)\mathrm{d}x \to 0 \quad \text{and} \quad \int_{\mathbb{R}^N} Q(x)|v_n|^\gamma \mathrm{d}x \to 0,$$

as $n \to \infty$. Therefore,

$$0 \le \int_{\mathbb{R}^N} (|\nabla_A v|^2 + V_{\mu_1}(x)|v|^2)\mathrm{d}x \le \lim_{n\to\infty}\int_{\mathbb{R}^N} (|\nabla_A v_n|^2 + V_{\mu_1}(x)|v_n|^2)\mathrm{d}x = 0.$$

Then we have $v = k\varphi_1$ for some constant k. By Brézis-Lieb Lemma,

$$\begin{aligned} 0 &= \lim_{n\to\infty}\int_{\mathbb{R}^N} Q(x)|v_n|^\gamma \mathrm{d}x \\ &= \int_{\mathbb{R}^N} Q(x)|k\varphi_1|^\gamma \mathrm{d}x + \lim_{n\to\infty}\int_{\mathbb{R}^N} Q(x)|v_n - k\varphi_1|^\gamma \mathrm{d}x \\ &\le \int_{\mathbb{R}^N} Q(x)|k\varphi_1|^\gamma \mathrm{d}x. \end{aligned}$$

However, $\int_{\mathbb{R}^N} Q(x)|\varphi_1|^\gamma \mathrm{d}x < 0$, it should have $k = 0$. So $v_n \to 0$ in $H^1_{A,V^+}(\mathbb{R}^N)$. Choose $R > 0$ so that $Q(x) < 0$ if $|x| \ge R$, then

$$0 = \lim_{n\to\infty}\int_{\mathbb{R}^N} Q(x)|v_n|^\gamma \mathrm{d}x = \lim_{n\to\infty}\int_{\{|x|\ge R\}} Q(x)|v_n|^\gamma \mathrm{d}x \le 0.$$

It yields $v_n \to 0$ in $L^\gamma(\mathbb{R}^N)$. Consequently, $v_n \to 0$ in E. However, $\|v_n\|_E = 1$, this contradiction leads to that $\{u_n\}$ is bounded in E. Assuming $u_n \rightharpoonup u$ in E, we deduce

$$\begin{aligned}\int_{\mathbb{R}^N}(|\nabla_A u|^2 + V_{\mu_1}(x)|u|^2)\mathrm{d}x &\le \liminf_{n\to\infty}\int_{\mathbb{R}^N}(|\nabla_A u_n|^2 + V_{\mu_1}(x)|u_n|^2)\mathrm{d}x \\ &= \liminf_{n\to\infty}\int_{\mathbb{R}^N} Q(x)|u_n|^\gamma \mathrm{d}x \\ &= \int_{\mathbb{R}^N} Q(x)|u|^\gamma \mathrm{d}x + \liminf_{n\to\infty}\int_{\mathbb{R}^N} Q(x)|u_n-u|^\gamma \mathrm{d}x \\ &\le \int_{\mathbb{R}^N} Q(x)|u|^\gamma \mathrm{d}x \\ &\le \int_{\mathbb{R}^N}(|\nabla_A u|^2 + V_{\mu_1}(x)|u|^2)\mathrm{d}x.\end{aligned}$$

The result follows.

Proposition 3.5 Suppose $\int_{\mathbb{R}^N} Q(x)|\varphi_1|^\gamma \mathrm{d}x < 0$ and $\lambda = \mu_1$, then problem (1.1) has a mountain-pass solution.

Proof Observing that $E \subset H^1_{A,V^+}(\mathbb{R}^N)$, let V denote the orthogonal complement of the subspace span$\{\varphi_1\}$ in $H^1_{A,V^+}(\mathbb{R}^N)$, that is

$$V = \left\{v \middle|\ v \in H^1_{A,V^+}(\mathbb{R}^N) \ \text{ and } \int_{\mathbb{R}^N} \nabla_A v \overline{\nabla_A \varphi_1} + V^+(x) v \overline{\varphi_1} \mathrm{d}x = 0\right\},$$

we decompose $u \in E$ as $u = t\varphi_1 + v$, where $v \in V \cap L^\gamma(\mathbb{R}^N)$. Choosing $R > 0$ such that $-Q(x) \ge -\frac{Q(\infty)}{2} = m > 0$ for $|x| \ge R$ and $\int_{\{|x|\le R\}} Q(x)|\varphi_1|^\gamma \mathrm{d}x < 0$, we have

$$\begin{aligned}J_{\mu_1}(u) &\ge \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla_A u|^2 + V_{\mu_1}(x)|u|^2)\mathrm{d}x - \frac{1}{\gamma}\int_{\{|x|\le R\}} Q(x)|u|^\gamma \mathrm{d}x + \frac{m}{\gamma}\int_{\{|x|\ge R\}}|u|^\gamma \mathrm{d}x \\ &\ge \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla_A u|^2 + V_{\mu_1}(x)|u|^2)\mathrm{d}x - \frac{1}{\gamma}\int_{\{|x|\le R\}} Q(x)(|u|^\gamma - |t\varphi_1|^\gamma)\mathrm{d}x \\ &\quad -\frac{1}{\gamma}\int_{\{|x|\le R\}} Q(x)|t\varphi_1|^\gamma \mathrm{d}x + \frac{m}{\gamma}\int_{\{|x|\ge R\}}|u|^\gamma \mathrm{d}x.\end{aligned}$$

Let $u_1 = u$, $u_2 = t\varphi_1$, $f(u_1) = |u_1|^\gamma$, $f(u_2) = |t\varphi_1|^\gamma$, then

$$f(u_1) - f(u_2) = \int_0^1 [f(su_1 + (1-s)u_2)]'_s\, ds.$$

Since $su_1 + (1-s)u_2 = sv + t\varphi_1$, we have $f(su_1 + (1-s)u_2) = |sv + t\varphi_1|^\gamma$ and

$$|sv + t\varphi_1|^\gamma = [(s\mathrm{Re}v + \mathrm{Re}(t\varphi_1))^2 + (s\mathrm{Im}v + \mathrm{Im}(t\varphi_1))^2|^{\frac{\gamma}{2}}.$$

It yields

$$\begin{aligned}(f(su_1 + (1-s)u_2))'_s &= (|sv + t\varphi_1|^\gamma)'_s \\ &= \frac{\gamma}{2}[(s\mathrm{Re}v + \mathrm{Re}(t\varphi_1))^2 + (s\mathrm{Im}v + \mathrm{Im}(t\varphi_1))^2]^{\frac{\gamma}{2}-1}[2(s\mathrm{Re}v \\ &\quad +\mathrm{Re}(t\varphi_1))\mathrm{Re}v + 2(s\mathrm{Im}v + \mathrm{Im}(t\varphi_1))\mathrm{Im}v] \\ &= \frac{\gamma}{2}[(s\mathrm{Re}v + \mathrm{Re}(t\varphi_1))^2 + (s\mathrm{Im}v + \mathrm{Im}(t\varphi_1))^2]^{\frac{\gamma}{2}-1}[2s|\mathrm{Re}v|^2\end{aligned}$$

$$
\begin{aligned}
&+2\mathrm{Re}(t\varphi_1)\mathrm{Re}v+2s|\mathrm{Im}v|^2+2\mathrm{Im}(t\varphi_1)\mathrm{Im}v]\\
\le&\ \frac{\gamma}{2}[2(s\mathrm{Re}v)^2+2(\mathrm{Re}(t\varphi_1))^2+2(s\mathrm{Im}v)^2+2(\mathrm{Im}(t\varphi_1))^2]^{\frac{\gamma}{2}-1}[2s|v|^2\\
&+2(\mathrm{Re}(t\varphi_1)\mathrm{Re}v+\mathrm{Im}(t\varphi_1)\mathrm{Im}v)]\\
\le&\ \frac{\gamma}{2}[2s^2|v|^2+2|t\varphi_1|^2]^{\frac{\gamma}{2}-1}[2s|v|^2+4|v||t\varphi_1|]\\
\le&\ c(|v|^{\gamma-2}+|t\varphi_1|^{\gamma-2})(|v|^2+|v||t\varphi_1|)\\
=&\ c(|v|^{\gamma}+|v|^{\gamma-1}|t\varphi_1|+|v|^2|t\varphi_1|^{\gamma-2}+|v||t\varphi_1|^{\gamma-1}).
\end{aligned}
$$

By Young's inequality,

$$
|v|^{\gamma-1}|t\varphi_1|\le c_1|v|^{(\gamma-1-\frac{1}{\gamma-1})\frac{\gamma}{\gamma-1-\frac{1}{\gamma-1}}}+c_2(|v|^{\frac{1}{\gamma-1}}|t\varphi_1|)^{\gamma-1}=c_1|v|^{\gamma}+c_2|v||t\varphi_1|^{\gamma-1}.
$$

$$
|v|^2|t\varphi_1|^{\gamma-2}\le c_1(|v|^{2-\frac{\gamma-2}{\gamma-1}})^{\frac{\gamma}{2-\frac{\gamma-2}{\gamma-1}}}+c_2(|v|^{\frac{\gamma-2}{\gamma-1}}|t\varphi_1|^{\gamma-2})^{\frac{\gamma-1}{\gamma-2}}=c_1|v|^{\gamma}+c_2|v||t\varphi_1|^{\gamma-1}.
$$

It follows $(f(su_1+(1-s)u_2))'_s\le c(|v|^{\gamma}+|v||t\varphi_1|^{\gamma-1})$. By Hölder inequality,

$$
\int_{\{|x|\le R\}}|v||t\varphi_1|^{\gamma-1}\mathrm{d}x\le ct^{\gamma-1}\|v\|.
$$

Thus

$$
\int_{\{|x|\le R\}}Q(x)(|u|^{\gamma}-|t\varphi_1|^{\gamma})\mathrm{d}x\le c\int_{\{|x|\le R\}}|v|^{\gamma}\mathrm{d}x+ct^{\gamma-1}\|v\|.
$$

On the other hand,

$$
\begin{aligned}
&\int_{\mathbb{R}^N}(|\nabla_A u|^2+V_{\mu_1}(x)|u|^2)\mathrm{d}x\\
=&\ t^2\int_{\mathbb{R}^N}(|\nabla_A\varphi_1|^2+V_{\mu_1}(x)|\varphi_1|^2)\mathrm{d}x+\int_{\mathbb{R}^N}(|\nabla_A v|^2+V_{\mu_1}(x)|v|^2)\mathrm{d}x\\
=&\int_{\mathbb{R}^N}(|\nabla_A v|^2+V_{\mu_1}(x)|v|^2)\mathrm{d}x\\
\ge&\ \Big(1-\frac{\mu_1}{\mu_2}\Big)\int_{\mathbb{R}^N}(|\nabla_A v|^2+V^+(x)|v|^2)\mathrm{d}x=\Big(1-\frac{\mu_1}{\mu_2}\Big)\|v\|^2,
\end{aligned}
$$

we obtain

$$
J_{\mu_1}(u)\ge\Big(1-\frac{\mu_1}{\mu_2}\Big)\|v\|^2+c|t|^{\gamma}-c\int_{\{|x|\le R\}}|v|^{\gamma}\mathrm{d}x-ct^{\gamma-1}\|v\|+\frac{m}{\gamma}\int_{\{|x|\ge R\}}|u|^{\gamma}\mathrm{d}x
$$

for some constant $c>0$. Introducing the following equivalent norm on E:

$$
\|u\|_E=\big(\max(|t|,\|v\|)^2+\|u\|_{\gamma}^2\big)^{\frac{1}{2}}:=\big(\|u\|_*^2+\|u\|_{\gamma}^2\big)^{\frac{1}{2}},
$$

we obtain that

$$
J_{\mu_1}(u)\ge c_0\delta^{\gamma}+\frac{m}{\gamma}\int_{\{|x|\ge R\}}|u|^{\gamma}\mathrm{d}x,
$$

for some constant $c_0>0$ and $\|u\|_*=\delta$ sufficiently small. If $\|u\|_E=\rho$ and $\|u\|_*=\delta\to 0$, then $\|u\|_{\gamma}\to\rho$. Hence

$$
\liminf_{\|u\|_E=\rho,\delta\to 0}\int_{|x|\ge R}|u|^{\gamma}\mathrm{d}x\ge c_1\rho^{\gamma},
$$

for some constant $c_1 > 0$. Therefore, $J_{\mu_1}(u) \geq \alpha$ for $\|u\|_E = \rho$ and $\alpha, \rho > 0$. Finally, let $\eta \in C_0^\infty(\mathbb{R}^N)$ be with $\mathrm{supp}\eta \subset \{x, Q(x) > 0\}$. Then for sufficiently large $t > 0$ we get $J_{\mu_1}(t\eta) < 0$. The result follows by the mountain pass lemma.

Proof of Theorem 1.1 The consequence of Theorem 1.1 follows from Lemma 2.2, Propositions 3.1, 3.3 and 3.5.

4 Sublinear case

In this section, we prove Theorem 1.2. We know from Section 3 that, $\int_{\mathbb{R}^N}(|\nabla_A u|^2 + V_\lambda(x)|u|^2)\mathrm{d}x \geq (\mu_1 - \lambda)\int_{\mathbb{R}^N} V^-(x)|u|^2\mathrm{d}x > 0$ for all $u \in E\backslash\{0\}$ provided that $\lambda < \mu_1$, so $L^+(\lambda) = \{u \in E : \|u\| = 1\}$ and $L^-(\lambda) = L^0(\lambda) = \emptyset$. If $\lambda > \mu_1$, $L^-(\lambda)$ becomes non-empty and gets larger as λ increases. So, to prove Theorem 1.2, we discuss the vital role played by the condition $L^-(\lambda) \subset B^-$ in determining the nature of the Nehari manifold, and we get a result similar to the Proposition 3.2. It is always true that $L^-(\lambda) \subset B^-$ if $\lambda < \mu_1$, and this may or may not be satisfied if $\lambda > \mu_1$.

Proposition 4.1 Suppose there exists $\hat{\lambda}$ such that for all $\lambda < \hat{\lambda}$, $L^-(\lambda) \subset B^-$. Then, for $\lambda < \hat{\lambda}$,

(i) $L^0(\lambda) \subset B^-$ and so $L^0(\lambda) \bigcap B^0 = \emptyset$;

(ii) S^+ is bounded;

(iii) $0 \notin \overline{S^-}$ and S^- is closed;

(iv) $\overline{S^+} \bigcap S^- = \emptyset$.

Proof (i) Suppose that the result is false. Then there would exist $u \in L^0(\lambda)$ and $u \notin B^-$. If $\lambda < \mu < \hat{\lambda}$, then $u \in L^-(\mu)$ and $L^-(\mu) \not\subset B^-$, which is a contradiction.

(ii) Suppose that S^+ is unbounded. Then there would exist $\{u_n\} \subset S^+$ such that $\|u_n\|_E \to \infty$ as $n \to \infty$. Let $v_n = \frac{u_n}{\|u_n\|_E}$. We may assume $v_n \rightharpoonup v$ in E and $\int_{\mathbb{R}^N} V^-(x)|v_n|^2\mathrm{d}x \to \int_{\mathbb{R}^N} V^-(x)|v|^2\mathrm{d}x$ as $n \to \infty$. Since $u_n \in S$,

$$\int_{\mathbb{R}^N}(|\nabla_A v_n|^2 + V_\lambda(x)|v_n|^2)\,dx = \frac{\int_{\mathbb{R}^N} Q(x)|v_n|^\gamma\mathrm{d}x}{\|u_n\|_E^{2-\gamma}}. \tag{4.1}$$

We deduce from

$$\int_{\mathbb{R}^N} Q(x)|v_n|^\gamma\mathrm{d}x \leq \sup_{x\in\mathbb{R}^N} Q(x)\int_{\mathbb{R}^N}|v_n|^\gamma\mathrm{d}x < \infty,$$

and (4.1) that

$$\int_{\mathbb{R}^N}|\nabla_A v_n|^2 + V_\lambda(x)|v_n|^2\mathrm{d}x \to 0$$

as $n \to \infty$. By the fact $u_n \in S^+$ and Brézis-Lieb Lemma, we deduce

$$\int_{\mathbb{R}^N} Q(x)|v|^\gamma\mathrm{d}x \geq 0. \tag{4.2}$$

We have $v_n \to v$ in $H^1_{A,V^+}(\mathbb{R}^N)$. Indeed, if not,

$$\int_{\mathbb{R}^N}(|\nabla_A v|^2 + V_\lambda(x)|v|^2)\mathrm{d}x < \liminf_{n\to\infty}\int_{\mathbb{R}^N}(|\nabla_A v_n|^2 + V_\lambda(x)|v_n|^2)\mathrm{d}x = 0.$$

Thus $v \neq 0$ and by (i), $\frac{v}{\|v\|} \in L^-(\lambda) \subset B^-$, a contradiction to (4.2). Hence,

$$\int_{\mathbb{R}^N} (|\nabla_A v|^2 + V_\lambda(x)|v|^2)\mathrm{d}x = \liminf_{n\to\infty} \int_{\mathbb{R}^N} (|\nabla_A v_n|^2 + \lambda V(x)|v_n|^2)\mathrm{d}x = 0.$$

We now distinguish two cases: (a) $v_n \to v$ in $L^\gamma(\mathbb{R}^N)$; (b) $v_n \not\to v$ in $L^\gamma(\mathbb{R}^N)$. If (a) occurs, then $v_n \to v$ in E and so $\|v\|_E = 1$, thus $v \neq 0$ and we have $\frac{v}{\|v\|} \in L^0(\lambda) \subset B^-$ which contradicts to (4.2). If (b) occurs, then by Brézis-Lieb Lemma, $\int_{\mathbb{R}^N} Q(x)|v|^\gamma \mathrm{d}x > 0$, again $v \neq 0$ and we have $\frac{v}{\|v\|} \in L^-(\lambda) \subset B^-$, which is impossible either. As a result, S^+ is bounded.

(iii) Suppose $0 \in \overline{S^-}$. Then there exists $\{u_n\} \subset S^-$ such that $u_n \to 0$ in E as $n \to \infty$. Let $v_n = \frac{u_n}{\|u_n\|_E}$. We may assume $v_n \rightharpoonup v$ in E and $\int_{\mathbb{R}^N} \lambda V^-(x)|v_n|^2\mathrm{d}x \to \int_{\mathbb{R}^N} \lambda V^-(x)|v|^2\mathrm{d}x$ as $n \to \infty$. Since $u_n \in S^-$, by (4.1),

$$\int_{\mathbb{R}^N} Q(x)|v_n|^\gamma \mathrm{d}x \to 0. \tag{4.3}$$

We have $v_n \to v$ in $L^\gamma(\mathbb{R}^N)$. Indeed, otherwise, by Brézis-Lieb Lemma and (**??**), we obtain

$$\int_{\mathbb{R}^N} Q(x)|v|^\gamma \mathrm{d}x > 0.$$

This implies $v \neq 0$ and $\frac{v}{\|v\|} \in B^+$. On the other hand,

$$\int_{\mathbb{R}^N} (|\nabla_A v|^2 + V_\lambda(x)|v|^2)\mathrm{d}x \leq \liminf_{n\to\infty} \int_{\mathbb{R}^N} (|\nabla_A v_n|^2 + V_\lambda(x)|v_n|^2)\mathrm{d}x \leq 0,$$

so $\frac{v}{\|v\|} \in L^0(\lambda)$ or $L^-(\lambda)$, but $\frac{v}{\|v\|} \in B^+$ which contradicts to (i). Therefore, $v_n \to v$ in $L^\gamma(\mathbb{R}^N)$.

If $v_n \to v$ in $H^1_{A,V^+}(\mathbb{R}^N)$, we get $v_n \to v$ in E. So $\|v\|_E = 1$ and $\frac{v}{\|v\|} \in L^0(\lambda) \cap B^0$ or $\frac{v}{\|v\|} \in L^-(\lambda) \cap B^0$, which is impossible. Hence $v_n \not\to v$ in $H^1_{A,V^+}(\mathbb{R}^N)$ and

$$\int_{\mathbb{R}^N} (|\nabla_A v|^2 + V_\lambda(x)|v|^2)\mathrm{d}x < \liminf_{n\to\infty} \int_{\mathbb{R}^N} (|\nabla_A v_n|^2 + V_\lambda(x)|v_n|^2)\mathrm{d}x \leq 0,$$

that is, $v \neq 0$ and $\frac{v}{\|v\|} \in L^-(\lambda) \cap B^0$, which again gives a contradiction. Thus $0 \notin \overline{S^-}$.

We now prove that S^- is closed. Suppose that $\{u_n\} \subset S^-$ and $u_n \to u$ in E as $n \to \infty$. Then $u \in \overline{S^-}$ and $u \neq 0$ by (iii). Moreover,

$$\int_{\mathbb{R}^N} (|\nabla_A u|^2 + V_\lambda(x)|u|^2)\mathrm{d}x = \int_{\mathbb{R}^N} Q(x)|u|^\gamma \mathrm{d}x \leq 0.$$

If both integrals equal 0, then $\frac{u}{\|u\|} \in L^0(\lambda) \bigcap B^0$, which contradicts (i). Hence both integrals must be negative and so $u \in S^-$. Thus S^- is closed.

(iv) If there is a $u \in \overline{S^+} \bigcap S^-$, then $u \neq 0$. Moreover,

$$\int_{\mathbb{R}^N} (|\nabla_A u|^2 + V_\lambda(x)|u|^2)\mathrm{d}x = \int_{\mathbb{R}^N} Q(x)|u|^\gamma \mathrm{d}x = 0$$

and so $\frac{u}{\|u\|} \in L^0(\lambda) \bigcap B^0$, a contradiction to (i).

Obviously, $J_\lambda(u) > 0$ on S^-. Moreover,

Proposition 4.2 Suppose there exists $\hat\lambda$ such that for all $\lambda < \hat\lambda$, $L^-(\lambda) \subset B^-$ and S^- is non-empty, then

(i) every minimizing sequence for J_λ on S^- is bounded;

(ii) $\inf\limits_{u\in S^-} J_\lambda(u) > 0$.

Proof (i) Let $\{u_n\}\subset S^-$ be a sequence such that $\lim\limits_{n\to\infty} J_\lambda(u_n) = \inf\limits_{u\in S^-} J_\lambda(u)$. Suppose by contradiction that $\{u_n\}$ is unbounded in E. We may assume that $\|u_n\|_E\to\infty$ as $n\to\infty$. Since $J_\lambda(u_n)$ is bounded and $\{u_n\}\subset S$, it follows that both $\{\int_{\mathbb{R}^N}(|\nabla_A u_n|^2+V_\lambda(x)|u_n|^2)\mathrm{d}x\}$ and $\{\int_{\mathbb{R}^N} Q(x)|u_n|^\gamma \mathrm{d}x\}$ are bounded. Let $v_n=\frac{u_n}{\|u_n\|_E}$, we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}(|\nabla_A v_n|^2+V_\lambda(x)|v_n|^2)\mathrm{d}x=\lim_{n\to\infty}\int_{\mathbb{R}^N} Q(x)|v_n|^\gamma\mathrm{d}x=0.$$

Similar to the proof of Proposition 4.1 (iii), we may get a contradiction.

(ii) We know that $\inf\limits_{u\in S^-} J_\lambda(u)\ge 0$. To show $\inf\limits_{u\in S^-} J_\lambda(u) > 0$, we may assume, on the contrary, that there exists a sequence $\{u_n\}\subset S^-$ such that $\lim\limits_{n\to\infty} J_\lambda(u_n)=\inf\limits_{u\in S^-} J_\lambda(u)=0$. By (i), $\{u_n\}$ is bounded in E. Then we may obtain a contradiction by the same argument as the proof of (i). The proof is completed.

Lemma 4.1 Suppose there exists a $\hat\lambda$ such that $L^-(\lambda)\subset B^-$ for all $\lambda<\hat\lambda$. Then for all $\lambda<\hat\lambda$,

(i) there exists a minimizer for J_λ on S^+;

(ii) there exists a minimizer for J_λ on S^- provided that S^- is non-empty.

Proof The proof of (i) is similar to that of Lemma 3.4, we sketch it. For $u\in S^+$, $J_\lambda(u)=(\frac12-\frac1\gamma)\int_{\mathbb{R}^N} Q(x)|u|^\gamma\mathrm{d}x<0$, thus $\inf\limits_{u\in S^+} J_\lambda(u)<0$. Let $\{u_n\}\subset S^+$ be a minimizing sequence, then

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} Q(x)|u_n|^\gamma\mathrm{d}x>0. \tag{4.4}$$

By Proposition 4.1, S^+ is bounded, we may assume that $u_n\rightharpoonup u$ in E. It yields

$$\int_{\mathbb{R}^N} Q(x)|u|^\gamma\mathrm{d}x>0. \tag{4.5}$$

So $u\neq 0$ and $\frac{u}{\|u\|}\in B^+$. By Proposition 4.1, $\frac{u}{\|u\|}\in L^+(\lambda)$. So there exists a $t(u)$ such that $t(u)u\in S^+$. Now, we may show $u_n\to u$ in E as in the proof of Lemma 3.4. The assertion then follows readily.

The idea of the proof of (ii) is similar to that of Lemma 3.3, we sketch it.

Let $\{u_n\}$ be a minimizing sequence of $\inf\limits_{u\in S^-} J_\lambda(u)$. By Proposition 4.2, $\{u_n\}$ is bounded in E. We assume $u_n\rightharpoonup u$ in E. By Proposition 4.2 (ii),

$$\Big(\frac12-\frac1\gamma\Big)\lim_{n\to\infty}\int_{\mathbb{R}^N} Q(x)|u_n|^\gamma\mathrm{d}x=\inf_{u\in S^-} J_\lambda(u)>0,$$

it follows that $\lim\limits_{n\to\infty}\int_{\mathbb{R}^N} Q(x)|u_n|^\gamma\mathrm{d}x<0$.

To complete the proof, it is sufficient to show $u_n\to u$ in E as $n\to\infty$.

First, we have $u_n\to u$ in $H^1_{A,V^+}(\mathbb{R}^N)$. Otherwise, we would have $u\neq 0$, $\frac{u}{\|u\|}\in L^-(\lambda)\bigcap B^-$ and there exists a $t(u)<1$ such that $t(u)u\in S^-$.

However, the map $J_\lambda(tu)$ attains its maximum at $t=1$ if $0\le t\le 1$ and $t(u)u\in S^-$. Then, we obtain

$$J_\lambda(t(u)u)<\lim_{n\to\infty} J_\lambda(t(u)u_n)\le J_\lambda(u_n)=\inf_{u\in S^-} J_\lambda(u),$$

a contradiction.

Next, we show $u_n \to u$ in $L^\gamma(\mathbb{R}^N)$. Otherwise, we may find a $t(u) < 1$ such that $t(u)u \in S^-$, again we may obtain a contradiction. The result then follows.

Lemma 4.2 Suppose $\int_{\mathbb{R}^N} Q(x)|\varphi_1|^\gamma \mathrm{d}x < 0$. Then there exists $\delta_1 > 0$ such that $u \in L^-(\lambda) \Rightarrow u \in B^-$ whenever $\mu_1 \leq \lambda < \mu_1 + \delta_1$.

The result can be proved similar to the proof of Lemma 3.2.

Corollary 4.1 Suppose $\int_{\mathbb{R}^N} Q(x)|\varphi_1|^\gamma \mathrm{d}x < 0$ and $\mu_1 < \lambda < \mu_1 + \delta_1$, then there exist minimizers u_λ and v_λ of J_λ on S^+ and S^-, respectively.

Proof We know that $\varphi_1 \in L^-(\lambda)$, so $L^-(\lambda)$ is non-empty if $\mu_1 < \lambda$. By Lemma 4.2, the hypotheses of Lemma 4.1 are satisfied with $\hat{\lambda} = \mu_1 + \delta_1$, the result follows.

Proof of Theorem 1.2 Since $L^-(\lambda)$ is empty for $\lambda < \mu_1$, it follows from Lemma 4.1 that J_λ has a minimizer on S^+ if $\lambda < \mu_1$. Theorem 1.2 is a direct consequence of Lemmas 2.2 and 4.1, and Corollary 4.1.

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