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A new approximate formulation of periodic gravity waves

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Abstract Periodic gravity waves travelling in irrotational deep water flows are examined. A new method of implicit approximation to the travelling waves along the free water surface rather than the explicit Stokes wave expansions along the calm water surface is formulated. Therefore, the approximate waves satisfy exactly the dynamic free surface boundary condition along the free water surface, while the corresponding Stokes waves satisfy the dynamic free surface boundary condition approximately along the calm water surface based on the Taylor expansion. The distinction between the proposed wave and the corresponding Stokes wave can be ignored at small wave steepness but becomes clear with the increment of the wave steepness due to the nonphysical form of the Stokes wave at large wave steepness. Approximation to the Stokes highest wave is demonstrated.

Keywords Free surface wave approximation · Nonlinear periodic gravity wave · Potential flow · Stokes waves

1 Introduction

As a fundamental problem in the field of free surface gravity waves, the study of two-dimensional periodic gravity waves travelling in an irrotational flow dates back to the early 19th century, presented by Gerstner [1], showing continuity and free surface pressure conditions of the gravity wave problem. Later, Stokes [2] employed the potential flow theory of gravity waves to propose a perturbation expansion approach to gravity wave solution in a power series of the dimensionless wave steepness $\epsilon = ak$ for wave number k and wave amplitude a of the associated linear wave. The convergence of the Stokes wave power series was given by Levi-Civita [3] and Struik [4] with respect to small steepness. An explicit formulation of the fifth order steady Stokes wave was provided by Fenton [5]. However, Schwartz [6] found that the Stokes expansion method is fundamentally invalid when ϵ is not small. Therefore he developed a different expansion method based on a more suitable expansion parameter. The further developments of the method were examined by Cokelet [7], Longuet-Higgins [8], Longuet-Higgins and Fox [9], Longuet-Higgins and Cleaver [10], Williams [11] and Maklakov [12].

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Actually, the numerical method of Schwartz [6] is based on a holomorphic function technique stemming from Stokes [13] so that the unknown fluid domain in the complex plane $x + \mathrm{i}y$ is conformally mapped onto the fixed domain $\psi < 0$ in the complex velocity potential plane and then conformally mapped onto a unit disc of a complex plane [14]. Therefore, the unknown free surface wave with a complete wave length L becomes the fixed unit circle $\{\mathrm{e}^{\mathrm{i}s};\ -\pi < s < \pi\}$ and the complex velocity reciprocal function $1/(u - \mathrm{i}v)$ is holomorphic in the unit disc. This holomorphic function technique also gives rise to the Nekrasov integral equation [14] of $\alpha(s)$, which denotes the angle of the wave surface inclined to the horizontal. With the use of the Nekrasov integral equation, Krasovskii [15] and Keady and Norbury [16] proved the existence of the periodic wave with angle $0° < \alpha < 30°$. The existence of the limiting wave $\alpha = 30°$ or the Stokes highest crest wave with crest angle 120° was proved by Toland [17] and Amick, Fraenkel and Toland [18].

Although the solutions of the periodic gravity wave problem have been well studied theoretically and numerically with respect to various steepness values, the Stokes wave is still fundamentally important in the understanding of the periodic gravity wave problem. The present paper aims to introduce a new approximation formulation scheme, comparable with the Stokes approximation expansion, of the periodic gravity wave without the requirement of the small steepness assumption. More precisely, the periodic gravity wave η can be approximated implicitly from the following dynamic free surface boundary condition:

$$
\begin{aligned}
k\eta &= -\frac{k}{g}\partial_t\phi - \frac{k}{2g}|\nabla\phi|^2 \\
&= \frac{\omega}{\sqrt{kg}}\sum_{m=1}^{n} m\epsilon^m a_m \mathrm{e}^{\mathrm{i}k\eta}\cos m\theta - \frac{1}{2}\sum_{m=1}^{n} m^2\epsilon^{2m}a_m^2\mathrm{e}^{2\mathrm{i}k\eta} \\
&\quad - \sum_{m=1}^{n-1}\sum_{j=1}^{n-m} m(j+m)\epsilon^{j+2m}\mathrm{e}^{(j+2m)k\eta}a_m a_{j+m}\cos j\theta
\end{aligned}
$$

for $\theta = kx - \omega t + \varphi$, provided that the nth order velocity potential

$$
\phi = \frac{g}{k}\frac{1}{\sqrt{kg}}\sum_{m=1}^{n}\epsilon^m a_m \mathrm{e}^{\mathrm{i}kz}\sin m\theta + O(\epsilon^{n+1})
$$

and the dispersion relation

$$
\frac{\omega}{\sqrt{kg}} = 1 + \sum_{m=1}^{n}\epsilon^m b_m + O(\epsilon^{n+1})
$$

are defined by the Stokes wave approximation [2], where g is the gravitational acceleration, ω is a wave frequency and φ is a phase constant. In contrast with the Stokes explicit expansion method approximating the wave solution through a power series expansion along the calm water surface, the proposed implicit method is to approach the wave solution along the original free surface wave boundary. However, for a gravity wave expanded in a Taylor series along the average water surface with respect to the small steepness ϵ, the proposed nth order approximate wave is equivalent to a corresponding nth order Stokes wave on ignoring a higher order term $O(\epsilon^{n+1})$.

The validity of the approximation formulation of the periodic gravity wave $\eta = \eta(\theta)$ is largely based on the following wave elevation bound:

$$
k\eta \le \frac{\omega^2}{2kg}\frac{(\partial_x\phi)^2}{|\nabla\phi|^2},
$$

which is also true for a periodic gravity wave of a three-dimensional flow travelling in the x-direction.

It should be mentioned that the waves obtained in the present formulation are always smooth, and thus it is necessary to choose a very high order form to approach the Stokes highest wave with non-smooth crest.

Similar to the Stokes waves, the first five order approximate waves are fundamentally important. Thus, the present investigation mainly focusses on the approximations with respect to the second and third orders, then the fourth and fifth orders. Finally, we introduce the general approximation formulation for arbitrary order waves.

The 2D periodic gravity waves can be considered as channel waves, which are generated from a wavemaker in a towing tank. For 2D gravity waves generated from a hydrofoil advancing at uniform speed, a dissipative free surface Green's function approach was developed by the author [19]. For the approximation of 3D Kelvin gravity waves (see Stoker [20]) generated from a moving vehicle, a harmonic function expansion formulation was recently introduced by the author [21].

2 Free surface wave equations and the wave upper bound

The pure gravity 2D flow considered is irrotational and incompressible. The fluid domain is bounded by the free surface wave elevation η and a horizontal impermeable bed $y = -h$. The origin of the Oxy frame is fixed on the calm water surface $y = 0$. The periodic wave travels from left to right. The velocity potential ϕ of the irrotational flow is subject to the Bernoulli equation (see, for example, Lamb [22] and Whitham [23])

$$\partial_t \phi + \frac{1}{2}|\nabla \phi|^2 + \frac{1}{\rho} p + gy = 0 \tag{1}$$

and the Laplace equation

$$\partial_x^2 \phi + \partial_y^2 \phi = 0 \tag{2}$$

in the fluid domain $-h < y < \eta$. Here ∇ denotes the gradient (∂_x, ∂_y), p is the pressure and ρ is the density of the fluid. In fact, if the frame is moving at the wave speed ω/k, this propagating periodic wave is identical to the steady periodic wave (see, for example, Fenton [5]).

The free surface gravity wave problem is to solve the velocity potential ϕ and the wave elevation $y = \eta(x, t)$. The main difficulty of the wave problem is the nonlinear free surface boundary, which can be defined by the kinematic free surface boundary condition (see, for example, Stoker [20] and Whitham [23])

$$(\partial_t + \nabla \phi \cdot \nabla)(y - \eta) = 0 \quad \text{on} \quad y = \eta \tag{3}$$

and the dynamic free surface boundary condition

$$\eta = -\frac{1}{g}\partial_t \phi - \frac{1}{2g}|\nabla \phi|^2 \quad \text{on} \quad y = \eta. \tag{4}$$

Here Eq. (4) is implied from Eq. (1) by assuming the atmospheric pressure $p = 0$ on the free surface. The kinematic boundary condition can be equivalently replaced by the assumption that the free surface is a streamline [5] for the steady wave problem.

To solve the wave problem, it is necessary to provide the free surface boundary condition on the velocity potential without involving η explicitly. This can be obtained from Eqs. (3) and (4), that is, the substitution of Eq. (4) into Eq. (3) gives the required free surface boundary condition as

$$\begin{aligned} 0 &= (\partial_t + \nabla \phi \cdot \nabla)\left(gy + \partial_t \phi + \frac{1}{2}|\nabla \phi|^2\right) \\ &= \partial_t^2 \phi + g\partial_y \phi + \nabla \phi \cdot \nabla \partial_t \phi + \frac{1}{2}\partial_t |\nabla \phi|^2 + \frac{1}{2}\nabla \phi \cdot \nabla |\nabla \phi|^2 \end{aligned}$$

on $y = \eta$, or

$$0 = \partial_t^2 \phi + g\partial_y \phi + \partial_t |\nabla \phi|^2 + \frac{1}{2}\nabla \phi \cdot \nabla |\nabla \phi|^2 \quad \text{on} \quad y = \eta. \tag{5}$$

Therefore, we have the complete gravity wave Eqs. (2), (4) and (5) by adding the impermeable bed condition

$$\partial_y \phi = 0 \quad \text{on} \quad y = -h. \tag{6}$$

To derive the upper bound of the gravity wave η, we employ Eq. (4) to produce the inequality

$$\eta \leq \max_{s>0}\left(-\frac{s}{g}\partial_t \phi - \frac{s^2}{2g}|\nabla \phi|^2\right) = \frac{(\partial_t \phi)^2}{2g|\nabla \phi|^2} \quad \text{whenever} \quad |\nabla \phi| \neq 0 \tag{7}$$

on the free surface boundary $y = \eta$. For the travelling wave $\eta = \eta(\theta)$, the velocity potential ϕ on the free surface is also in the travelling form and hence

$$\partial_t \phi = -\frac{\omega}{k}\partial_x \phi,$$

since $\theta = kx - \omega t + \varphi$. This together with Eq. (7) implies that

$$k\eta \le \frac{\omega^2}{2kg}\frac{(\partial_x \phi)^2}{|\nabla\phi|^2} \quad \text{whenever} \quad |\nabla\phi| \ne 0, \tag{8}$$

which reduces to the wave upper bound

$$k\eta \le \frac{\omega^2}{2kg}. \tag{9}$$

3 Approximate waves of order 1, 2 and 3

3.1 Initial formulation

The gravity wave problem governed by Eqs. (2), (4), (5) and (6) can be approximately formulated with respect to the wave steepness $\epsilon = ak$. Classifying the nonlinear free surface boundary Eqs. (4) and (5) with respect to the order $O(\epsilon^n)$ on the free surface boundary $y = \eta$ rather than on the calm water surface $y = 0$, we obtain the following nonlinear approximation wave equations at first order:

$$\left.\begin{array}{l} \partial_x^2\phi + \partial_y^2\phi = 0, \\ \partial_t^2\phi + g\partial_y\phi\big|_{y=\eta} = O(\epsilon^2), \\ \partial_y\phi\big|_{y=-h} = 0, \end{array}\right\} \tag{10}$$

$$\eta = -\frac{1}{g}\partial_t\phi\big|_{y=\eta}, \tag{11}$$

at second order:

$$\left.\begin{array}{l} \partial_x^2\phi + \partial_y^2\phi = 0, \\ \partial_t^2\phi + g\partial_y\phi + \partial_t|\nabla\phi|^2\big|_{y=\eta} = O(\epsilon^3), \\ \partial_y\phi|_{y=-h} = 0, \end{array}\right\} \tag{12}$$

$$\eta = -\frac{1}{g}\partial_t\phi - \frac{1}{2g}|\nabla\phi|^2\big|_{y=\eta}, \tag{13}$$

and at third order:

$$\left.\begin{array}{l} \partial_x^2\phi + \partial_y^2\phi = 0, \\ \partial_t^2\phi + g\partial_y\phi + \partial_t|\nabla\phi|^2 + \frac{1}{2}\nabla\phi\cdot\nabla|\nabla\phi|^2\big|_{y=\eta} = O(\epsilon^4), \\ \partial_y\phi|_{y=-h} = 0, \end{array}\right\} \tag{14}$$

$$\eta = -\frac{1}{g}\partial_t\phi - \frac{1}{2g}|\nabla\phi|^2\big|_{y=\eta}. \tag{15}$$

3.2 Additional formulation for infinite water depth $h \to \infty$

When the water depth is infinite $h \to \infty$, the approximation problem can be simplified significantly. Therefore, from now on, we examine the approximation problem based on the deep water assumption, $h \to \infty$.

In order to be comparable with the Stokes waves, we choose the same velocity potential with respect to the Stokes waves up to the third order in the following form (see Fenton [5]):

$$\phi = a\delta\sqrt{\frac{g}{k}}\mathrm{e}^{ky}\sin\theta \quad \text{with} \quad \theta = kx - \omega t + \varphi, \tag{16}$$

where δ is defined as

$$\delta = \begin{cases} 1 & \text{for order } 1, \\ 1 & \text{for order } 2, \\ 1 - \frac{1}{2}\epsilon^2 & \text{for order } 3. \end{cases}$$

The velocity potential ϕ is a suitable solution to the boundary value problems (10), (12) and (14). Especially, upon substitution of Eq. (16) into the free surface boundary conditions (10), (12) and (14), the first two order free surface equations are

$$0 = \partial_t^2\phi + g\partial_y\phi + \partial_t|\nabla\phi|^2\Big|_{y=\eta} = \partial_t^2\phi + g\partial_y\phi\Big|_{y=\eta}$$
$$= \left(-\omega^2 + kg\right)a\sqrt{\frac{g}{k}}\mathrm{e}^{k\eta}\sin\theta,$$

which gives the linear dispersion relation

$$\frac{\omega^2}{kg} = 1, \tag{17}$$

while the third order free surface equation is

$$0 = \partial_t^2\phi + g\partial_y\phi + \partial_t|\nabla\phi|^2 + \frac{1}{2}\nabla\phi\cdot\nabla|\nabla\phi|^2\Big|_{y=\eta} + O(\epsilon^4)$$
$$= \left(-\omega^2 + kg + \epsilon^2\delta^2 kg\mathrm{e}^{2k\eta}\right)a\delta\sqrt{\frac{g}{k}}\mathrm{e}^{k\eta}\sin\theta + O(\epsilon^4)$$
$$= \left(-\omega^2 + kg + \epsilon^2\delta^2 kg\right)a\delta\sqrt{\frac{g}{k}}\mathrm{e}^{k\eta}\sin\theta + O(\epsilon^4),$$

which gives the dispersion relation

$$\frac{\omega^2}{kg} = 1 + \epsilon^2\delta^2 = 1 + \epsilon^2\left(1 - \frac{1}{2}\epsilon^2\right)^2. \tag{18}$$

Here the third order approximation relies on the observation

$$\mathrm{e}^{2k\eta} = 1 + O(\epsilon).$$

Therefore, the combination of the velocity potential ϕ given by Eq. (16), the dynamic free surface boundary conditions (11), (13) and (15) and the dispersion relations (17) and (18) produce the implicit expression of the periodic wave solutions in the first order form

$$\eta = a\mathrm{e}^{k\eta}\cos\theta, \qquad \frac{\omega^2}{kg} = 1, \tag{19}$$

the second order form

$$\eta = \frac{\omega}{\sqrt{kg}}a\mathrm{e}^{k\eta}\cos\theta - \frac{a^2k}{2}\mathrm{e}^{2k\eta}, \qquad \frac{\omega^2}{kg} = 1, \tag{20}$$

and the third order form

$$\eta = \frac{\omega}{\sqrt{kg}}a\delta\mathrm{e}^{k\eta}\cos\theta - \frac{a^2k}{2}\delta^2\mathrm{e}^{2k\eta}, \qquad \frac{\omega^2}{kg} = 1 + \epsilon^2\delta^2. \tag{21}$$

3.3 The Stokes waves and the approximate waves

The relationship between the Stokes waves [5,2] and the approximate waves (20), (21) can be derived by applying the Taylor expansion to the approximate waves along the calm water surface.

Indeed, the Taylor series of the waves (20), (21) with respect to small $k\eta$ is

$$\begin{aligned} k\eta &= \frac{\omega}{\sqrt{kg}}\epsilon\delta e^{k\eta}\cos\theta - \frac{\epsilon^2}{2}\delta^2 e^{2k\eta} \\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\partial_s^n\left[\frac{\omega}{\sqrt{kg}}\epsilon\delta e^s\cos\theta - \frac{\epsilon^2}{2}\delta^2 e^{2s}\right]_{s=0} k^n\eta^n, \end{aligned}$$

which implies that

$$\begin{aligned} k\eta &= \epsilon\delta\frac{\omega}{\sqrt{kg}}\cos\theta - \frac{\epsilon^2}{2}\delta^2 \\ &\quad + \left(\epsilon\delta\frac{\omega}{\sqrt{kg}}\cos\theta - \epsilon^2\delta^2\right)\left(\epsilon\delta\frac{\omega}{\sqrt{kg}}\cos\theta - \frac{1}{2}\epsilon^2\delta^2 + \epsilon^2\delta^2\frac{\omega^2}{kg}\cos^2\theta\right) \\ &\quad + \frac{1}{2}\left(\epsilon\delta\frac{\omega}{\sqrt{kg}}\cos\theta - 2\epsilon^2\delta^2\right)\epsilon^2\delta^2\frac{\omega^2}{kg}\cos^2\theta + O(\epsilon^4). \end{aligned}$$

By the linear dispersion relation (17) for the second order wave and the nonlinear dispersion relation (18) for the third order wave or

$$\frac{\omega}{\sqrt{kg}} = \sqrt{1+\epsilon^2\delta^2} = 1 + \frac{1}{2}\epsilon^2 + O(\epsilon^4), \tag{22}$$

we have

$$\begin{aligned} k\eta &= \epsilon\delta\frac{\omega}{\sqrt{kg}}\cos\theta - \frac{1}{2}\epsilon^2 + \epsilon^2\frac{\omega^2}{kg}\cos^2\theta - \frac{3}{2}\epsilon^3\frac{\omega}{\sqrt{gk}}\cos\theta \\ &\quad + \epsilon^3\left(\frac{\omega^2}{kg}\right)^3\cos^3\theta + \frac{1}{2}\epsilon^3\left(\frac{\omega^2}{kg}\right)^3\cos^3\theta + O(\epsilon^4) \\ &= \epsilon\left(1-\frac{1}{2}\epsilon^2\right)\left(1+\frac{1}{2}\epsilon^2\right)\cos\theta - \frac{1}{2}\epsilon^2 + \epsilon^2\cos^2\theta - \frac{3}{2}\epsilon^3\cos\theta + \frac{3}{2}\epsilon^3\cos^3\theta + O(\epsilon^4) \\ &= \epsilon\cos\theta + \frac{1}{2}\epsilon^2\cos 2\theta + \frac{3}{8}\epsilon^3(\cos 3\theta - \cos\theta) + O(\epsilon^4). \end{aligned}$$

This gives the linear wave

$$k\zeta_1 = \epsilon\cos\theta, \qquad \frac{\omega^2}{kg} = 1, \tag{23}$$

the second order Stokes wave (see [2,5])

$$k\zeta_2 = \epsilon\cos\theta + \frac{1}{2}\epsilon^2\cos 2\theta, \qquad \frac{\omega^2}{kg} = 1, \tag{24}$$

and the third order Stokes wave (see [2,5])

$$k\zeta_3 = \epsilon\cos\theta + \frac{\epsilon^2}{2}\cos 2\theta + \frac{3}{8}\epsilon^3\left(\cos 3\theta - \cos\theta\right), \tag{25}$$

together with the dispersion relation defined by Eq. (22).

4 Numerical simulation of the second and the third order waves

It is proved in Appendix A that the second order wave (20) and the third order wave (21) exist uniquely. The existence analysis also shows the convergence of the following elementary iteration scheme:

$$\eta_{n+1} = \frac{\omega}{\sqrt{kg}} a\delta e^{k\eta_n} \cos\theta - \frac{a^2 k}{2}\delta^2 e^{2k\eta_n} \quad \text{for} \quad n \geq 0, \tag{26}$$

for any initial wave elevation η_0.

Let the contraction constant be defined as

$$\tau = \max\left\{\sqrt{1+\epsilon^2\delta^2}\epsilon\delta + \epsilon^2\delta^2,\ 0.633\right\} \quad \text{for} \quad \epsilon < 0.75,$$

due to Eqs. (44), (45) and (46). The error between the analytic solution η from Eqs. (20) and (21) and the numerical solution η_n is estimated as, for $-\pi \leq \theta < \pi$,

$$\begin{aligned} |\eta(\theta) - \eta_n(\theta)| &\leq \sum_{m=n}^{\infty} |\eta_{m+1}(\theta) - \eta_m(\theta)| \\ &\leq \sum_{m=n}^{\infty} \tau^m |\eta_1(\theta) - \eta_0(\theta)| \leq \frac{\tau^{n+1}}{1-\tau} \end{aligned}$$

for $\epsilon \leq 0.6$, if the round-off error is ignored and $\eta_0 = 0$. Here we have used Eq. (41).

For example, for $\epsilon \leq 0.6$, the following estimate:

$$\sqrt{1+\epsilon^2\delta^2}\epsilon\delta + \epsilon^2\delta^2 < 0.8,$$

is true and hence

$$\tau = \max\{0.8,\ 0.633\} = 0.8, \qquad \epsilon \leq 0.6,$$

and so the convergence is estimated as

$$\max_{-\pi<\theta<\pi} |\eta(\theta) - \eta_n(\theta)| \leq \frac{\tau^{n+1}}{1-\tau} = 5 \times 0.8^{n+1}.$$

The existence of the analytic solution given in the previous section is subject to the bound $\epsilon \leq 0.75$. Actually, this bound can be increased if the rigorous analysis in Appendix A is replaced by numerical computation.

For presentation purpose, selected numerical results are shown in Fig. 1 for propagation wave in the time domain [0, 4] and steady wave in the spatial domain $[2\pi, 6\pi]$. Especially, the grid points for the space and time domains are between 101 and 201, and the iteration step number is 40 since the wave difference $\eta_{n+1} - \eta_n = O(10^{-16})$ for $n \geq 40$ can be ignored. A double-precision FORTRAN 90 code is used. The CPU time to achieve convergence is always less than 0.1 s.

Define the upper bound function

$$\xi = \frac{1}{k}\frac{\omega^2}{2kg}\frac{(\partial_x\phi)^2}{|\nabla\phi|^2}.$$

Then Eq. (8) gives

$$k\eta \leq k\xi. \tag{27}$$

It is readily seen that the velocity potential expressed by Eq. (16) implies the simple formulation of the upper bound function forms

$$\xi_2 \equiv \xi = \frac{1}{2k}\cos^2(kx - \omega t) \quad \text{for the second order wave} \tag{28}$$

and

$$\xi_3 \equiv \xi = \frac{1+\epsilon^2\delta^2}{2k}\cos^2(kx - \omega t) \quad \text{for the third order wave.} \tag{29}$$

The comparisons of the approximate waves, the associated Stokes waves and the upper bound waves are displayed respectively in Figs. 1 and 2 with respect to different wave number k and different linear wave amplitude a.

Figures 1 and 2 illustrate that the gaps between the upper bound functions and the approximate waves become narrower as the steepness ϵ increases. In particular, for the third order wave, the inequality (27) reduces to an identity around the crests when the steepness $\epsilon \geq 0.7$. Therefore, it follows from Eq. (29) and Fig. 1 that the third order wave (21) crest is subject to the limit relation

$$k\eta_{\text{crest}} \approx \frac{\omega^2}{2kg} = \frac{1+\epsilon^2(1-\frac{1}{2}\epsilon^2)^2}{2} \quad \text{for} \ \ \epsilon = 0.7, \ 0.8.$$

To understand the approximation, it is necessary to provide comparison between the approximation wave and the exact water wave solution to Eqs. (2), (4–6) or the exact gravity wave equations

$$\left.\begin{aligned} &0 = \partial_x^2\phi + \partial_y^2\phi, \\ &0 = \partial_t^2\phi + g\partial_y\phi + \partial_t|\nabla\phi|^2 + \frac{1}{2}\nabla\phi\cdot\nabla|\nabla\phi|^2\Big|_{y=\eta}, \\ &\eta = -\frac{1}{g}\partial_t\phi - \frac{1}{2g}|\nabla\phi|^2\Big|_{y=\eta}, \\ &0 = \lim_{y\to-\infty}\partial_y\phi. \end{aligned}\right\} \tag{30}$$

The existence of the analytic wave solution to this problem was proved in [15–18]. Its numerical wave elevation was simulated by Schwartz [6] in a dimensionless form based on the nonlinear wave steepness H/L, where the wave height H denotes the vertical distance between a wave crest and a wave trough and the wave length $L = 2\pi/k$.

The nonlinear wave steepness value

$$H/L \approx 0.1412$$

was measured by Schwartz [6] numerically for the Stokes highest wave with crest angle 120°. The numerical solution comparison between the third order wave and the gravity wave of Eq. (30) is displayed in Fig. 3, which shows excellent agreement for $H/L = 0.1$ and $H/L = 0.13$, but not good enough for the Stokes highest wave with $H/L = 0.1412$. The comparison in Fig. 3 indicates the trend that the crest gap between the present third order wave and the exact wave enlarges with increment of the ratio H/L.

5 A modified third order approximation

Actually, the wave solutions within the same order of the approximation are not unique. For a better approximation result, the third order wave (21) is now modified as

$$k\eta = \frac{\omega}{\sqrt{kg}}\epsilon\left(1-\frac{1}{2}\epsilon^2\right)e^{k\eta}\cos\theta - \frac{\epsilon^2}{2}(1-2\epsilon^2)e^{2k\eta}, \tag{31}$$

with the same dispersion relation

$$\frac{\omega^2}{kg} = 1+\epsilon^2\left(1-\frac{1}{2}\epsilon^2\right)^2 \tag{32}$$

and the same velocity potential ϕ given by Eq. (16). Therefore, this is the third order approximate wave satisfying the following equations:

$$\left.\begin{aligned} &0 = \partial_x^2\phi + \partial_y^2\phi, \\ &0 = \partial_t^2\phi + g\partial_y\phi + \partial_t|\nabla\phi|^2 + \frac{1}{2}\nabla\phi\cdot\nabla|\nabla\phi|^2 + O(\epsilon^4)\Big|_{y=\eta}, \\ &\eta = -\frac{1}{g}\partial_t\phi - \frac{1}{2g}|\nabla\phi|^2\Big|_{y=\eta} + O(\epsilon^4), \\ &0 = \lim_{y\to-\infty}\partial_y\phi. \end{aligned}\right\}$$

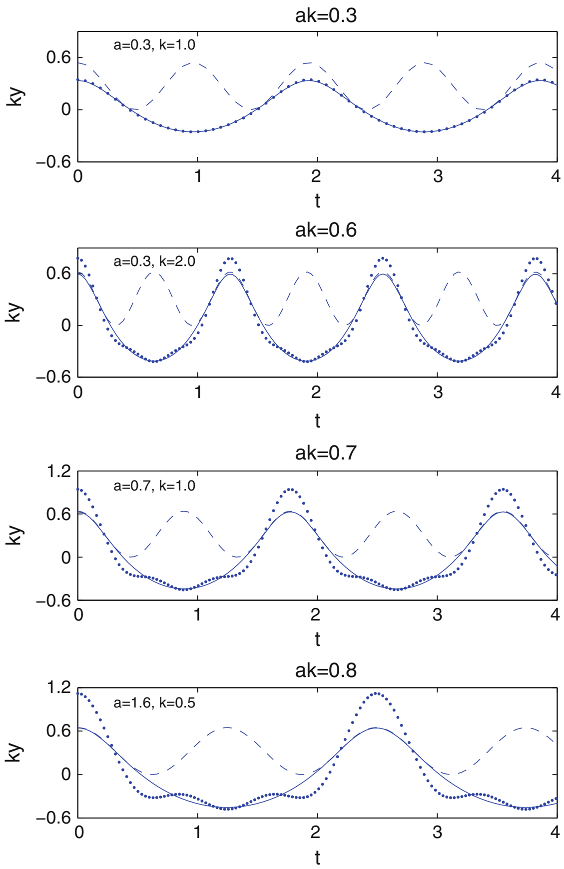

Fig. 1 Comparison of the third order wave $k\eta$ (21) (*continuous line*), the upper bound wave $k\xi_3$ (29) (*dashed line*) and the third order Stokes wave $k\zeta_3$ (25) (*dotted line*), with respect to various values of wave steepness $\epsilon = ak$, travelling from the position $x = 0$

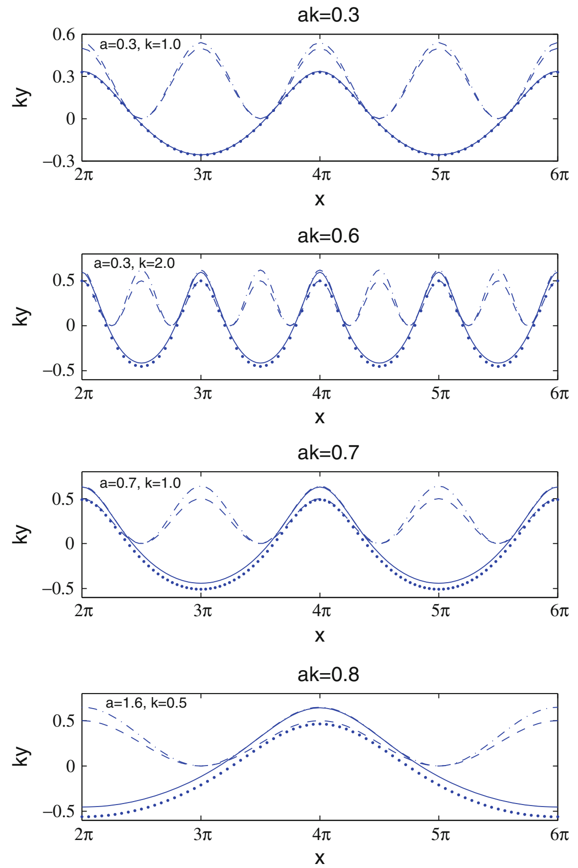

Fig. 2 Comparison of the third order wave $k\eta$ (21) (*continuous line*), the upper bound wave $k\xi_3$ (29) (*dashed dotted line*), the second order wave $k\eta$ (20) (*dotted line*) and the upper bound wave $k\xi_2$ (28) (*dashed line*), with respect to various values of the wave steepness $\epsilon = ak$, at the initial stage $t = 0$

Applying the iteration scheme

$$\eta_{n+1} = \frac{\omega}{\sqrt{kg}} a \left(1 - \frac{1}{2}\epsilon^2\right) \mathrm{e}^{k\eta_n} \cos\theta - \frac{a^2 k}{2}(1 - 2\epsilon^2)\mathrm{e}^{2k\eta_n}$$

with the initial wave $\eta_0 = 0$ and the same computation parameters as in the third order problem in the previous section, we produce the results displayed in Fig. 4, which improves the original third order approximation (Fig. 3) significantly. More precisely, the agreement between the modified wave and the exact wave remains excellent for the ratio $H/L = 0.1$, while the crest gaps between them (Fig. 4) with respect to the ratios $H/L = 0.13$ and $H/L = 0.1412$ are substantially narrowed compared with the corresponding crest gaps illustrated in Fig. 3.

6 The fourth and the fifth order approximate waves

This section contributes to the further approximate wave problem defined by the free surface boundary condition

$$0 = \partial_t^2\phi + g\partial_y\phi + \partial_t|\nabla\phi|^2 + \frac{1}{2}\nabla\phi\cdot\nabla|\nabla\phi|^2 + O(\epsilon^{n+1}) \tag{33}$$

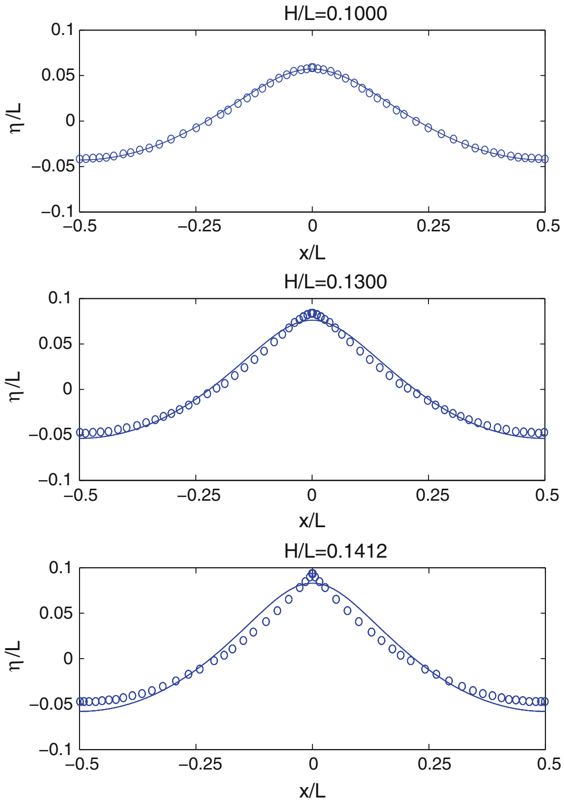

Fig. 3 Comparison of the third order approximate waves (14), (15) and (16) (*solid lines*) and the nonlinear exact waves (30) (*circles lines*) from Schwartz [6, Fig. 9] with respect to moderate ratio of wave height to wave length $H/L = 0.1$, large ratio $H/L = 0.13$ and the largest ratio $H/L = 0.1412$

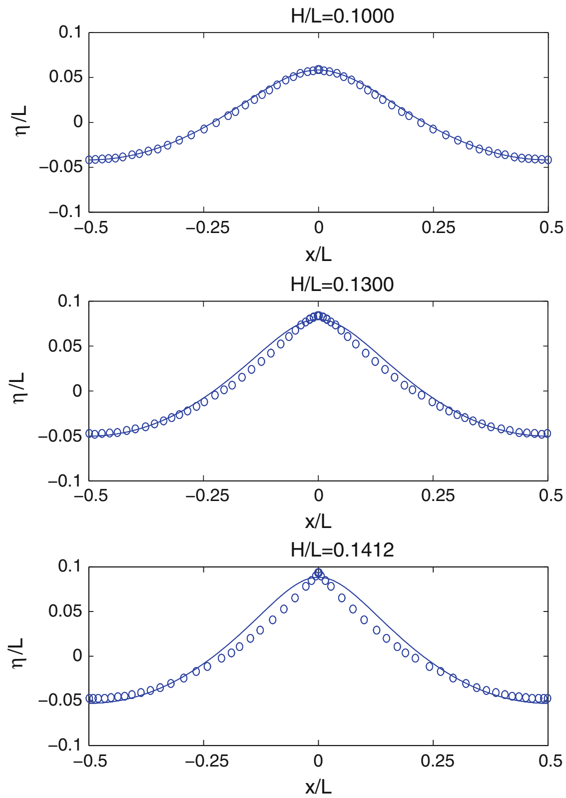

Fig. 4 Comparison of the modified third order approximate waves (16), (31) and (32) (*solid lines*) and the nonlinear exact waves (30) (*circles lines*) from Schwartz [6, Fig. 9]

for the orders $n = 4$ and $n = 5$. Recall that $\epsilon = ak$ and $\theta = kx - \omega t + \varphi$. It follows from Fenton [5] that the free surface boundary condition (33) is satisfied by the velocity potential

$$\phi = \sqrt{\frac{g}{k^3}}\left(\left(\epsilon - \frac{1}{2}\epsilon^3\right)\mathrm{e}^{k\eta}\sin\theta + \frac{\epsilon^4}{2}\mathrm{e}^{2k\eta}\sin 2\theta\right) \tag{34}$$

together with the dispersion relation

$$\frac{\omega}{\sqrt{kg}} = 1 + \frac{1}{2}\epsilon^2 + \frac{1}{8}\epsilon^4 \tag{35}$$

for the order $n = 4$, and is satisfied by the velocity potential

$$\phi = \sqrt{\frac{g}{k^3}}\left(\left(\epsilon - \frac{1}{2}\epsilon^3 - \frac{37}{24}\epsilon^5\right)\mathrm{e}^{k\eta}\sin\theta + \frac{\epsilon^4}{2}\mathrm{e}^{2k\eta}\sin 2\theta + \frac{\epsilon^5}{12}\mathrm{e}^{3k\eta}\sin 3\theta\right) \tag{36}$$

together with the same dispersion relation (35) for the order $n = 5$.

For the fourth order velocity potential (34), we see that

$$-\frac{k}{g}\partial_t\phi = \frac{\omega}{\sqrt{kg}}\left(\left(\epsilon - \frac{1}{2}\epsilon^3\right)\mathrm{e}^{k\eta}\cos\theta + \epsilon^4\mathrm{e}^{2k\eta}\cos 2\theta\right)$$

and

$$\frac{k}{g}|\nabla\phi|^2 = \left(\left(\epsilon - \frac{1}{2}\epsilon^3\right) e^{k\eta}\cos\theta + \epsilon^4 e^{2k\eta}\cos 2\theta\right)^2 + \left(\left(\epsilon - \frac{1}{2}\epsilon^3\right) e^{k\eta}\sin\theta + \epsilon^4 e^{2k\eta}\sin 2\theta\right)^2$$
$$= \left(\epsilon - \frac{1}{2}\epsilon^3\right)^2 e^{2k\eta} + 2\left(\epsilon - \frac{1}{2}\epsilon^3\right)\epsilon^4 e^{3k\eta}\cos\theta + \epsilon^8 e^{4k\eta}.$$

Thus, the dynamic free surface boundary condition (4) together with the dispersion relation (35) gives rise to the fourth order approximate wave equation

$$\begin{aligned} k\eta &= -\frac{k}{g}\partial_t\phi - \frac{k}{2g}|\nabla\phi|^2 \\ &= \left(1 + \frac{1}{2}\epsilon^2 + \frac{1}{8}\epsilon^4\right)\left(\left(\epsilon - \frac{1}{2}\epsilon^3\right) e^{k\eta}\cos\theta + \epsilon^4 e^{2k\eta}\cos 2\theta\right) \\ &\quad -\frac{1}{2}\left(\left(\epsilon - \frac{1}{2}\epsilon^3\right)^2 e^{2k\eta} + 2\left(\epsilon - \frac{1}{2}\epsilon^3\right)\epsilon^4 e^{3k\eta}\cos\theta + \epsilon^8 e^{4k\eta}\right). \end{aligned} \tag{37}$$

Iterating the following equation:

$$\begin{aligned} k\eta_{n+1} &= \left(1 + \frac{1}{2}\epsilon^2 + \frac{1}{8}\epsilon^4\right)\left(\left(\epsilon - \frac{1}{2}\epsilon^3\right) e^{k\eta_n}\cos\theta + \epsilon^4 e^{2k\eta_n}\cos 2\theta\right) \\ &\quad -\frac{1}{2}\left(\left(\epsilon - \frac{1}{2}\epsilon^3\right)^2 e^{2k\eta_n} + 2\left(\epsilon - \frac{1}{2}\epsilon^3\right)\epsilon^4 e^{3k\eta_n}\cos\theta + \epsilon^8 e^{4k\eta_n}\right) \end{aligned}$$

up to 40 steps with 101 grid points in a single wave length, we obtain the numerical results with respect to the nonlinear steepness values $H/L = 0.1, 0.13$ and 0.1412 displayed in Fig. 5, which shows very good agreement between the fourth order wave and exact wave for $H/L = 0.1$ and $H/L = 0.13$, and the difference between these waves at $H/L = 0.1412$ is also further reduced.

Similarly, for the fifth order velocity potential (36), we see that

$$\begin{aligned} -\frac{k}{g}\partial_t\phi &= \frac{\omega}{\sqrt{kg}}\left(\left(\epsilon - \frac{1}{2}\epsilon^3 - \frac{37}{24}\epsilon^5\right) e^{k\eta}\cos\theta + \epsilon^4 e^{2k\eta}\cos 2\theta + \frac{\epsilon^5}{4} e^{3k\eta}\cos 3\theta\right) \\ &= \left(1 + \frac{1}{2}\epsilon^2 + \frac{1}{8}\epsilon^4\right)\left(\left(\epsilon - \frac{1}{2}\epsilon^3 - \frac{37}{24}\epsilon^5\right) e^{k\eta}\cos\theta + \epsilon^4 e^{2k\eta}\cos 2\theta\right) \\ &\quad + \left(1 + \frac{1}{2}\epsilon^2 + \frac{1}{8}\epsilon^4\right)\frac{\epsilon^5}{4} e^{3k\eta}\cos 3\theta \end{aligned}$$

and

$$\begin{aligned} \frac{k}{g}|\nabla\phi|^2 &= \left(\left(\epsilon - \frac{1}{2}\epsilon^3 - \frac{37}{24}\epsilon^5\right) e^{k\eta}\cos\theta + \epsilon^4 e^{2k\eta}\cos 2\theta + \frac{\epsilon^5}{4} e^{3k\eta}\cos 3\theta\right)^2 \\ &\quad + \left(\left(\epsilon - \frac{1}{2}\epsilon^3 - \frac{37}{24}\epsilon^5\right) e^{k\eta}\sin\theta + \epsilon^4 e^{2k\eta}\sin 2\theta + \frac{\epsilon^5}{4} e^{3k\eta}\sin 3\theta\right)^2 \\ &= \left(\epsilon - \frac{1}{2}\epsilon^3 - \frac{37}{24}\epsilon^5\right)^2 e^{2k\eta} + 2\left(\epsilon - \frac{1}{2}\epsilon^3 - \frac{37}{24}\epsilon^5\right)\epsilon^4 e^{3k\eta}\cos\theta \\ &\quad + \frac{\epsilon^9}{2} e^{5k\eta}\cos\theta + \left(\epsilon - \frac{1}{2}\epsilon^3 - \frac{37}{24}\epsilon^5\right)\frac{\epsilon^5}{2} e^{4k\eta}\cos 2\theta + \epsilon^8 e^{4k\eta} + \frac{\epsilon^{10}}{16} e^{6k\eta}. \end{aligned}$$

Thus, the dynamic free surface boundary condition (33) becomes the following fifth order approximate wave equation:

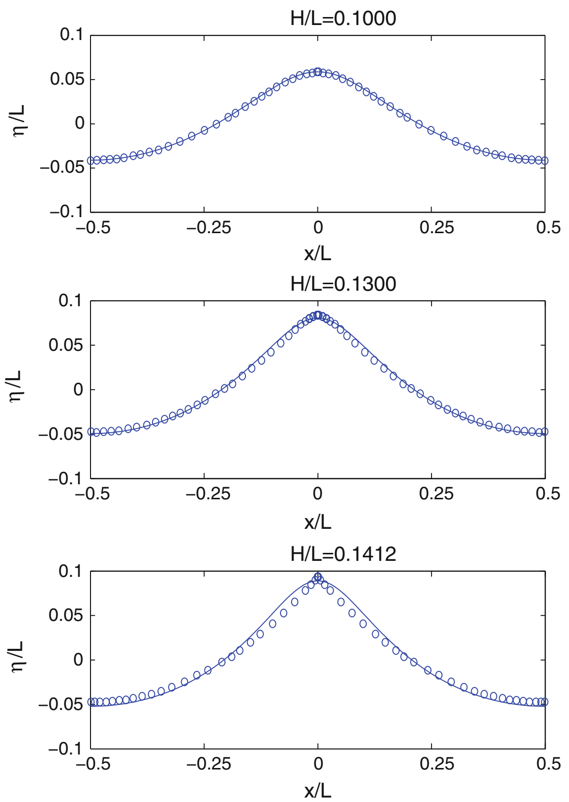

Fig. 5 Comparison of the fourth order approximate waves (37) (*solid lines*) and the nonlinear exact waves (30) (*circles lines*) from Schwartz [6, Fig. 9]

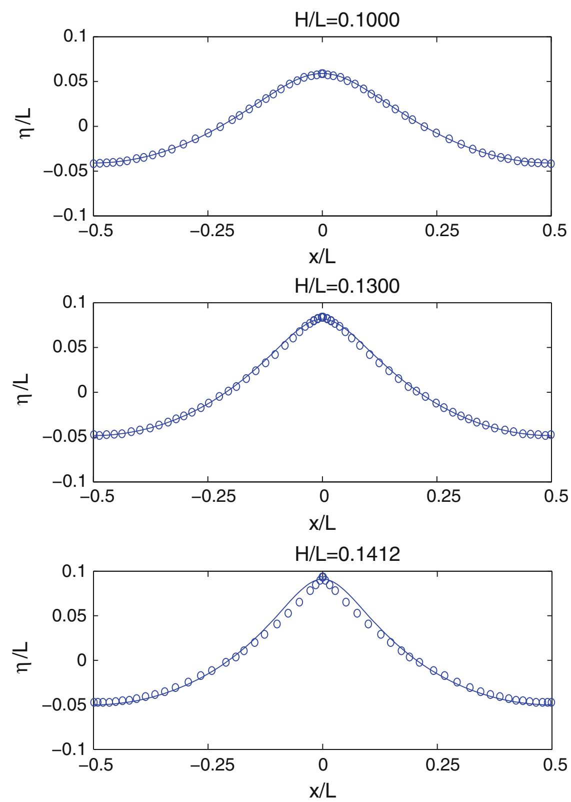

Fig. 6 Comparison of the fifth order approximate waves (38) (*solid lines*) and the nonlinear exact waves (30) (*circles lines*) from Schwartz [6, Fig. 9]

$$
\begin{aligned}
k\eta &= -\frac{k}{g}\partial_t\phi - \frac{k}{2g}|\nabla\phi|^2 \\
&= \left(1+\frac{1}{2}\epsilon^2+\frac{1}{8}\epsilon^4\right)\left(\left(\epsilon-\frac{1}{2}\epsilon^3-\frac{37}{24}\epsilon^5\right)\mathrm{e}^{k\eta}\cos\theta+\epsilon^4\mathrm{e}^{2k\eta}\cos 2\theta\right) \\
&\quad+\left(1+\frac{1}{2}\epsilon^2+\frac{1}{8}\epsilon^4\right)\frac{\epsilon^5}{4}\mathrm{e}^{3k\eta}\cos 3\theta \\
&\quad-\frac{1}{2}\left(\left(\epsilon-\frac{1}{2}\epsilon^3-\frac{37}{24}\epsilon^5\right)^2\mathrm{e}^{2k\eta}+2\left(\epsilon-\frac{1}{2}\epsilon^3-\frac{37}{24}\epsilon^5\right)\epsilon^4\mathrm{e}^{3k\eta}\cos\theta\right. \\
&\quad\left.+\frac{\epsilon^9}{2}\mathrm{e}^{5k\eta}\cos\theta+\left(\epsilon-\frac{1}{2}\epsilon^3-\frac{37}{24}\epsilon^5\right)\frac{\epsilon^5}{2}\mathrm{e}^{4k\eta}\cos 2\theta+\epsilon^8\mathrm{e}^{4k\eta}+\frac{\epsilon^{10}}{16}\mathrm{e}^{6k\eta}\right).
\end{aligned}
\tag{38}
$$

Employing the same iteration scheme as in the fourth order wave by setting $\eta = \eta_n$ on the right-hand side of Eq. (38) and $\eta = \eta_{n+1}$ on the left-hand side of Eq. (38) and choosing the same computation parameters as in the fourth order problem, we obtain numerical results for the values $H/L = 0.1, 0.13$ and 0.1412 displayed in Fig. 6, which shows that very little difference can be observed even for the limit case $H/L = 0.1412$. The CPU time to achieve the result in the FORTRAN 90 code is still less than 0.1 s.

The comparison of the approximate waves at $H/L = 0.1412$ and the Stokes highest wave in Fig. 7 shows that the wave crests are sharpened as the approximation order increases. This gives the convergence tendency of the

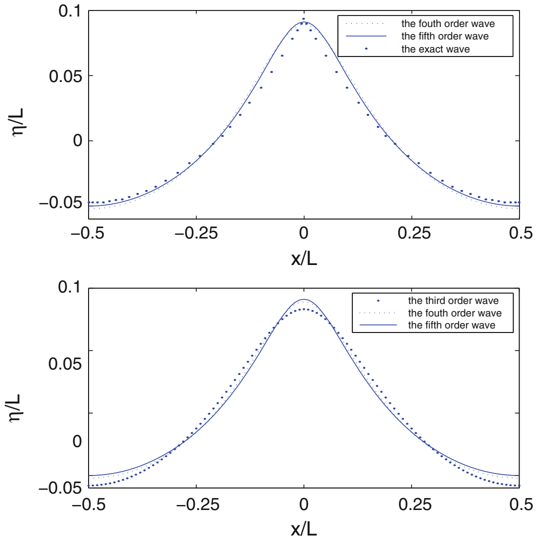

Fig. 7 Comparison of the approximate waves and the nonlinear exact wave from Schwartz [6, Fig. 9] for $H/L = 0.1412$

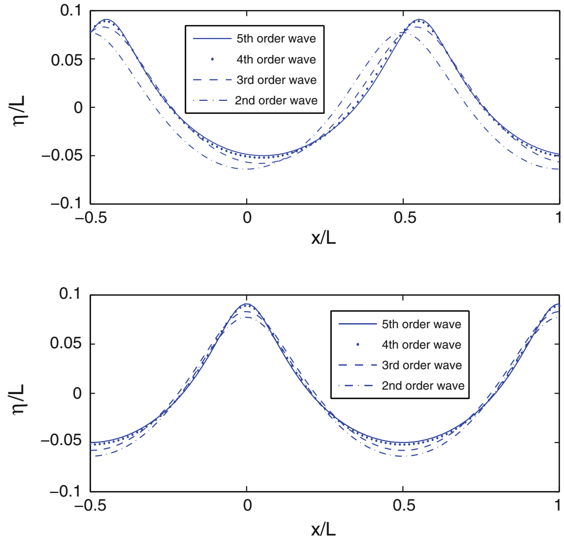

Fig. 8 Approximate waves for $H/L = 0.1412$ at the initial time $t = 0$ (*bottom*) and at time $t = 1$ (*top*)

approximate waves to the Stokes highest wave, although higher order approximate waves are necessary to further approximate the non-smooth crest of the Stokes highest wave.

Figure 8 shows that lower order approximate waves travel slowly and implies the tendency that the exact wave defined by Eq. (30) travels at the highest speed of the approximate waves.

By following the expansion analysis in Subsect. 3.3 to the third order wave, it is not difficult to prove that the Taylor expansion of the fifth order wave η in Eq. (38) with respect to ϵ is identical to the fifth order Stokes wave [5] in the following sense:

$$\eta = \text{ the fifth order Stokes wave } + O(\epsilon^6).$$

This also implies the relationship between the fourth order wave η in Eq. (37) and the fourth order Stokes wave [5] in the following form:

$$\eta = \text{ the fourth order Stokes wave } + O(\epsilon^5).$$

However, the existence of the analytic solutions to Eqs. (37) and (38) are much more complicated and is not discussed herein.

Fortunately, the numerical iteration scheme for the fourth and the fifth orders is always convergent for all the periodic gravity waves including the highest one with steepness $H/L = 0.1412$. For the understanding of the iteration convergence, numerical simulation of the fifth order wave is applied on the half non-dimensional wave length interval $[-0.5, 0]$ with 31 grid points. The numerical error

$$\frac{1}{L}|\eta_n - \eta_{n-1}| = \frac{1}{L}\left(\sum_{i=1}^{31}|\eta_n(i) - \eta_{n-1}(i)|^2\right)^{1/2}$$

with regard to the iteration number $n \le 50$ and the wave difference $\eta_n - \eta_{50}$ are displayed in Fig. 9, which shows $\eta_n = \eta_{n+1}$ for $n > 30$ since the error $\frac{1}{L}|\eta_n - \eta_{n-1}|$ with $n > 30$ almost equals the round-off error of the double-precision FORTRAN programming.

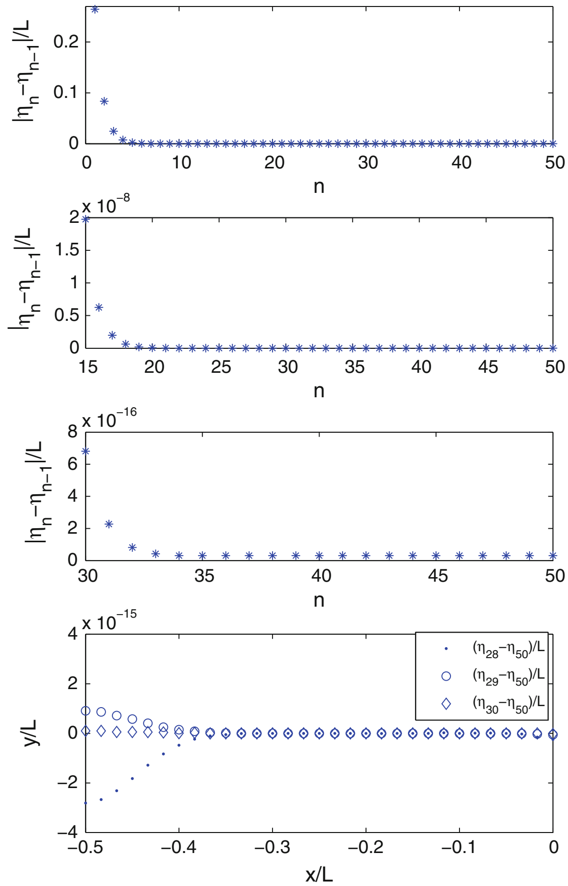

Fig. 9 Errors of the fifth order approximate wave with $H/L = 0.1412$ and $t = 0$

7 Approximation formulation in higher orders

Following the approximation scheme in the previous section, we can produce a general approximation formulation for arbitrary orders. For $n \geq 6$, let the velocity potential

$$\phi = \sqrt{\frac{g}{k^3}} \sum_{m=1}^{n} \epsilon^m a_m \mathrm{e}^{mky} \sin(mkx - m\omega t + m\varphi)$$

and the dispersion relation

$$\frac{\omega}{\sqrt{kg}} = 1 + \sum_{m=1}^{n} \epsilon^m b_m$$

be defined by the nth order Stokes wave [20], where the coefficients a_m are polynomials of the steepness $\epsilon = ak$, the coefficients b_m are independent of ϵ and $b_m = 0$ for odd number m. Therefore, the nth order free surface boundary condition (33) holds true.

Upon the observation

$$-\frac{k}{g}\partial_t \phi = \frac{\omega}{\sqrt{kg}} \sum_{m=1}^{n} m\epsilon^m a_m \mathrm{e}^{mky} \cos m\theta$$

and

$$\frac{k}{g}|\nabla\phi|^2 = \left(\sum_{m=1}^{n} m\epsilon^m a_m e^{mky}\cos m\theta\right)^2 + \left(\sum_{m=1}^{n} m\epsilon^m a_m e^{mky}\sin m\theta\right)^2$$
$$= 2\sum_{m=1}^{n-1}\sum_{j=1}^{n-m} m(j+m)\epsilon^{j+2m} e^{(j+2m)ky} a_m a_{j+m}\cos j\theta$$
$$+ \sum_{m=1}^{n} m^2\epsilon^{2m} a_m^2 e^{2mky},$$

we see that the nth order approximate wave is defined by the dynamic free surface boundary condition (4) as

$$k\eta = -\frac{k}{g}\partial_t\phi - \frac{k}{2g}|\nabla\phi|^2\Big|_{y=\eta}$$
$$= \left(1 + \sum_{m=1}^{n}\epsilon^m b_m\right)\sum_{m=1}^{n} m\epsilon^m a_m e^{mk\eta}\cos m\theta - \frac{1}{2}\sum_{m=1}^{n} m^2\epsilon^{2m} a_m^2 e^{2mk\eta}$$
$$- \sum_{m=1}^{n-1}\sum_{j=1}^{n-m} m(j+m)\epsilon^{j+2m} e^{(j+2m)k\eta} a_m a_{j+m}\cos j\theta,$$

after the dispersion relation is taken into account.

8 Discussion

A new approximation scheme for travelling gravity waves based on an implicit formulation along the dynamic free surface boundary Eq. (4) is obtained. The approximation wave satisfies exactly Eq. (4), while the nth order Stokes wave satisfies Eq. (4) up to nth order. In particular, it can be obtained from the Taylor expansion of the dynamic free surface boundary equation along the average water surface in terms of the steepness $\epsilon = ak$ that the nth order approximate wave becomes the nth order Stokes waves if a higher order term $O(\epsilon^{n+1})$ is ignored. The approximate wave is not in a Fourier expansion form and thus the smallness requirement on the steepness ϵ as in Stokes wave expansion can be reduced.

The proposed scheme up to the fifth order approximation provides a simple numerical computation method for the periodic wave problem. Although Fig. 7 shows the convergence tendency with respect to the Stokes highest wave with $H/L = 0.1412$ produced by the approximation method of Schwartz [6], the numerical simulations restricted to the fifth order are not enough to present the Stokes highest wave by the present scheme. For the convergence of the present approximation scheme to the Stokes highest wave, numerical simulations of higher order waves are necessary, and then the approximation scheme becomes complicated as the order number increases. This problem will be discussed elsewhere. The present approximation scheme is comparable with the Stokes expansion scheme, but is quite different to the traditional approximation method of Schwartz [6] based on conformal mappings and the Laurent expansion.

The present scheme relies on the assumption that there is an approximation velocity potential ϕ satisfying the nth order free surface boundary condition

$$\partial_t^2\phi + g\partial_y\phi + \partial_t|\nabla\phi|^2 + \frac{1}{2}\nabla\phi\cdot\nabla|\nabla\phi|^2 = O(\epsilon^{n+1})$$

on the calm water surface $y = 0$. Fortunately, this velocity potential ϕ can be derived from the Stokes wave expansion scheme.

It should be mentioned that nonlinear term $-\frac{1}{2}|\nabla\phi|^2$ in the dynamic free surface boundary Eq. (4) is not only for the physics of the problem, but also for the convergence of the numerical iteration in the derivation of approximate waves.

Appendix A: Analytic solutions of the deep water wave equations (19–21)

The velocity potentials for the first three order waves are in a simple form. The unique existence for the first three order waves can be now proved.

The derivation of the analytic solution to the first order problem (19) is quite straightforward, and the solution can be determined by the iteration scheme

$$\eta_{n+1} = \frac{\epsilon}{k} e^{k\eta_n} \cos\theta \quad \text{for} \quad n \geq 0$$

initially from the calm water line $\eta_0 = 0$. Therefore, detailed derivations are only provided for the more interesting wave solutions (20) and (21).

Now we use the iteration scheme

$$\eta_{n+1} = \frac{\epsilon\omega}{k\sqrt{kg}} \delta e^{k\eta_n} \cos\theta - \frac{\epsilon^2}{2k} \delta^2 e^{2k\eta_n} \quad \text{for} \quad n \geq 0 \tag{39}$$

to show the unique existence of the analytic solutions (20), (21) under the steepness bound assumption

$$\epsilon \leq 0.75. \tag{40}$$

To do so, we let X_+ be the space of continuous functions $k\eta \leq \omega^2/(2kg)$ defined over the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ where $\cos\theta \geq 0$, and let X_- be the space of continuous functions $\eta \leq 0$ defined over the interval $[\frac{\pi}{2}, \frac{3\pi}{2}]$ where $\cos\theta \leq 0$.

Therefore, the definition of the spaces $X_\pm$ implies that $\eta_{n+1} \in X_\pm$ whenever $\eta_n \in X_\pm$ in terms of the bound (9).

In order to show the convergence of the iteration scheme in terms of the Banach contraction principle, it remains to prove the contraction property

$$|\eta_{n+1}(\theta) - \eta_{m+1}(\theta)| \leq \tau |\eta_n(\theta) - \eta_m(\theta)| \quad \text{with} \;\; \eta_n,\ \eta_m \in X_\pm, \;\; n,\ m \geq 0 \tag{41}$$

for a contraction constant $\tau < 1$ independent of θ, n, m, η_n and η_m. Indeed, it follows from Eq. (39) that

$$|\eta_{n+1}(\theta) - \eta_{m+1}(\theta)| \leq \int_0^1 |\Phi| \mathrm{d}s \, |\eta_n(\theta) - \eta_m(\theta)|, \tag{42}$$

where

$$\Phi = \frac{\omega}{\sqrt{kg}} \epsilon\delta \cos\theta e^{k[\eta_m(\theta)+s(\eta_n(\theta)-\eta_m(\theta))]} - \epsilon^2\delta^2 e^{2k[\eta_m(\theta)+s(\eta_n(\theta)-\eta_m(\theta))]}. \tag{43}$$

For $\theta \in [\frac{\pi}{2}, \frac{3\pi}{2}]$, we see that $\cos\theta \leq 0$ and hence

$$\int_0^1 |\Phi| \mathrm{d}s \leq \frac{\omega}{\sqrt{kg}} \epsilon\delta + \epsilon^2\delta^2 \leq \epsilon\delta\sqrt{1+\epsilon^2\delta^2} + \epsilon^2\delta^2, \tag{44}$$

which is a straightforward consequence of Eq. (42) and the dispersion relations Eqs. (17) and (18), observing that the inequality

$$\epsilon\delta\sqrt{1+\epsilon^2\delta^2} + \epsilon^2\delta^2 < 1$$

or

$$\sqrt{\frac{1}{\epsilon^2\delta^2} + 1} < \frac{1}{\epsilon^2\delta^2} - 1$$

is equivalent to the bound

$$\epsilon\delta = \epsilon\left(1 - \frac{1}{2}\epsilon^2\right) < \frac{1}{\sqrt{3}},$$

which is true for $\epsilon \leq 0.75$.

Next, we show the contraction property in the space X_+ when $\epsilon\delta \le 1/\sqrt{3}$.

Since the functions in the space X_+ are nonnegative, it suffices to show the contraction property of the sequence $\hat{\eta}_n$ instead of that of the sequence η_n. Here $\hat{\eta}$ denotes the cut-off function

$$\hat{\eta} = \begin{cases} \eta \text{ for } & \eta \ge 0, \\ 0 \text{ for } & \eta < 0. \end{cases}$$

Therefore for $\eta_n, \eta_m \in X_+$, we may suppose $\eta_{n+1}(\theta) \ge 0$ and $\eta_{m+1}(\theta) \ge 0$. Eq. (39) with η_n replaced by $\hat{\eta}_n$ produces

$$|\hat{\eta}_{n+1}(\theta) - \hat{\eta}_{m+1}(\theta)| = |\eta_{n+1}(\theta) - \eta_{m+1}(\theta)| \le \int_0^1 |\Phi| \mathrm{d}s\, |\hat{\eta}_n(\theta) - \hat{\eta}_m(\theta)|.$$

To estimate the function Φ, we begin with the assumption $\Phi \le 0$. Without loss of generality, we may suppose that $\eta_n(\theta) - \eta_m(\theta) \ge 0$. It follows from Eqs. (39), (43) that

$$\begin{aligned} |\Phi| &\le \frac{1}{2}\epsilon^2\delta^2 \mathrm{e}^{2k[\eta_m(\theta)+s(\eta_n(\theta)-\eta_m(\theta))]} - \epsilon\delta\frac{\omega}{\sqrt{kg}}\mathrm{e}^{k[\eta_m(\theta)+s(\eta_n(\theta)-\eta_m(\theta))]}\cos\theta \\ &\le \frac{1}{2}\epsilon^2\delta^2 \mathrm{e}^{\frac{\omega^2}{kg}}. \end{aligned}$$

The contraction constant is now derived as

$$\int_0^1 |\Phi| \mathrm{d}s \le \frac{1}{2}\epsilon^2\delta^2 \mathrm{e}^{1+\epsilon^2\delta^2} \le 0.633 \tag{45}$$

in terms of the dispersion relations (17), (18) and the bound $\epsilon\delta \le 1/\sqrt{3}$.

For the remaining case $\Phi \ge 0$, we find that

$$\begin{aligned} |\Phi| &\le \max_{\sigma>0}\left(\sigma\epsilon\delta\frac{\omega}{\sqrt{kg}}\cos\theta\,\mathrm{e}^{k[\eta_m(\theta)+s(\eta_n(\theta)-\eta_m(\theta))]} - \sigma^2\epsilon^2\delta^2\mathrm{e}^{2k[\eta_m(\theta)+s[\eta_n(\theta)-\eta_m(\theta))]}\right) \\ &\le \frac{\omega^2}{4kg}\cos^2\theta. \end{aligned}$$

This gives the contraction constant

$$\int_0^1 |\Phi| \mathrm{d}s \le \frac{\omega^2}{4kg} \le \frac{1+\epsilon^2\delta^2}{4} \le \frac{1}{3} \tag{46}$$

due to the dispersion relations (17), (18) and bound $\epsilon\delta \le 1/\sqrt{3}$.

Consequently, the iteration sequences η_n in the space X_- and $\hat{\eta}_n$ in the space X_+ are convergent uniquely for any initial functions $\eta_0 \in X_-$ and $\hat{\eta}_0 \in X_+$. The convergence of the sequences is also confirmed rigorously by the Banach contraction principle. Thus, the unique existence of the desired analytic solution η is defined to be $\eta(\theta) = \lim_{n\to\infty} \eta_n(\theta)$ for $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and $\eta(\theta) = \lim_{n\to\infty} \hat{\eta}_n(\theta)$ for $\theta \in [\frac{\pi}{2}, \frac{3\pi}{2}]$. The analytic solution is continuously defined in the whole interval $[-\frac{\pi}{2}, \frac{3\pi}{2}]$ due to the uniqueness.

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