

A FORMULA FOR THE DISCRIMINANT OF NUMBER FIELDS

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ABSTRACT. We obtain a formula for the discriminant $D_{\kappa/\mathbb{Q}}$ of an algebraic number field κ in terms of a ratio of the first two coefficients of the Taylor series of ζ_{κ} at $1/2$.

Let κ be a number field of degree $n = r_1 + 2r_2$, where r_1, r_2 are the number of real, complex places respectively. The Dedekind ζ -function of the number field κ is defined by the series

$$\zeta_{\kappa}(s) = \sum_{\mathfrak{a}} \frac{1}{\mathcal{N}(\mathfrak{a})^s},$$

where $\mathfrak{a}$ varies over the non-zero integral ideals of κ and $\mathcal{N}(\mathfrak{a})$ denotes the absolute norm of $\mathfrak{a}$. Denote by $D_{\kappa/\mathbb{Q}}$ the discriminant of κ .

The Dedekind function $\zeta_{\kappa}(s)$ admits a holomorphic continuation with the exclusion of a simple pole at $s = 1$ and satisfies the following functional equation

$$(1) \quad \zeta_{\kappa}(1-s) = A(s)\zeta_{\kappa}(s),$$

where

$$A(s) = |D_{\kappa/\mathbb{Q}}|^{s-\frac{1}{2}} \left(\cos \frac{\pi s}{2}\right)^{r_1+r_2} \left(\sin \frac{\pi s}{2}\right)^{r_2} 2^{(1-s)n} \pi^{-sn} \Gamma^n(s).$$

A straightforward computation gives $A(1/2) = 1$. Furthermore, let $\beta_{1,\kappa}$ ($\neq 0$), $\gamma_{1,\kappa}$ be defined by the Taylor expansion of $\zeta_{\kappa}(s)$ at $s = 1/2$, i.e.,

$$(2) \quad \zeta_{\kappa}(s) = \beta_{1,\kappa} \left(s - \frac{1}{2}\right)^{\mu} + \gamma_{1,\kappa} \left(s - \frac{1}{2}\right)^{\mu+1} + \cdots.$$

It turns out from (1) that μ is a non-negative even integer. It is well-known ([2]) that

$$|D_{\kappa/\mathbb{Q}}|^{1/n} \geq (4\pi e^{\gamma+1})^{r_1/n} (4\pi e^{\gamma})^{2r_2/n} - O(n^{-2/3}),$$

and further, if the Generalized Riemann Hypothesis is assumed, a much stronger inequality states that

$$(3) \quad |D_{\kappa/\mathbb{Q}}|^{1/n} \geq (8\pi e^{\gamma+\pi/2})^{r_1/n} (8\pi e^{\gamma})^{2r_2/n} - O(\log^{-2} n),$$

where γ is the Euler constant. We refer the reader to the survey paper [2] for the history of these bounds. Many mathematicians, such as A. Odlyzko, G. Poitou,

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J.-P. Serre and H. Stark, have contributed to this theory. Moreover, explicit forms of these inequalities have proven quite useful in several types of applications in algebraic number theory, as described in [2] for example.

In this short paper, we prove the following theorem.

Theorem. *Let κ be a number field of degree n and let $D_{\kappa/\mathbb{Q}}$, r_1 , r_2 , $\beta_{1,\kappa}$ and $\gamma_{1,\kappa}$ be defined as above. Then we have*

$$|D_{\kappa/\mathbb{Q}}|^{1/n} = (8\pi e^{\gamma+\pi/2})^{r_1/n} (8\pi e^{\gamma})^{2r_2/n} e^{-2\gamma_{1,\kappa}/(n\beta_{1,\kappa})}.$$

Proof. It is clear from (2) that

$$\frac{\zeta'_\kappa(s)}{\zeta_\kappa(s)} = \frac{\mu}{s - \frac{1}{2}} + \frac{\gamma_{1,\kappa}}{\beta_{1,\kappa}} + O((s - \frac{1}{2})) \quad \text{and} \quad -\frac{\zeta'_\kappa(1-s)}{\zeta_\kappa(1-s)} = \frac{\mu}{s - \frac{1}{2}} - \frac{\gamma_{1,\kappa}}{\beta_{1,\kappa}} + O((s - \frac{1}{2})).$$

Logarithmically differentiating (1) gives

$$-\frac{\gamma_{1,\kappa}}{\beta_{1,\kappa}} = \log |D_{\kappa/\mathbb{Q}}| - \frac{r_1\pi}{2} - n \log(2\pi) + n \frac{\Gamma'(\frac{1}{2})}{\Gamma(\frac{1}{2})} + \frac{\gamma_{1,\kappa}}{\beta_{1,\kappa}}.$$

By using the fact (e.g. [1], page 482) that $\Gamma'(\frac{1}{2})/\Gamma(\frac{1}{2}) = -\gamma - \log 4$, we obtain

$$\frac{\gamma_{1,\kappa}}{\beta_{1,\kappa}} = \frac{n}{2} \{\gamma + \log(8\pi)\} + \frac{r_1\pi}{4} - \frac{1}{2} \log |D_{\kappa/\mathbb{Q}}|,$$

which completes the proof. $\square$

Clearly, the theorem provides us with a formula to compute the discriminant of individual number fields. Further, it is worth pointing out that the main term in (3) appears exactly in the theorem. Thus, the Serre-Odlyzko inequality (3) can be achieved without assuming the Generalized Riemann Hypothesis if one can give an absolute upper bound for $\gamma_{1,\kappa}/\beta_{1,\kappa}$. Indeed, even a slightly weaker bound on the ratio $\gamma_{1,\kappa}/\beta_{1,\kappa}$ could still improve the unconditional bound.

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