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New families of travelling wave solutions for Boussinesq–Burgers equation and (3 + 1)-dimensional Kadomtsev–Petviashvili equation

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Abstract

In this Letter, Boussinesq–Burgers equation and (3 + 1)-dimensional Kadomtsev–Petviashvili equation are studied by using a generalized algebraic method. A variety of explicit exact travelling wave solutions, including solitary wave, Jacobi and Weierstrass elliptic function periodic, triangular periodic and rational solutions, are obtained. Many new results are presented. The proposed method can be also extended to construct more new exact travelling wave solutions of some nonlinear evolution equations with variable coefficients in mathematical physics.

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1. Introduction

In the past several decades, a great number of efforts have been made to study various nonlinear wave equations. The traditional methods of solving nonlinear wave equations include inverse scattering theory [1,2], Bäcklund transformation [3–5], Darboux transformation [6], Hirota bilinear method [7] and Painlevé expansion method [8], etc.

With the rapid development of nonlinear science, some new powerful solving methods have been developed, such as homogeneous balance method [9], tanh method [10], Jacobi elliptic function method [11], method of bifurcation [12], F-expansion method [13,14], ADM method [15] and method of auxiliary equation [16], etc. Fan [17] found a series of travelling wave solutions for nonlinear evolution equations by applying a direct approach with computerized symbolic computations. The Fan's method, in comparison with most existing symbolic computation methods, not only gives new and more general exact travelling wave solutions, but also provides a guideline to classify various types of the solution according to some parameters. Yan et al. [18–21] improved this algebraic method such that it can be used to seek more types of solutions.

In this Letter, we attempt to develop the Fan's method on the basis of [17–21] and to study further Boussinesq–Burgers equation [22]

$$u_t = -2uu_x + \frac{1}{2}v_x, \qquad v_t = \frac{1}{2}u_{xxx} - 2(uv)_x, \tag{1}$$

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and (3 + 1)-dimensional Kadomtsev–Petviashvili equation [23–25]

$$u_{xt} - 6u_x^2 + 6uu_{xx} - u_{xxxx} - u_{yy} - u_{zz} = 0. \tag{2}$$

Li et al. [22] gave some soliton solutions of Eq. (1) by means of a new Darboux transformation with multi-parameters based on the resulting lax pairs. Eq. (2) is known as (3 + 1)-dimensional KP equation. Chen et al. [23] obtained more new exact solutions for Eq. (2) by using a new generalized transformation in homogeneous balance method. El-Sayed et al. [24] studied Eq. (2) by considering the decomposition scheme. Hu [25] obtained some travelling wave solutions by applying Fan's algebraic method.

2. The generalized algebraic method based on the symbolic computation

For a given nonlinear PDEs with some physical fields $u_i(x_j, t)$ $(i = 1, 2, \ldots; j = 1, 2, \ldots)$ in independent variables x_j and t,

$$N_i(u_i, u_{i,t}, u_{i,x_1}, u_{i,x_2}, u_{i,x_3}, u_{i,tt}, u_{i,x_1t}, u_{i,x_2t}, u_{i,x_3t}, u_{i,x_1x_1}, u_{i,x_2x_2}, u_{i,x_3x_3}, \ldots) = 0, \tag{3}$$

(3) can be turned to an ODE by the travelling wave transformation $u_i(x_j, t) = u_i(\zeta)$, $\zeta = k(x_1 + l_1x_2 + \cdots + \lambda t)$,

$$M_i(u_i, u_i', u_i'', \ldots) = 0, \tag{4}$$

where k is the wave number and λ is the wave speed.

We assume that Eq. (3) has solutions in the forms

$$u_i(\zeta) = \sum_{j=0}^{r_i} a_{ij}\phi^j + \sum_{j=1}^{r_i} \phi^{-j}\bigl[b_{ij} + |\phi'|\bigl(c_{ij}\phi^{2j-1} + d_{ij}\bigr)\bigr], \tag{5}$$

where r_i is a positive integer, a_{ij}, b_{ij}, c_{ij} and d_{ij} $(i = 1, 2, \ldots; j = 0, 1, 2, \ldots, r_i)$ are constants to be determined later, and $\phi = \phi(\zeta)$ satisfies the following elliptic equation

$$\phi' = \frac{d\phi}{d\zeta} = \varepsilon\sqrt{\sum_{i=0}^{s}\lambda_i\phi^i}, \tag{6}$$

where $\varepsilon = \pm 1$, s is a positive integer and λ_i $(i = 0, 1, 2, \ldots, s)$ are constants to be determined later. r and s can be determined by balancing the nonlinear term and the highest order derivative term in Eq. (4) (if r is not a positive integer, we firstly make the transformation $u = v^r$). Substitute (5) into (4) along with (6), and then set all coefficients of

$$\phi^{w_1}\left(\sqrt{\sum_{i=0}^{s}\lambda_i\phi^i}\right)^{w_2} \quad (w_1 = 0, \pm 1, \pm 2, \ldots; w_2 = 0, 1)$$

to be zero to get an over-determined system of nonlinear algebraic equations with respect to k, λ, a_{ij}, b_{ij}, c_{ij}, d_{ij} and λ_i $(i = 0, 1, 2, \ldots; j = 0, 1, 2, \ldots, r_i)$. Solving the system of nonlinear equations by using the Wu elimination method with the aid of symbolic computation packages like *Maple* or *Mathematica*, we can derive a series of fundamental solutions. Eq. (6) has many different solutions which are listed in Table 1 (here we take $s = 4$).

Remark 1. e is a modulus of the Jacobi elliptic functions, $f_2 = -4\lambda_1/\lambda_3$ and $f_3 = -4\lambda_0/\lambda_3$ are called invariants of Weierstrass elliptic function. The more detailed notations for the Jacobi and Weierstrass elliptic functions can be found in [26–28].

In Sections 3 and 4, we will discuss Eqs. (1) and (2) using the proposed method, respectively.

3. The exact travelling wave solutions of Eq. (1)

We firstly make the following travelling wave transformation

$$u(x, t) = u(\zeta), \qquad v(x, t) = v(\zeta), \qquad \zeta = x + \lambda t, \tag{7}$$

where λ is a constant to be determined. Substituting (7) into (1) and integrating it once read

$$v = 2(\lambda u + u^2 - g_1), \tag{8a}$$

$$\lambda v = \frac{1}{2}u'' - 2uv + g_2, \tag{8b}$$

Table 1
Different solutions of Eq. (6)

λ_0	λ_1	λ_2	λ_3	λ_4	ϕ
0	0	>0	0	<0	$\varepsilon\sqrt{-\lambda_2/\lambda_4}\,\mathrm{sech}\big(\sqrt{\lambda_2}\zeta\big)$
0	0	>0	$\neq 0$	0	$(-\lambda_2/\lambda_3)\,\mathrm{sech}^2\big(\sqrt{\lambda_2}\zeta/2\big)$
$\lambda_2^2/(4\lambda_4)$	0	<0	0	>0	$\varepsilon\sqrt{-\lambda_2/(2\lambda_4)}\,\tanh\big(\sqrt{-\lambda_2/2}\zeta\big)$
0	0	>0			$\dfrac{\lambda_2\,\mathrm{sech}^2(0.5\sqrt{\lambda_2}\zeta)}{2\varepsilon\sqrt{\lambda_2\lambda_4}\tanh(0.5\sqrt{\lambda_2}\zeta)-\lambda_3}$
$\lambda_2^2e^2/\big[\lambda_4(e^2+1)^2\big]$	0	<0	0	>0	$\varepsilon\sqrt{-\lambda_2e^2/[\lambda_4(e^2+1)]}\,\mathrm{sn}\big[\sqrt{-\lambda_2/(e^2+1)}\zeta\big]$
$\lambda_2^2e^2(e^2-1)/\big[\lambda_4(2e^2-1)^2\big]$	0	>0	0	<0	$\varepsilon\sqrt{\lambda_2e^2/[\lambda_4(1-2e^2)]}\,\mathrm{cn}\big[\sqrt{\lambda_2/(2e^2-1)}\zeta\big]$
$\lambda_2^2(1-e^2)/\big[\lambda_4(2-e^2)^2\big]$	0	>0	0	<0	$\varepsilon\sqrt{\lambda_2/[\lambda_4(e^2-2)]}\,\mathrm{dn}\big[\sqrt{\lambda_2/(2-e^2)}\zeta\big]$
		0	>0	0	$\wp\big(\sqrt{\lambda_3}\zeta/2,\,f_2,\,f_3\big)$
0	0	<0	0	>0	$\varepsilon\sqrt{-\lambda_2/\lambda_4}\,\sec\big(\sqrt{-\lambda_2}\zeta\big)$
$\lambda_2^2/(4\lambda_4)$	0	>0	0	>0	$\varepsilon\sqrt{\lambda_2/(2\lambda_4)}\,\tan\big(\sqrt{\lambda_2/2}\zeta\big)$
0	0	<0	$\neq 0$	0	$(-\lambda_2/\lambda_3)\,\sec^2\big(\sqrt{-\lambda_2}\zeta/2\big)$
0	0	<0			$\dfrac{-\lambda_2\sec^2(0.5\sqrt{-\lambda_2}\zeta)}{2\varepsilon\sqrt{-\lambda_2\lambda_4}\tan(0.5\sqrt{-\lambda_2}\zeta)+\lambda_3}$
$\lambda_1^2/(4\lambda_2)$		>0	0	0	$e^{\varepsilon\sqrt{\lambda_2}\zeta}-\lambda_1/(2\lambda_2)$
0	$\neq 0$	<0	0	0	$\lambda_1\big[\varepsilon\sin(\sqrt{-\lambda_2}\zeta)-1\big]/(2\lambda_2)$
0	$\neq 0$	>0	0	0	$\lambda_1\big[\varepsilon\sinh(2\sqrt{\lambda_2}\zeta)-1\big]/(2\lambda_2)$
0	0	0	0	>0	$-\varepsilon/\big(\sqrt{\lambda_4}\zeta\big)$
0	0	0	$\neq 0$	0	$1/(\lambda_3\zeta^2)$
>0	0	0	0	0	$\varepsilon\sqrt{\lambda_0}\zeta$
	$\neq 0$	0	0	0	$\lambda_1\zeta^2/4-\lambda_0/\lambda_1$

where g_1 and g_2 are integration constants. Substituting (8a) into (8b) yields

$$\frac{1}{2}u''-2(u^2+\lambda u-g_1)(\lambda+2u)+g_2=0. \tag{9}$$

We suppose that (9) has the following formal solution:

$$u(\zeta)=a_{00}+a_{01}\phi+b_{01}\phi^{-1}+c_{01}|\phi'|+d_{01}\phi^{-1}|\phi'|, \tag{10}$$

and $\phi=\phi(\zeta)$ satisfies Eq. (6), where a_{00}, a_{01}, b_{01}, c_{01} and d_{01} are constants to be determined later.

With the aid of *Maple*, substituting (10) into (9) along with (6) and setting the coefficients of

$$\phi^{w_1}\left(\sqrt{\sum_{i=0}^{s}\lambda_i\phi^i}\right)^{w_2}\quad (w_1=0,\pm1,\pm2,\ldots;\ w_2=0,1)$$

to zero yield a system of nonlinear algebraic equations with respect to a_{00}, a_{01}, b_{01}, c_{01}, d_{01}, λ, λ_0, λ_1, λ_2, λ_3, λ_4, g_1 and g_2.

By use of *Maple*, we obtain five classes of solutions of the nonlinear algebraic system, namely,

Case 1.

$$\lambda_2=24a_{00}(\lambda+a_{00})-8g_1+4\lambda^2,\quad \lambda_3=8a_{01}(2a_{00}+\lambda),\quad \lambda_4=4a_{01}^2,$$
$$g_2=-\frac{1}{4}a_{01}\lambda_1+2a_{00}(\lambda^2+2a_{00}^2-2g_1+3\lambda a_{00})-2\lambda g_1,\quad b_{01}=c_{01}=d_{01}=0.$$

Case 2.

$$\lambda_0=4b_{01}^2,\quad \lambda_1=8b_{01}(2a_{00}+\lambda),\quad \lambda_2=24a_{00}(\lambda+a_{00})-8g_1+4\lambda^2,$$
$$g_2=-\frac{1}{4}b_{01}\lambda_3+2a_{00}(\lambda^2+2a_{00}^2-2g_1+3\lambda a_{00})-2\lambda g_1,\quad a_{01}=c_{01}=d_{01}=0.$$

Case 3.

$$\lambda_0 = 4b_{01}^2, \quad \lambda_1 = 8b_{01}(2a_{00}+\lambda), \quad \lambda_2 = 24(\lambda a_{00} + a_{01}b_{01} + a_{00}^2) - 8g_1 + 4\lambda^2, \quad \lambda_3 = 8a_{01}(2a_{00}+\lambda),$$
$$\lambda_4 = 4a_{01}^2, \quad g_2 = 8a_{01}b_{01}(2a_{00}+\lambda) + 2a_{00}(\lambda^2 + 2a_{00}^2 - 2g_1 + 3\lambda a_{00}) - 2\lambda g_1, \quad c_{01} = d_{01} = 0.$$

Case 4.

$$a_{00} = -\frac{1}{2}\lambda, \quad d_{01} = \pm\frac{1}{2}, \quad g_1 = \frac{1}{4}(\lambda_2 - \lambda^2), \quad \lambda_1 = \lambda_3 = a_{01} = b_{01} = c_{01} = g_2 = 0.$$

Case 5.

$$a_{00} = -\frac{1}{2}\lambda, \quad d_{01} = \pm\frac{1}{4}, \quad \lambda_0 = 16b_{01}^2, \quad \lambda_4 = 16a_{01}^2, \quad g_1 = -\frac{1}{4}\lambda^2 + \frac{1}{16}\lambda_2 + 6a_{01}b_{01},$$
$$g_2 = \frac{1}{2}(a_{01}\lambda_1 + b_{01}\lambda_3), \quad c_{01} = 0.$$

Using solutions in Table 1, a series of explicit exact solutions can be constructed, which are shown as follows. To avoid lengthiness, we only list the expressions of u and omit the ones of v in this Letter.

Family 1. According to Case 1, we get the following exact solutions of Eq. (1):

$$u_{11} = -\frac{\lambda}{2} + \frac{\varepsilon}{2}\sqrt{\lambda^2+4g_1}\tanh\left(\sqrt{\lambda^2+4g_1}\,\zeta\right), \quad \lambda_0 = \lambda_2^2/(4\lambda_4),\ \lambda_1 = \lambda_3 = 0,\ \lambda_2 < 0,\ \lambda_4 > 0,\ g_2 = 0.$$
$$u_{12} = -\frac{\lambda}{2} + \frac{\varepsilon}{2}\sqrt{-\lambda^2-4g_1}\tan\left(\sqrt{-\lambda^2-4g_1}\,\zeta\right), \quad \lambda_0 = \lambda_2^2/(4\lambda_4),\ \lambda_1 = \lambda_3 = 0,\ \lambda_2 > 0,\ \lambda_4 > 0,\ g_2 = 0.$$
$$u_{13} = a_{00} + \frac{a_{01}(6\lambda a_{00} - 2g_1 + \lambda^2 + 6a_{00}^2)\operatorname{sech}^2\left(\sqrt{6\lambda a_{00} - 2g_1 + \lambda^2 + 6a_{00}^2}\,\zeta\right)}{2\left[\varepsilon\sqrt{6\lambda a_{00} - 2g_1 + \lambda^2 + 6a_{00}^2}\tanh\left(\sqrt{6\lambda a_{00} - 2g_1 + \lambda^2 + 6a_{00}^2}\,\zeta\right) - a_{01}(2a_{00}+\lambda)\right]}, \quad \lambda_0 = \lambda_1 = 0,\ \lambda_2 > 0.$$
$$u_{14} = a_{00} + \frac{a_{01}(2g_1 - 6\lambda a_{00} - \lambda^2 - 6a_{00}^2)\sec^2\left(\sqrt{2g_1 - 6\lambda a_{00} - \lambda^2 - 6a_{00}^2}\,\zeta\right)}{2\left[\varepsilon\sqrt{2g_1 - 6\lambda a_{00} - \lambda^2 - 6a_{00}^2}\tan\left(\sqrt{2g_1 - 6\lambda a_{00} - \lambda^2 - 6a_{00}^2}\,\zeta\right) + a_{01}(2a_{00}+\lambda)\right]}, \quad \lambda_0 = \lambda_1 = 0,\ \lambda_2 < 0.$$
$$u_{15} = -\frac{\lambda}{2} + \varepsilon\sqrt{\frac{(\lambda^2+4g_1)e^2}{2(e^2+1)}}\operatorname{sn}\left[\sqrt{\frac{2(\lambda^2+4g_1)}{e^2+1}}\,\zeta\right], \quad \lambda_0 = \lambda_2^2 e^2/\left[\lambda_4(e^2+1)^2\right],\ \lambda_1 = \lambda_3 = 0,\ \lambda_2 < 0,\ \lambda_4 > 0,\ g_2 = 0.$$
$$u_{16} = -\frac{\lambda}{2} + \frac{\varepsilon}{2}\sqrt{2\lambda^2+8g_1}\sec\left(\sqrt{2\lambda^2+8g_1}\,\zeta\right), \quad \lambda_0 = \lambda_1 = \lambda_3 = 0,\ \lambda_2 < 0,\ \lambda_4 > 0,\ g_2 = 0.$$

Family 2. Case 2 leads to

$$u_{21} = -\frac{\lambda}{2} + \frac{1}{2}\varepsilon\sqrt{\lambda^2+4g_1}\tanh^{-1}\left(\sqrt{\lambda^2+4g_1}\,\zeta\right), \quad \lambda_0 = \lambda_2^2/(4\lambda_4), \quad \lambda_1 = \lambda_3 = 0,\ \lambda_2 < 0,\ \lambda_4 > 0,\ g_2 = 0.$$
$$u_{22} = -\frac{\lambda}{2} + \frac{1}{2}\varepsilon\sqrt{-\lambda^2-4g_1}\tan^{-1}\left(\sqrt{-\lambda^2-4g_1}\,\zeta\right), \quad \lambda_0 = \lambda_2^2/(4\lambda_4),\ \lambda_1 = \lambda_3 = 0,\ \lambda_2 > 0,\ \lambda_4 > 0,\ g_2 = 0.$$
$$u_{23} = -\frac{\lambda}{2} + \varepsilon\sqrt{\frac{\lambda^2+4g_1}{2(e^2+1)}}\operatorname{ns}\left[\sqrt{\frac{2(\lambda^2+4g_1)}{e^2+1}}\,\zeta\right], \quad \lambda_0 = \lambda_2^2 e^2/\left[\lambda_4(e^2+1)^2\right],\ \lambda_1 = \lambda_3 = 0,\ \lambda_2 < 0,\ \lambda_4 > 0,\ g_2 = 0.$$
$$u_{24} = -\frac{\lambda}{2} + \varepsilon\sqrt{\frac{(\lambda^2+4g_1)(e^2-1)}{2(2e^2-1)}}\operatorname{nc}\left[\sqrt{\frac{2(\lambda^2+4g_1)}{1-2e^2}}\,\zeta\right], \quad \lambda_0 = \frac{\lambda_2^2 e^2(e^2-1)}{\lambda_4(2e^2-1)^2},\ \lambda_1 = \lambda_3 = 0,\ \lambda_2 > 0,\ \lambda_4 < 0,\ g_2 = 0.$$
$$u_{25} = -\frac{\lambda}{2} + \varepsilon\sqrt{\frac{(\lambda^2+4g_1)(1-e^2)}{2(2-e^2)}}\operatorname{nd}\left[\sqrt{\frac{2(\lambda^2+4g_1)}{e^2-2}}\,\zeta\right], \quad \lambda_0 = \frac{\lambda_2^2(1-e^2)}{\lambda_4(2-e^2)^2},\ \lambda_1 = \lambda_3 = 0,\ \lambda_2 > 0,\ \lambda_4 < 0,\ g_2 = 0.$$
$$u_{26} = a_{00} + b_{01}\wp^{-1}\left[\sqrt{\lambda_3}\,\zeta/2, -32b_{01}(\lambda+2a_{00})\lambda_3^{-1}, -16b_{01}^2\lambda_3^{-1}\right], \quad \lambda_2 = \lambda_4 = 0,\ \lambda_3 > 0,\ g_1 = \left[\lambda^2 + 6a_{00}(\lambda+a_{00})\right]/2.$$
$$u_{27} = a_{00} + b_{01}\left[e^{2\varepsilon\sqrt{6\lambda a_{00} - 2g_1 + \lambda^2 + 6a_{00}^2}\,\zeta} - \frac{b_{01}(\lambda+2a_{00})}{6a_{00}(\lambda+a_{00}) - 2g_1 + \lambda^2}\right]^{-1}, \quad \lambda_0 = \lambda_1^2/(4\lambda_2),\ \lambda_2 > 0,\ \lambda_3 = \lambda_4 = 0.$$

Family 3. Case 3 gives the following explicit solutions:

$$u_{31}=\frac{-\lambda}{2}+\frac{\varepsilon}{2}\sqrt{\lambda^2+4g_1-12a_{01}b_{01}}\tanh\Bigl(\sqrt{\lambda^2+4g_1-12a_{01}b_{01}}\zeta\Bigr)+\frac{2b_{01}\varepsilon|a_{01}|}{\sqrt{\lambda^2+4g_1-12a_{01}b_{01}}\tanh(\sqrt{\lambda^2+4g_1-12a_{01}b_{01}}\zeta)},\quad \lambda_0=\lambda_2^2/(4\lambda_4),\ \lambda_1=\lambda_3=0,\ \lambda_2<0,\ \lambda_4>0,\ g_2=0.$$

$$u_{32}=\frac{-\lambda}{2}+\frac{\varepsilon}{2}\sqrt{12a_{01}b_{01}-\lambda^2-4g_1}\tan\Bigl(\sqrt{12a_{01}b_{01}-\lambda^2-4g_1}\zeta\Bigr)+\frac{2b_{01}\varepsilon|a_{01}|\tan^{-1}(\sqrt{12a_{01}b_{01}-\lambda^2-4g_1}\zeta)}{\sqrt{12a_{01}b_{01}-\lambda^2-4g_1}},$$

$$\lambda_0=\lambda_2^2/(4\lambda_4),\ \lambda_1=\lambda_3=0,\ \lambda_2>0,\ \lambda_4>0,\ g_2=0.$$

$$u_{33}=a_{00}+\frac{(6\lambda a_{00}-2g_1+\lambda^2+6a_{00}^2)\operatorname{sech}^2\Bigl(\sqrt{6\lambda a_{00}-2g_1+\lambda^2+6a_{00}^2}\zeta\Bigr)}{2\varepsilon\sqrt{6\lambda a_{00}-2g_1+\lambda^2+6a_{00}^2}\tanh\Bigl(\sqrt{6\lambda a_{00}-2g_1+\lambda^2+6a_{00}^2}\zeta\Bigr)-2(2a_{00}+\lambda)},\quad \lambda_0=\lambda_1=0,\ \lambda_2>0.$$

$$u_{34}=a_{00}+\frac{(2g_1-6\lambda a_{00}-\lambda^2-6a_{00}^2)\sec^2\Bigl(\sqrt{2g_1-6\lambda a_{00}-\lambda^2-6a_{00}^2}\zeta\Bigr)}{2\varepsilon\sqrt{2g_1-6\lambda a_{00}-\lambda^2-6a_{00}^2}\tan\Bigl(\sqrt{2g_1-6\lambda a_{00}-\lambda^2-6a_{00}^2}\zeta\Bigr)+2(2a_{00}+\lambda)},\quad \lambda_0=\lambda_1=0,\ \lambda_2<0.$$

$$u_{35}=-\frac{\lambda}{2}+a_{01}|b_{01}|\varepsilon\sqrt{\frac{2(e^2+1)}{\lambda^2+4g_1-12a_{01}b_{01}}}\operatorname{sn}\Biggl[\sqrt{\frac{2(\lambda^2+4g_1-12a_{01}b_{01})}{e^2+1}}\zeta\Biggr]+\varepsilon\sqrt{\frac{\lambda^2+4g_1-12a_{01}b_{01}}{2(e^2+1)}}\operatorname{ns}\Biggl[\sqrt{\frac{2(\lambda^2+4g_1-12a_{01}b_{01})}{e^2+1}}\zeta\Biggr],$$

$$\lambda_0=\lambda_2^2e^2/\bigl[\lambda_4(e^2+1)^2\bigr],\ \lambda_1=\lambda_3=0,\ \lambda_2<0,\ \lambda_4>0,\ g_2=0.$$

$$u_{36}=\frac{-b_{01}(\lambda+2a_{00})}{6\lambda a_{00}-2g_1+\lambda^2+6a_{00}^2}+e^{2\varepsilon\sqrt{6\lambda a_{00}-2g_1+\lambda^2+6a_{00}^2}\zeta},\quad \lambda_0=\lambda_1^2/(4\lambda_2),\ \lambda_2>0,\ \lambda_3=\lambda_4=0.$$

$$u_{37}=\frac{-\lambda}{2}+\varepsilon\sqrt{2g_1-6\lambda a_{00}-\lambda^2-6a_{00}^2}\sec\Bigl(2\sqrt{2g_1-6\lambda a_{00}-\lambda^2-6a_{00}^2}\zeta\Bigr),\quad \lambda_0=\lambda_1=\lambda_3=0,\ \lambda_2<0,\ \lambda_4>0,\ g_2=0.$$

Family 4. According to Case 4, we obtain the following solutions:

$$u_{41}=\frac{-\lambda}{2}\pm\frac{1}{2}\sqrt{\lambda_2}\bigl|\tanh\bigl(\sqrt{\lambda_2}\zeta\bigr)\bigr|,\quad \lambda_0=\lambda_1=\lambda_3=0,\ \lambda_2>0,\ \lambda_4<0.$$

$$u_{42}=\frac{-\lambda}{2}\pm\frac{1}{2}\sqrt{-\lambda_2}\bigl|\tan\bigl(\sqrt{-\lambda_2}\zeta\bigr)\bigr|,\quad \lambda_0=\lambda_1=\lambda_3=0,\ \lambda_2<0,\ \lambda_4>0.$$

$$u_{43}=\frac{-\lambda}{2}\pm\frac{1}{4}\sqrt{-2\lambda_2}\operatorname{sech}^2\bigl(\sqrt{-\lambda_2/2}\zeta\bigr)\tanh^{-1}\bigl(\sqrt{-\lambda_2/2}\zeta\bigr),\quad \lambda_0=\lambda_2^2/(4\lambda_4),\ \lambda_1=\lambda_3=0,\ \lambda_2<0,\ \lambda_4>0.$$

$$u_{44}=\frac{-\lambda}{2}\pm\frac{1}{4}\sqrt{2\lambda_2}\bigl[1+\tan^2\bigl(\sqrt{\lambda_2/2}\zeta\bigr)\bigr]\tan^{-1}\bigl(\sqrt{\lambda_2/2}\zeta\bigr),\quad \lambda_0=\lambda_2^2/(4\lambda_4),\ \lambda_1=\lambda_3=0,\ \lambda_2>0,\ \lambda_4>0.$$

$$u_{45}=\frac{-\lambda}{2}\pm\frac{1}{4}\lambda^{1/2}\bigl|2\tanh\bigl(0.5\sqrt{\lambda_2}\zeta\bigr)+\operatorname{sech}^2\bigl(0.5\sqrt{\lambda_2}\zeta\bigr)\tanh^{-2}\bigl(0.5\sqrt{\lambda_2}\zeta\bigr)\bigr|,\quad \lambda_0=\lambda_1=\lambda_3=0,\ \lambda_2>0.$$

$$u_{46}=\frac{-\lambda}{2}\pm\frac{1}{4}\sqrt{-\lambda_2}\bigl|\tan\bigl(0.5\sqrt{-\lambda_2}\zeta\bigr)\bigr|\bigl|2-\bigl[1+\tan^2\bigl(0.5\sqrt{-\lambda_2}\zeta\bigr)\bigr]\tan^2\bigl(0.5\sqrt{-\lambda_2}\zeta\bigr)\bigr|,\quad \lambda_0=\lambda_1=\lambda_3=0,\ \lambda_2<0.$$

$$u_{47}=\frac{-\lambda}{2}\pm\frac{1}{2}\Bigl|\operatorname{sn}'\Bigl[\sqrt{-\lambda_2/(e^2+1)}\zeta\Bigr]\Bigr|\operatorname{ns}\Bigl[\sqrt{-\lambda_2/(e^2+1)}\zeta\Bigr],\quad \lambda_0=\lambda_2^2e^2/\bigl[\lambda_4(e^2+1)^2\bigr],\ \lambda_1=\lambda_3=0,\ \lambda_2<0,\ \lambda_4>0.$$

$$u_{48}=\frac{-\lambda}{2}\pm\frac{1}{2}\Bigl|\operatorname{cn}'\Bigl[\sqrt{\lambda_2/(2e^2-1)}\zeta\Bigr]\Bigr|\operatorname{nc}\Bigl[\sqrt{\lambda_2/(2e^2-1)}\zeta\Bigr],$$

$$\lambda_0=\lambda_2^2e^2(e^2-1)/\bigl[\lambda_4(2e^2-1)^2\bigr],\ \lambda_1=\lambda_3=0,\ \lambda_2>0,\ \lambda_4<0.$$

$$u_{49}=\frac{-\lambda}{2}\pm\frac{1}{2}\Bigl|\operatorname{dn}'\Bigl\{\sqrt{-\lambda_2/\bigl[\lambda_4(2-e^2)\bigr]}\zeta\Bigr\}\Bigr|\operatorname{nd}\Bigl\{\sqrt{-\lambda_2/\bigl[\lambda_4(2-e^2)\bigr]}\zeta\Bigr\},\quad \lambda_0=\frac{\lambda_2^2(1-e^2)}{\lambda_4(2-e^2)^2},\ \lambda_1=\lambda_3=0,\ \lambda_2>0,\ \lambda_4<0.$$

Family 5. From Case 5, we obtain the following solutions of Eq. (1):

$$u_{51}=\frac{-\lambda}{2}\pm\frac{1}{4}\sqrt{\lambda_2}\bigl|\tanh\bigl(0.5\sqrt{\lambda_2}\zeta\bigr)\bigr|,\quad \lambda_0=\lambda_1=\lambda_4=0,\ \lambda_2>0,\ \lambda_3\neq 0.$$

$$u_{52}=\frac{-\lambda}{2}\pm\frac{1}{4}\sqrt{-\lambda_2}\left|\tan\left(0.5\sqrt{-\lambda_2}\zeta\right)\right|,\quad \lambda_0=\lambda_1=\lambda_4=0,\ \lambda_2<0,\ \lambda_3\neq 0.$$

$$u_{53}=\frac{-\lambda}{2}\pm\frac{1}{4}\sqrt{-\lambda_2/2}\tanh\left(\sqrt{-\lambda_2/2}\zeta\right)\pm\tanh^{-1}\left(\sqrt{-\lambda_2/2}\zeta\right)\left[4b_{01}\sqrt{-2a_{01}^2/\lambda_2}\pm\frac{1}{8}\sqrt{-2\lambda_2}\operatorname{sech}^2\left(\sqrt{-\lambda_2/2}\zeta\right)\right],$$
$$\lambda_0=\lambda_2^2/(4\lambda_4),\ \lambda_1=\lambda_3=0,\ \lambda_2<0,\ \lambda_4>0.$$

$$u_{54}=\frac{-\lambda}{2}\pm\frac{1}{4}\sqrt{\lambda_2/2}\tan\left(\sqrt{\lambda_2/2}\zeta\right)\pm\tan^{-1}\left(\sqrt{\lambda_2/2}\zeta\right)\left\{4b_{01}\sqrt{2a_{01}^2/\lambda_2}\pm\frac{1}{8}\sqrt{2\lambda_2}\left[1+\tan^2\left(\sqrt{\lambda_2/2}\zeta\right)\right]\right\},$$
$$\lambda_0=\lambda_2^2/(4\lambda_4),\ \lambda_1=\lambda_3=0,\ \lambda_2>0,\ \lambda_4>0.$$

$$u_{55}=\frac{-\lambda}{2}+\frac{a_{01}\lambda_2\operatorname{sech}^2(0.5\sqrt{\lambda_2}\zeta)}{8\varepsilon a_{01}\sqrt{\lambda_2}\tanh(0.5\sqrt{\lambda_2}\zeta)-\lambda_3}\pm\frac{\sqrt{\lambda_2}}{4}\left|\tanh\left(0.5\sqrt{\lambda_2}\zeta\right)\pm\frac{4a_{01}\operatorname{sech}^2(0.5\sqrt{\lambda_2}\zeta)}{8\varepsilon a_{01}\sqrt{\lambda_2}\tanh(0.5\sqrt{\lambda_2}\zeta)-\lambda_3}\right|,$$
$$\lambda_0=\lambda_1=0,\ \lambda_2>0.$$

$$u_{56}=\frac{-\lambda}{2}+\frac{a_{01}\lambda_2\sec^2(0.5\sqrt{-\lambda_2}\zeta)}{8\varepsilon a_{01}\sqrt{-\lambda_2}\tan(0.5\sqrt{-\lambda_2}\zeta)+\lambda_3}\pm\frac{1}{4}\sqrt{-\lambda_2}\left|\frac{4\varepsilon|a_{01}|[1+\tan^2(0.5\sqrt{-\lambda_2}\zeta)]}{8\varepsilon|a_{01}|\sqrt{-\lambda_2}\tan(0.5\sqrt{-\lambda_2}\zeta)+\lambda_3}-\tan\left(0.5\sqrt{-\lambda_2}\zeta\right)\right|,$$
$$\lambda_0=\lambda_1=0,\ \lambda_2<0.$$

$$u_{57}=\frac{-\lambda}{2}\pm a_{01}\sqrt{-\lambda_2e^2/\left[\lambda_4(e^2+1)\right]}\operatorname{sn}\left[\sqrt{-\lambda_2/(e^2+1)}\zeta\right]\pm\operatorname{ns}\left[\sqrt{-\lambda_2/(e^2+1)}\zeta\right]$$
$$\times\left\{b_{01}\sqrt{-\lambda_4(e^2+1)/\lambda_2e^2}\pm\frac{1}{4}\left|\operatorname{sn}'\left[\sqrt{-\lambda_2/(e^2+1)}\zeta\right]\right|\right\},$$
$$\lambda_0=\lambda_2^2e^2/\left[\lambda_4(e^2+1)^2\right],\ \lambda_1=\lambda_3=0,\ \lambda_2<0,\ \lambda_4>0.$$

$$u_{58}=\frac{-\lambda}{2}+\wp^{-1}\left(\sqrt{\lambda_3}\zeta/2,f_2,f_3\right)\left[b_{01}\pm\frac{1}{4}\left|\wp'\left(\sqrt{\lambda_3}\zeta/2,f_2,f_3\right)\right|\right],\quad \lambda_2=\lambda_4=0,\ \lambda_3>0.$$

$$u_{59}=\frac{-\lambda}{2}+a_{01}\left(e^{\varepsilon\sqrt{\lambda_2}\zeta}-\frac{\lambda_1}{2\lambda_2}\right)+\left(e^{\varepsilon\sqrt{\lambda_2}\zeta}-\frac{\lambda_1}{2\lambda_2}\right)^{-1}\left(b_{01}\pm\frac{1}{4}\sqrt{\lambda_2}e^{\varepsilon\sqrt{\lambda_2}\zeta}\right),\quad \lambda_0=\lambda_1^2/(4\lambda_2),\ \lambda_2>0,\ \lambda_3=\lambda_4=0.$$

$$u_{510}=\frac{-\lambda}{2}\pm\frac{1}{4}\sqrt{-\lambda_2}\left|\cos\left(\sqrt{-\lambda_2}\zeta\right)\right|\left[\varepsilon\sin\left(\sqrt{-\lambda_2}\zeta\right)-1\right]^{-1},\quad \lambda_0=\lambda_3=\lambda_4=0,\ \lambda_1\neq 0,\ \lambda_2<0.$$

$$u_{511}=\frac{-\lambda}{2}\pm\frac{1}{2}\sqrt{\lambda_2}\left|\cosh\left(2\sqrt{\lambda_2}\zeta\right)\right|\left[\varepsilon\sinh\left(2\sqrt{\lambda_2}\zeta\right)-1\right]^{-1},\quad \lambda_0=\lambda_3=\lambda_4=0,\ \lambda_1\neq 0,\ \lambda_2>0.$$

$$u_{512}=\frac{-\lambda}{2}\pm\frac{1}{4}\sqrt{-\lambda_2}\left[\sec\left(\sqrt{-\lambda_2}\zeta\right)\pm\left|\tan\left(\sqrt{-\lambda_2}\zeta\right)\right|\right],\quad \lambda_0=\lambda_1=\lambda_3=0,\ \lambda_2<0,\ \lambda_4>0,\ g_2=0,$$

where $\zeta=x+\lambda t$.

Remark 2. For rational solutions are usually meaningless in physics and mechanics, we omit them in this Letter. Compared with [22], the solutions obtained here are more abundant. Many new solutions of Eq. (1) have been found. The graphics of some solutions are shown in Figs. 1–4 to explain the properties of these solutions.

Remark 3. When $e\to 1$, the Jacobi elliptic function periodic solutions will be degenerative to the corresponding solitary wave solutions, which are shown in Table 2.

Table 2

Jacobi periodic solutions	u_{15}	u_{23}	u_{35}	u_{47}	u_{48}	u_{57}
$e\to 1$	↓	↓	↓	↓	↓	↓
solitary wave solutions	u_{11}	u_{21}	u_{31}	u_{43}	u_{41}	u_{53}

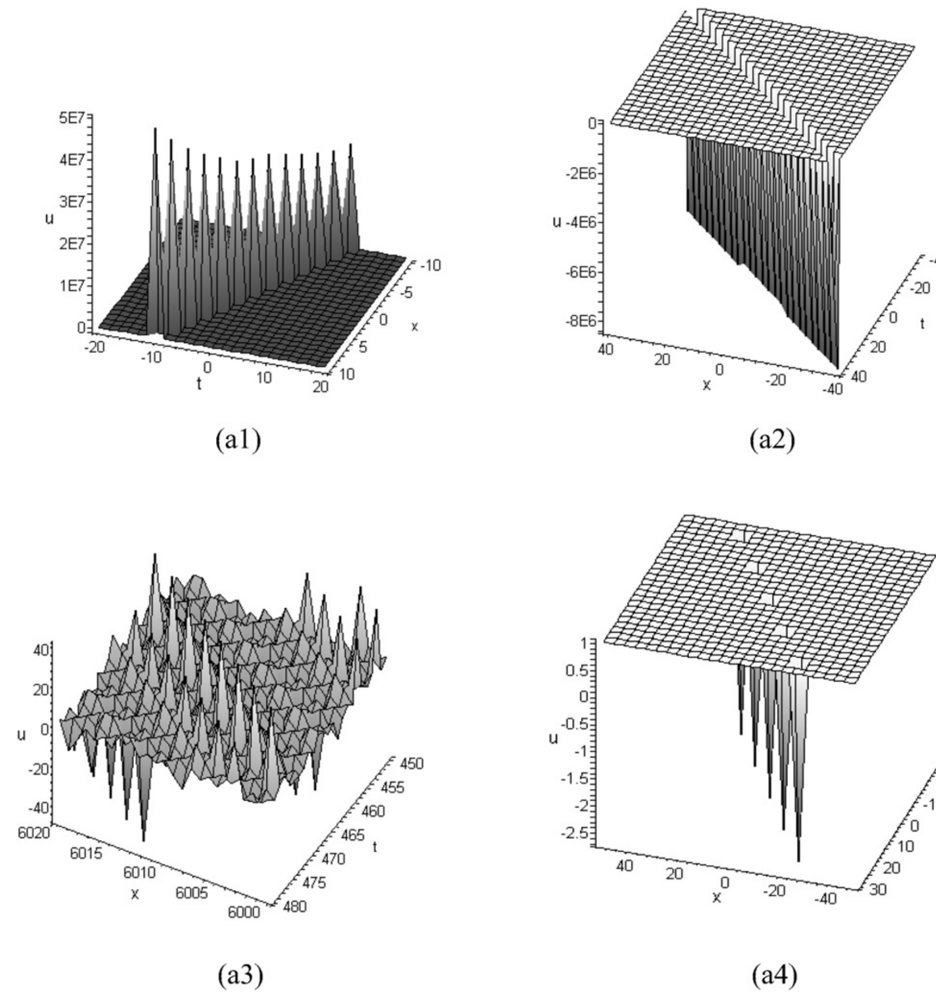

Fig. 1. (a1) $u_{21}(\lambda = g_1 = 1, \varepsilon = 1)$; (a2) $u_{22}(\lambda = g_1 = 1, \varepsilon = 1)$; (a3) $u_{23}(\lambda = g_1 = 1, e = 0.0004, \varepsilon = 1)$; (a4) $u_{27}(a_{00} = b_{01} = \lambda = g_1 = 1, \varepsilon = 1)$.

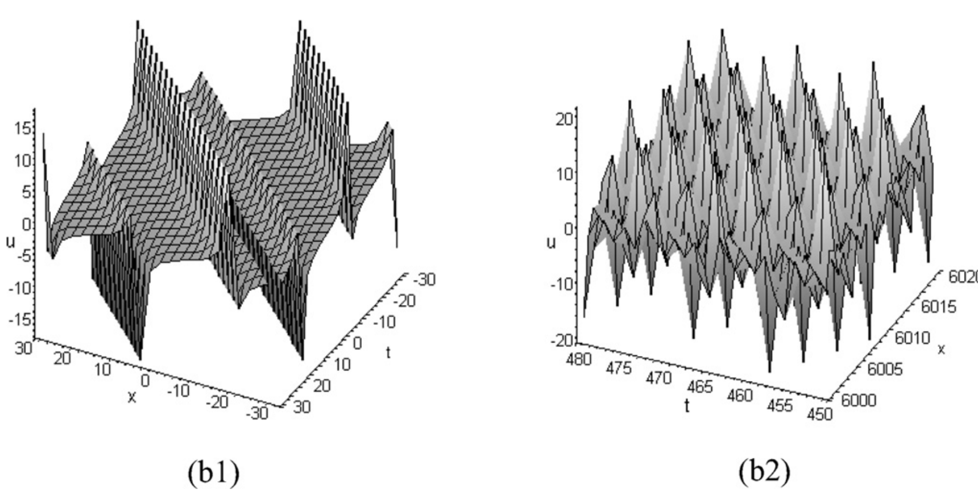

Fig. 2. (b1) $u_{31}(a_{01} = b_{01} = \lambda = g_1 = 1, \varepsilon = 1)$; (b2) $u_{35}(a_{01} = b_{01} = 0.1, \lambda = g_1 = 1, e = 0.4, \varepsilon = 1)$.

4. The exact travelling wave solutions of Eq. (2)

Let us now consider Eq. (2). According to the steps in Section 2, we firstly make the following formal travelling wave transformation:

$$u(x, y, z, t) = u(\zeta), \quad \zeta = \alpha x + \beta y + gz - \lambda t, \tag{11}$$

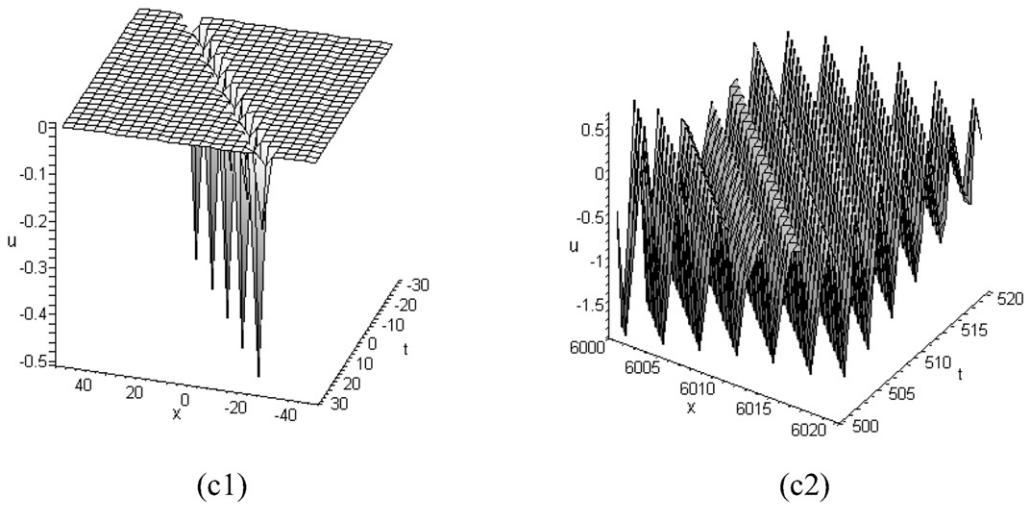

Fig. 3. (c1) $u_{41}(\lambda = \lambda_2 = 1, \varepsilon = 1)$; (c2) $u_{48}(\lambda = \lambda_2 = 1, e = 0.8, \varepsilon = 1)$.

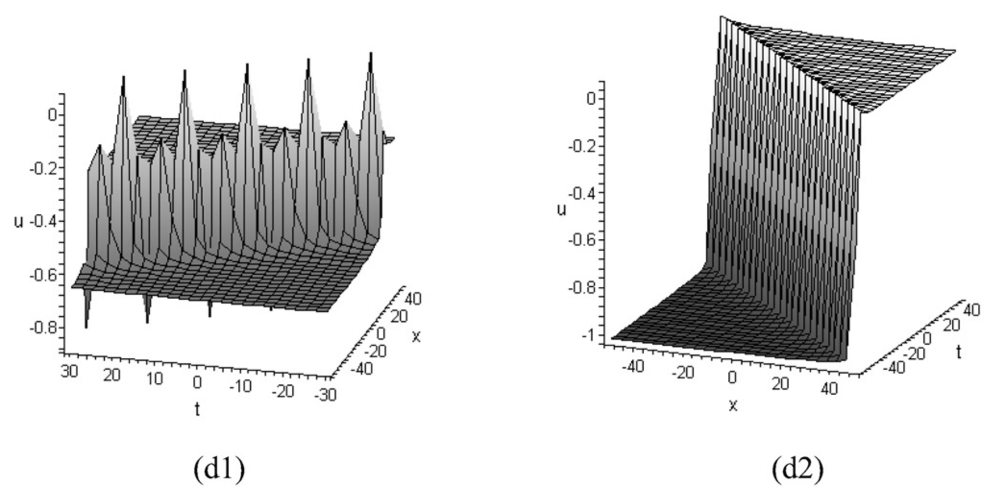

Fig. 4. (d1) $u_{510}(\lambda = \lambda_1 = 1, \lambda_2 = -1, \varepsilon = 1)$; (d2) $u_{511}(\lambda = \lambda_1 = \lambda_2 = 1, \varepsilon = 1)$.

where α, β, g and λ are undetermined constants. Substituting (11) into (2) gives rise to

$$\alpha^4 u^{(4)} + 6\big[(\alpha u')^2 - \alpha^2 u u''\big] + \big(\beta^2 + g^2 + \alpha\lambda\big)u'' = 0. \tag{12}$$

We assume that (12) has the solution in the form

$$u(\zeta) = a_0 + a_1\phi + a_2\phi^2 + b_1\phi^{-1} + b_2\phi^{-2} + |\phi'|\big(c_1 + c_2\phi + d_1\phi^{-1} + d_2\phi^{-2}\big), \tag{13}$$

and $\phi = \phi(\zeta)$ satisfies Eq. (6), where a_0, a_1, a_2, b_1, b_2, c_1, c_2, d_1 and d_2 are constants to be determined later.

Substituting (13) into (12) along with (6), we can get a set of over-determined algebraic equations. Solving the obtained system leads to the following cases (to avoid more complicated discussions, we only restrict ourselves to several interesting cases). From the above discussion, we can see that sec, tan and $\sec^2$ type solutions appear in pairs with sech, tanh and sech^2 type solutions respectively. For the limit of length, we omit some solutions and only list some interesting ones here.

Case I.

$$\lambda_0 = \lambda_3 = \lambda_4 = 0, \quad \lambda_2 < 0,$$
$$a_0 = \big(2\alpha\lambda\lambda_2 + 2\lambda_2 g^2 + 2\lambda_2\beta^2 + 8\alpha^4\lambda_2^2 \pm 3\alpha^2 c_1\lambda_1\sqrt{-\lambda_2}i\big)/\big(12\alpha^2\lambda_2\big), \quad a_1 = \pm 2\sqrt{-\lambda_2}c_1 i,$$
$$a_2 = -3\alpha^2 c_1^2\lambda_2/\big(\alpha\lambda + g^2 + \beta^2 - 6\alpha^2 a_0 + 4\alpha^4\lambda_2\big), \quad b_1 = -d_1,\ b_2 = -d_2,\ c_2 = \pm 3\sqrt{-\lambda_2}c_1^2\alpha^2 i.$$

In such case, Eq. (2) has the following solution

$$u_1 = \frac{1}{12\alpha^2\lambda_2}\left(2\alpha\lambda\lambda_2 + 2\lambda_2 g^2 + 2\lambda_2\beta^2 + 8\alpha^4\lambda_2^2 \pm 3\alpha^2 c_1\lambda_1\sqrt{-\lambda_2}i\right) \pm c_1\lambda_1\left[\varepsilon\sin\left(\sqrt{-\lambda_2}\zeta\right) - 1\right]$$
$$\times\left\{\frac{1}{\lambda_2}\sqrt{-\lambda_2}i - \frac{3\alpha^2 c_1\lambda_1[\varepsilon\sin(\sqrt{-\lambda_2}\zeta) - 1]}{4\lambda_2(\alpha\lambda + g^2 + \beta^2 - 6\alpha^2 a_0 + 4\alpha^4\lambda_2)}\right\} - \frac{2\lambda_2}{[\varepsilon\sin(\sqrt{-\lambda_2}\zeta) - 1]}\left\{\frac{d_1}{\lambda_1} - \frac{2d_2\lambda_2}{\lambda_1^2[\varepsilon\sin(\sqrt{-\lambda_2}\zeta) - 1]}\right\}$$
$$+\left|\cos\left(\sqrt{-\lambda_2}\zeta\right)\right|\left\{\frac{c_1\sqrt{-\lambda_2}|\lambda_1|}{2\lambda_2} \pm \frac{3c_1^2\alpha^2\lambda_1^2 i[\varepsilon\sin(\sqrt{-\lambda_2}\zeta) - 1]}{4\lambda_2} \pm \frac{d_1\sqrt{-\lambda_2}}{\varepsilon\sin(\sqrt{-\lambda_2}\zeta) - 1} \pm \frac{2d_2(-\lambda_2)^{2/3}}{\lambda_1[\varepsilon\sin(\sqrt{-\lambda_2}\zeta) - 1]^2}\right\}.$$

Case II.

$$\lambda_0 = \lambda_3 = \lambda_4 = 0, \quad \lambda_2 > 0,$$
$$a_0 = \left(2\alpha\lambda\lambda_2 + 2\lambda_2 g^2 + 2\lambda_2\beta^2 + 8\alpha^4\lambda_2^2 \pm 3\alpha^2 c_1\lambda_1\sqrt{\lambda_2}\right)/\left(12\alpha^2\lambda_2\right), \quad a_1 = \pm 2\sqrt{\lambda_2}c_1,$$
$$a_2 = -3\alpha^2 c_1^2\lambda_2/\left(\alpha\lambda + g^2 + \beta^2 - 6\alpha^2 a_0 + 4\alpha^4\lambda_2\right), \quad b_1 = -d_1,\ b_2 = -d_2,\ c_2 = \pm 3\sqrt{\lambda_2}c_1^2\alpha^2.$$

The solution of Eq. (2) is

$$u_2 = \frac{1}{12\alpha^2\lambda_2}\left(2\alpha\lambda\lambda_2 + 2\lambda_2 g^2 + 2\lambda_2\beta^2 + 8\alpha^4\lambda_2^2 \pm 3\alpha^2 c_1\lambda_1\sqrt{\lambda_2}\right) \pm c_1\lambda_1\left[\varepsilon\sinh\left(2\sqrt{\lambda_2}\zeta\right) - 1\right]$$
$$\times\left\{\lambda_2^{-1/2} - \frac{3\alpha^2 c_1\lambda_1[\varepsilon\sinh(2\sqrt{\lambda_2}\zeta) - 1]}{4\lambda_2(\alpha\lambda + g^2 + \beta^2 - 6\alpha^2 a_0 + 4\alpha^4\lambda_2)}\right\} - \frac{2\lambda_2}{\lambda_1[\varepsilon\sinh(2\sqrt{\lambda_2}\zeta) - 1]}\left\{d_1 - \frac{2d_2\lambda_2}{\lambda_1[\varepsilon\sinh(2\sqrt{\lambda_2}\zeta) - 1]}\right\}$$
$$+\left|\cosh\left(2\sqrt{\lambda_2}\zeta\right)\right|\left\{\frac{c_1\lambda_1}{\sqrt{\lambda_2}} \pm \frac{3c_1^2\alpha^2\lambda_1^2[\varepsilon\sinh(2\sqrt{\lambda_2}\zeta) - 1]}{2\lambda_2} + \frac{2d_1\sqrt{\lambda_2}}{\varepsilon\sinh(2\sqrt{\lambda_2}\zeta) - 1} + \frac{4d_2\lambda_2^{2/3}}{\lambda_1[\varepsilon\sinh(2\sqrt{\lambda_2}\zeta) - 1]^2}\right\}.$$

Case III.

$$\lambda_0 = \lambda_3 = \lambda_4 = a_2 = c_2 = 0, \quad \lambda_2 < 0, \quad a_0 = \left(g^2 + 5\alpha^4\lambda_2 + \alpha\lambda - \beta^2\right)/\left(6\alpha^2\right), \quad a_1 = \pm\sqrt{-\lambda_2}c_1 i,$$
$$b_1 = -d_1, \quad b_2 = -d_2, \quad c_1 = \pm 4\sqrt{-\lambda_2}\alpha^2\lambda_2 i/(3\lambda_1).$$

Eq. (2) has the following solution

$$u_3 = \frac{g^2 + 5\alpha^4\lambda_2 + \alpha\lambda - \beta^2}{6\alpha^2} \pm \frac{\lambda_1}{2\lambda_2}\left[\varepsilon\sin\left(\sqrt{-\lambda_2}\zeta\right) - 1\right]\left\{\sqrt{-\lambda_2}c_1 i - \frac{\lambda_1 d_2[\varepsilon\sin(\sqrt{-\lambda_2}\zeta) - 1]}{2\lambda_2^2}\right\} - \frac{2d_1\lambda_2}{\lambda_1[\varepsilon\sin(\sqrt{-\lambda_2}\zeta) - 1]}$$
$$\pm\left|\cos\left(\sqrt{-\lambda_2}\zeta\right)\right|\left\{\frac{2\alpha^2\lambda_2 i}{3} \pm \frac{d_1\sqrt{-\lambda_2}}{\varepsilon\sin(\sqrt{-\lambda_2}\zeta) - 1} \pm \frac{2d_2(-\lambda_2)^{2/3}}{\lambda_1[\varepsilon\sin(\sqrt{-\lambda_2}\zeta) - 1]^2}\right\}.$$

Case IV.

$$\lambda_0 = \lambda_3 = \lambda_4 = a_2 = c_2 = 0, \quad \lambda_2 > 0, \quad a_0 = \left(g^2 + 5\alpha^4\lambda_2 + \alpha\lambda - \beta^2\right)/\left(6\alpha^2\right), \quad a_1 = \pm\sqrt{\lambda_2}c_1,$$
$$b_1 = -d_1, \quad b_2 = -d_2, \quad c_1 = \pm 4\sqrt{\lambda_2}\alpha^2\lambda_2/(3\lambda_1).$$

We can write the solution of Eq. (2) in the form

$$u_4 = \frac{g^2 + 5\alpha^4\lambda_2 + \alpha\lambda - \beta^2}{6\alpha^2} \pm \frac{\sqrt{\lambda_2}c_1\lambda_1[\varepsilon\sinh(2\sqrt{\lambda_2}\zeta) - 1]}{2\lambda_2} - \frac{2\lambda_2}{\lambda_1[\varepsilon\sinh(2\sqrt{\lambda_2}\zeta) - 1]}\left\{d_1 - \frac{2d_2\lambda_2}{\lambda_1[\varepsilon\sinh(2\sqrt{\lambda_2}\zeta) - 1]}\right\}$$
$$\pm 2\left|\cosh\left(2\sqrt{\lambda_2}\zeta\right)\right|\left\{\frac{2\alpha^2\lambda_2}{3} + \frac{d_1\sqrt{\lambda_2}}{\varepsilon\sinh(2\sqrt{\lambda_2}\zeta) - 1} + \frac{2d_2\lambda_2^{2/3}}{\lambda_1[\varepsilon\sinh(2\sqrt{\lambda_2}\zeta) - 1]^2}\right\}.$$

Case V.

$$\lambda_3 = \lambda_4 = 0, \quad \lambda_1 = a_1^2/(2c_1c_2), \quad \lambda_2 = a_1^2/(4c_1^2),$$
$$a_0 = \left(2\beta^2 c_1^2 c_2 + 2a_1^2\alpha^4 c_2 + 2g^2 c_1^2 c_2 + 3a_1\alpha^2 c_1^3 + 2\alpha\lambda c_1^2 c_2\right)/\left(12\alpha^2 c_1^2 c_2\right),$$
$$a_2 = c_2 a_1/(2c_1), \quad b_1 = -d_1, \quad b_2 = -d_2.$$

With this choice, Eq. (2) has the following solution

$$u_5 = \frac{1}{12\alpha^2 c_1^2 c_2} 2\beta^2 c_1^2 c_2 + 2a_1^2\alpha^4 c_2 + 2g^2 c_1^2 c_2 + 3a_1\alpha^2 c_1^3 + 2\alpha\lambda c_1^2 c_2 + \frac{1}{2\lambda_2}\lambda_1\left[\varepsilon \sinh\left(2\sqrt{\lambda_2}\zeta\right) - 1\right]$$
$$\times \left\{a_1 + \frac{\lambda_1[\varepsilon\sinh(2\sqrt{\lambda_2}\zeta) - 1]}{2\lambda_2}\left(\frac{c_2 a_1}{2c_1} - d_2\right)\right\} - \frac{2d_1\lambda_2}{\lambda_1[\varepsilon\sinh(2\sqrt{\lambda_2}\zeta) - 1]}$$
$$+ \left|\cosh(2\sqrt{\lambda_2}\zeta)\right|\left\{\frac{c_1\lambda_1}{\sqrt{\lambda_2}} + \frac{c_2\lambda_1^2[\varepsilon\sinh(2\sqrt{\lambda_2}\zeta) - 1]}{2\lambda_2\sqrt{\lambda_2}} + \frac{2d_1\sqrt{\lambda_2}}{\varepsilon\sinh(2\sqrt{\lambda_2}\zeta) - 1} + \frac{4d_2\lambda_2^{2/3}}{\lambda_1[\varepsilon\sinh(2\sqrt{\lambda_2}\zeta) - 1]^2}\right\}.$$

Case VI.

$$\lambda_3 = \lambda_4 = c_2 = a_2 = 0, \quad \lambda_2 = a_1^2/c_1^2,$$
$$a_0 = \left(\beta^2 c_1^2 a_1 + a_1^3\alpha^4 + g^2 c_1^2 a_1 + 3\lambda_1\alpha^2 c_1^4 + \alpha\lambda c_1^2 a_1\right)/\left(6\alpha^2 c_1^2 a_1\right), \quad b_1 = -d_1, \quad b_2 = -d_2.$$

From this case, we can obtain the following solution:

$$u_6 = \frac{\beta^2 c_1^2 a_1 + a_1^3\alpha^4 + g^2 c_1^2 a_1 + 3\lambda_1\alpha^2 c_1^4 + \alpha\lambda c_1^2 a_1}{6\alpha^2 c_1^2 a_1} + \frac{\lambda_1[\varepsilon\sinh(2\sqrt{\lambda_2}\zeta) - 1]}{2\lambda_2}\left\{a_1 - \frac{d_2\lambda_1[\varepsilon\sinh(2\sqrt{\lambda_2}\zeta) - 1]}{2\lambda_2}\right\}$$
$$- \frac{2d_1\lambda_2}{\lambda_1[\varepsilon\sinh(2\sqrt{\lambda_2}\zeta) - 1]} + \left|\cosh\left(2\sqrt{\lambda_2}\zeta\right)\right|\left\{\frac{c_1\lambda_1}{\sqrt{\lambda_2}} + \frac{2d_1\sqrt{\lambda_2}}{\varepsilon\sinh(2\sqrt{\lambda_2}\zeta) - 1} + \frac{4d_2\lambda_2^{2/3}}{\lambda_1[\varepsilon\sinh(2\sqrt{\lambda_2}\zeta) - 1]^2}\right\},$$

where $\zeta = \alpha x + \beta y + gz - \lambda t$.

Remark 4. Due to our more generalized method, we can not only recover the solutions in [23,25] easily but also obtain more abundant solutions.

5. Conclusions

In summary, a generalized algebraic method for seeking more types of exact travelling wave solutions of Boussinesq–Burgers equation and (3 + 1)-dimensional Kadomtsev–Petviashvili equation is implemented. By using this scheme, rich new families of exact solutions are obtained. The method proposed in Section 2 can be also extended to solve some nonlinear evolution equations with variable coefficients, such as (3 + 1)-dimensional Kadomtsev–Petviashvili equation with variable coefficients [29]. For the limit of length, we do not list them here.

Moreover, the ansatz (5) in Section 2 can be extended to a more general transformation to find more types of solutions for nonlinear evolution equations with variable coefficients. For example, we can assume that the solution is in the form

$$u_i(\zeta) = \sum_{j=0}^{r_i} a_{ij}(x)\phi^j + \sum_{j=1}^{r_i}\phi^{-j}\left\{b_{ij}(x) + |\phi'|\left[c_{ij}(x)\phi^{2j-1} + d_{ij}(x)\right]\right\} + \sum_{j=1}^{r}|\phi'|^{-j}\left\{e_{ij}(x) + \phi^{-j}\left[h_{ij}(x)\phi^{2j} + l_{ij}(x)\right]\right\},$$
$$x \equiv (x_0 \equiv t, x_1, x_2, \ldots x_n),$$

where $a_{ij}(x)$, $b_{ij}(x)$, $c_{ij}(x)$, $d_{ij}(x)$, $e_{ij}(x)$, $h_{ij}(x)$ and $l_{ij}(x)$ are all constants to be determined later, and $\phi = \phi(\zeta)$ satisfies the elliptic equation in [30]. We will study these cases in another paper.

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