Disentanglement, Bell-nonlocality violation and teleportation capacity of the decaying tripartite states

Ming-Liang Hu

School of Science, Xi'an University of Posts and Telecommunications, Xi'an 710061, China

ABSTRACT

Dynamics of disentanglement as measured by the tripartite negativity and Bell nonlocality as measured by the extent of violation of the multipartite Bell-type inequalities are investigated in this work. It is shown definitively that for the initial three-qubit Greenberger–Horne–Zeilinger (GHZ) or W class state preparation, the Bell nonlocality suffers sudden death under the influence of thermal reservoirs. Moreover, all the Bell-nonlocal states are useful for nonclassical teleportation, while there are entangled states that do not violate any Bell-type inequalities, but still yield nonclassical teleportation fidelity.

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1. Introduction

Entanglement and Bell nonlocality are two basic ingredients of a quantum state which are intimately related to each other [1]. Their relations have been the focus of research interest for many years and still remain an important subject deserving to be investigated. Entanglement, which describes correlations between two or more subsystems, refers to the state of a composite system that cannot be written as products of states of each subsystem. As a physical resource, entanglement is crucial for nearly all applications related to quantum information processing (QIP). One such application is quantum teleportation [2], by which an unknown state can be transmitted from the sender to a distant receiver with the help of local operations and classical communication (LOCC). However, not all the states that are entangled can be used for teleportation with fidelity (see sections below) better than that achievable via classical communication alone, and the fidelity is even not a monotonic function of the degree of entanglement of the resource [3–5]. This demonstrates that entanglement may only reveal certain aspects of a quantum state.

Bell nonlocality corresponds to another quantum correlation of a quantum state that cannot be reproduced by any classical local hidden variable models. This nonlocal property is manifested

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E-mail addresses: mingliang0301@163.com, mingliang0301@xupt.edu.cn.

unambiguously by the violation of different Bell-type inequalities, which plays a fundamental role in better understanding of the subtle aspects of quantum mechanics [1,6]. Historically, the violation of Bell inequalities has been considered as a means of determining whether there is entanglement between two qubits, for the inseparability of a bipartite pure state corresponds to the violation of the Bell inequality in the Clauser–Horne–Shimony–Holt (CHSH) form, and vice versa [6]. But this is not the case for the mixed states (which are in practice the ones always encountered). As demonstrated initially by Werner [7], there exist bipartite mixed states which are entangled but do not violate any Bell-type inequalities. Further studies also showed that the maximal violation of a Bell inequality does not behave monotonously under LOCC [3,8].

The ability for transmitting information reveals also an important aspect of nonseparability of a quantum state which is intimately related to entanglement and Bell-nonlocal correlations [3]. For bipartite two-qubit systems, it has been demonstrated that all states that violate the CHSH form of Bell inequality can be used for teleporting an arbitrary one-qubit state with nonclassical fidelity [4], while there are entangled mixed states which do not violate any Bell-type inequalities, but still yield nonclassical teleportation fidelity [3]. For the tripartite systems, as we know, there are two distinct classes of multipartite entangled states, i.e., the Greenberger–Horne–Zeilinger (GHZ) class and the *W* class, which bear incompatible multipartite correlations in the sense that they cannot be transformed into each other under stochastic local operations and classical communication [9]. Particularly, it has been demonstrated that besides the Bell states, the three-qubit GHZ and *W* class states in the form of $|\psi_{GHZ}\rangle = (|000\rangle + |111\rangle)/\sqrt{2}$ and $|\psi_W\rangle = (\sqrt{2}|001\rangle + |010\rangle + |100\rangle)/2$ can also be adopted as quantum channels for perfect teleportation under ideal circumstance [10,11]. Since they both are unavoidably disturbed by the surrounding environments, it is natural to ask for their robustness against decoherence in terms of their teleportation capacity, and its possible relations with the degrees of entanglement and Bell-nonlocality violation.

From an application point of view, one may hope that quantum correlations which are crucial for QIP can be maintained for sufficiently a long time to permit designed tasks to be fulfilled. But in practice, every quantum system is open and susceptible to the unavoidable interaction with its surroundings [12–14]. This may lead to decoherence and destruction of correlations. Particularly, under certain circumstances, the entanglement of a bipartite state can even terminate abruptly in a finite time, a phenomenon termed entanglement sudden death (ESD) by Yu and Eberly [13] and has been recently confirmed experimentally [15]. Moreover, due to its practical applications in QIP [16,17], the research interest in the decay dynamics of Bell-type correlations has been renewed. Analogous to ESD, it has also been demonstrated that an initial nonlocal quantum state may lose its nonlocal property in a finite timescale under the influence of external forces. This phenomenon has been named Bell-nonlocality sudden death (BNSD) [18,19] and many theoretical efforts have been devoted to it with different contexts [20–25]. For instance, Ann and Jaeger have demonstrated the occurrence of BNSD for the initial three-qubit *W* class state as measured by its violation of the MABK inequality [18], as well as for the initial generic class of tripartite state as measured by its violation of the Svetlichny and WWZB inequalities [19].

2. Theoretical framework

In this paper, we would like to investigate the phenomena of disentanglement and BNSD for tripartite states, and their relation with fidelity of quantum teleportation. Our system consists of three qubits (here labeled as $K = A, B, C$), each embedded in a thermal reservoir. To focus exclusively on the decay of entanglement and Bell nonlocality as they arise from the influence of thermal noise, we assume that the qubits are separated by spatial distances large enough and thus there are no direct interactions between them, that is, every qubit interacts only and independently with its own environment. Then under the condition of Markovian approximation, the reduced dynamics of the system state ρ can be described by a general master equation of Lindblad form [12]

$$\frac{d\rho}{dt} = \frac{1}{2} \sum_{K,m} \gamma_K (2\mathcal{L}_{K,m}\rho\mathcal{L}_{K,m}^\dagger - \mathcal{L}_{K,m}^\dagger\mathcal{L}_{K,m}\rho - \rho\mathcal{L}_{K,m}^\dagger\mathcal{L}_{K,m}), \tag{1}$$

where $\gamma_K (K = A, B, C)$ are the damping rates of the qubits due to their coupling to the reservoir. The generators of decoherence are given by $\mathcal{L}_{K,1} = \sqrt{\bar{n} + 1}\sigma_K^-$ and $\mathcal{L}_{K,2} = \sqrt{\bar{n}}\sigma_K^+$, with $\sigma_K^\pm$ being the raising and lowering operators. $\mathcal{L}_{K,1}$ and $\mathcal{L}_{K,2}$ describe, respectively, decay and excitation processes, with rates which depend on the temperature, here parameterized by the average thermal photons $\bar{n}$ in the reservoir. For the limiting case of vanishing temperature (i.e., $\bar{n} = 0$), only the spontaneous decay term survives, leading to a purely dissipative process which drives all initial states to a unique asymptotic pure state, in which the three qubits are in their ground states. For the opposite case of infinite temperature (i.e., $\bar{n} \to \infty$), decay and excitation occur at exactly the same rate, and the noise induced by the transitions between the two levels brings the system into a stationary, maximally mixed state. Physically, the above models can describe, e.g., three two-level atoms with interatomic separations larger than the spatial correlation length of the reservoir such that the collective damping and the collective shift of the atomic levels are negligible [12].

Solutions of the above master equation with arbitrary initial conditions can be derived exactly in several different ways [26,27], and here we use the operator-sum representation [26]. In many situations of physical interest, this representation allows a transparent analysis of system dynamics without invoking the explicit forms of the initial conditions. In the standard basis expanded by the eigenstates of the product Pauli spin operator $\sigma_A^z \otimes \sigma_B^z \otimes \sigma_C^z$, the reduced density matrix for the three qubits together is given by the following completely positive and trace preserving (CPTP) map [26]

$$\rho(t) = \mathcal{L}[\rho(0)] = \sum_{i,j,k=1}^{4} G_{ijk}\rho(0)G_{ijk}^\dagger, \tag{2}$$

where the time-dependent tensor-product superoperator $\mathcal{L} = \mathcal{L}_A \otimes \mathcal{L}_B \otimes \mathcal{L}_C$ contains 64 terms. The Kraus operator G_{ijk} describes the interaction of the qubits with the thermal reservoir and satisfies the CPTP relation $\sum_{ijk} G_{ijk}^\dagger G_{ijk} = 1$ for all t. Since the thermal noise operates locally on individual subsystems, G_{ijk} can be expressed in terms of the tensor products of E_i^A, E_j^B and E_k^C as $G_{ijk} = E_i^A \otimes E_j^B \otimes E_k^C$. Here E_i^A, E_j^B and E_k^C are the Kraus operators describing time evolution of each qubit alone, and individually satisfy the usual completeness condition for the operator-sum decomposition of CPTP maps [26]. Their explicit forms are as follows

$$E_1^K = \sqrt{\frac{\bar{n} + 1}{2\bar{n} + 1}} \begin{pmatrix} p_K & 0 \\ 0 & 1 \end{pmatrix},$$

$$E_2^K = \sqrt{\frac{\bar{n}}{2\bar{n} + 1}} \begin{pmatrix} 1 & 0 \\ 0 & p_K \end{pmatrix},$$

$$E_3^K = \sqrt{\frac{\bar{n}}{2\bar{n} + 1}} \begin{pmatrix} 0 & \sqrt{1 - p_K^2} \\ 0 & 0 \end{pmatrix},$$

$$E_4^K = \sqrt{\frac{\bar{n} + 1}{2\bar{n} + 1}} \begin{pmatrix} 0 & 0 \\ \sqrt{1 - p_K^2} & 0 \end{pmatrix}, \tag{3}$$

where the time-dependent factors $p_K (K = A, B, C)$ appearing in the above equations are given by $p_K = e^{-(2\bar{n}+1)\gamma_K t/2}$. Note that when $\bar{n} = 0$, only E_1^K and E_4^K survive, so solution (2) has simple analytical form. We will apply solution (2) to analyze the effects of the thermal reservoir on the decay dynamics of entanglement and Bell-inequality violation for the initial GHZ and W class states. For simplicity, we will take the noise properties to be the same for the three qubits such that $\gamma_A = \gamma_B = \gamma_C = \gamma$ and $p_A = p_B = p_C = p = e^{-(2\bar{n}+1)\gamma t/2}$.

After obtaining explicit forms of the reduced density matrix $\rho(t)$, we can discuss dynamics of disentanglement, Bell-nonlocality violation and the ability of $\rho(t)$ for quantum teleportation. To describe the disentanglement process, we need a concrete measure of entanglement contained in a quantum state. The tripartite negativity N, which was introduced by Sabín and García-Alcaine [28] is particularly convenient for the case of current interest. It can be calculated explicitly from the density

matrix $\rho(t)$ as

$$N = (N_{A-BC}N_{B-CA}N_{C-AB})^{1/3}, \tag{4}$$

where $N_{A-BC} = -\sum_i \mu_i^A$, $N_{B-CA} = -\sum_i \mu_i^B$ and $N_{C-AB} = -\sum_i \mu_i^C$ are the negativities introduced by Vidal and Werner [29], and the sums are taken over all the negative eigenvalues μ_i^K of the partial transpose $\rho^{T_K}(t)$ of $\rho(t)$ with respect to the subsystem K. Note that for the mixed states, N is not able to quantify multipartite entanglement fully, but its positivity ensures that the state under consideration is not separable [28]. Thus if the Bell nonlocality of a quantum state dies out before that of the tripartite negativity, one can say that it dies out before that of entanglement.

Moreover, we will consider Bell-type nonlocality in different contexts and we use two classes of multipartite Bell-type inequalities to detect the existence of nonlocal correlations as measured by the extent of their violations. The first one we are interested in is the Svetlichny inequality $|\langle \mathcal{S} \rangle_{\rho(t)}| \leqslant 4$ [30], which distinguishes genuinely tripartite Bell nonlocality associated with $\rho(t)$. Here $\langle \mathcal{S} \rangle_{\rho(t)} = \mathrm{tr}[\mathcal{S}\rho(t)]$ is the expectation value of the Svetlichny operator given by

$$
\begin{aligned}
\mathcal{S} = {} & M_A M_B M_C + M_A M_B M_C' + M_A M_B' M_C + M_A' M_B M_C \\
& - M_A' M_B' M_C' - M_A' M_B' M_C - M_A' M_B M_C' - M_A M_B' M_C',
\end{aligned} \tag{5}
$$

where the measurement operators M_K and M_K' correspond to the measurements on each of the subsystems K, while the primed and unprimed terms denote the two different directions in which the corresponding party measures (the same applies also to the WWZB operators). A quantum state $\rho(t)$ violates the Svetlichny inequality whenever $|\langle \mathcal{S} \rangle_{\rho(t)}| > 4$, and in quantum mechanics the Svetlichny inequality is violated up to $|\langle \mathcal{S} \rangle_{\rho(t)}| = 4\sqrt{2}$, which is achieved only when the system is prepared in the maximally entangled GHZ state [9].

The second inequality convenient for our purpose is the WWZB Bell-type inequalities [31,32]. For the three-qubit system, there are 256-element sets of such inequalities, which belong to five distinct classes due to some basic symmetries, and the behavior of a single class is identical to that of all members of that class, i.e., the violation of even a single element of each class is sufficient for Bell nonlocality of that class [31]. Thus we only need to consider one inequality from each of the five distinct classes:

$$
\begin{aligned}
\mathcal{B}_{P1} = {} & 2M_A M_B M_C, \\
\mathcal{B}_{P2} = {} & \frac{1}{2}(-M_A M_B M_C + M_A M_B M_C' + M_A M_B' M_C + M_A' M_B M_C \\
& + M_A M_B' M_C' + M_A' M_B M_C' + M_A' M_B' M_C + M_A' M_B' M_C'), \\
\mathcal{B}_{P3} = {} & [M_A(M_B + M_B') + M_A'(M_B - M_B')]M_C, \\
\mathcal{B}_{P4} = {} & M_A M_B(M_C + M_C') - M_A' M_B'(M_C - M_C'), \\
\mathcal{B}_{P5} = {} & M_A M_B M_C' + M_A M_B' M_C + M_A' M_B M_C - M_A' M_B' M_C'.
\end{aligned} \tag{6}
$$

For nonlocal quantum states, there should be at least one of the $|\langle \mathcal{B}_{PI} \rangle_{\rho(t)}| > 2$ ($I = 1, 2, 3, 4, 5$), where $\langle \mathcal{B}_{PI} \rangle_{\rho(t)} = \mathrm{tr}[\mathcal{B}_{PI}\rho(t)]$. For the behavior of a system to be describable by a fully local hidden variable model, all of the WWZB set of inequalities must be satisfied jointly.

To characterize the quality of the teleported output state ρ_{out} under the influence of thermal reservoir, we calculate the average fidelity (the fidelity $F(\theta, \phi) = \langle \varphi_{\mathrm{in}}|\rho_{\mathrm{out}}|\varphi_{\mathrm{in}}\rangle$ averaged over all pure input states on the Bloch sphere) [1], defined as

$$F_{\mathrm{av}} = \frac{1}{4\pi}\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta F(\theta, \phi), \tag{7}$$

where $|\varphi_{\mathrm{in}}\rangle = \cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle$ is the input state needs to be teleported, with $0 \leqslant \theta \leqslant \pi$ and $0 \leqslant \phi \leqslant 2\pi$ being the polar and azimuthal angles, respectively. When the aforementioned initial GHZ class state $|\psi_{\mathrm{GHZ}}\rangle = (|000\rangle + |111\rangle)/\sqrt{2}$ or the initial W class state $|\psi_W\rangle = (\sqrt{2}|001\rangle + |010\rangle + |100\rangle)/2$ is used as a quantum channel, by following the methodology of

Ref. [14], the average teleportation fidelity at an arbitrary time t can be derived readily as $F_{\mathrm{av}}(\rho_{\mathrm{GHZ}}) = (1 + \rho_{\mathrm{GHZ}}^{11+44+55+88} + 2\rho_{\mathrm{GHZ}}^{18})/3$ and $F_{\mathrm{av}}(\rho_W) = (1 + \rho_W^{22+33+44+88+35-46} + 2\sqrt{2}\rho_W^{23})/3$, where the abbreviations $\rho_{\Pi}^{ij \pm kl \pm mn \pm \dots} = \rho_{\Pi}^{ij}(t) \pm \rho_{\Pi}^{kl}(t) \pm \rho_{\Pi}^{mn}(t) \pm \dots$, with $\rho_{\Pi}^{ij}(t)$ (Π = GHZ or W) being the elements of the density matrix $\rho_{\Pi}(t)$. Since the noise imposed by the thermal reservoir gives rise both to decoherence and to entanglement losses, unit fidelity cannot be achieved for this case.

3. Entanglement, Bell-inequality violation and teleportation dynamics

Now we begin our discussion about disentanglement dynamics and teleportation capacity for the system prepared initially in the GHZ class state $|\psi_{\mathrm{GHZ}}\rangle$. The density matrix at an arbitrary time t can be obtained directly from Eqs. (2) and (3), and the nonvanishing elements are $\rho_{\mathrm{GHZ}}^{11-88}(t)$ (i.e., the diagonal elements) and $\rho_{\mathrm{GHZ}}^{18,81}(t)$. Combining this with Eqs. (4) and (7), one can derive the tripartite negativity $N(\rho_{\mathrm{GHZ}})$ and the average fidelity $F_{\mathrm{av}}(\rho_{\mathrm{GHZ}})$ analytically as

$$N(\rho_{\mathrm{GHZ}}) = \frac{1}{2} \max\{0, \ \sqrt{\alpha^2 + p^6} - \beta\},$$

$$F_{\mathrm{av}}(\rho_{\mathrm{GHZ}}) = \frac{1}{2} + \frac{p^3}{3} + \frac{p^4}{6} + \frac{(1-p^2)^2}{6(2\bar{n}+1)^2}, \tag{8}$$

where the corresponding parameters α and β appeared in the above equations are given by

$$\alpha = \frac{(1-p^2)[2\bar{n}(\bar{n}+1)(3p^4-1)+2p^4-p^2]}{2(2\bar{n}+1)^3},$$

$$\beta = \frac{(1-p^2)[2\bar{n}(\bar{n}+1)(p^2+1)+p^2]}{2(2\bar{n}+1)^2}, \tag{9}$$

with the factor p being defined below Eq. (3).

Plots of the tripartite negativity $N(\rho_{\mathrm{GHZ}})$ and average fidelity $F_{\mathrm{av}}(\rho_{\mathrm{GHZ}})$ versus the rescaled time γt are displayed in Fig. 1 with different $\bar{n}$. One can observe that the evolution of $N(\rho_{\mathrm{GHZ}})$ shows an exponentially decaying behavior, and it is very sensitive to the variations of the reservoir temperature. From Eq. (8) one can see that the entanglement measured by the tripartite negativity disappears if $\alpha^2 + p^6 \leqslant \beta^2$. When $\bar{n} = 0$, this simplifies to $1 - p^2 \geqslant 1$. Since for all finite values of γ, the factor p approaches zero exponentially only in the infinite time limit, the sudden death of $N(\rho_{\mathrm{GHZ}})$ does not happen for this special case. This behavior is clearly illustrated by the black solid curve shown in Fig. 1. When $\bar{n} \neq 0$, however, $N(\rho_{\mathrm{GHZ}})$ behaves very differently. As can be seen from Fig. 1, it ceases to exist in a finite timescale τ_E which is shortened gradually by increasing the values of $\bar{n}$. This implies that the devastating effects of the thermal reservoir on entanglement of the system becomes severe and severe with increasing temperature. Furthermore, it should be noted that for the initial GHZ state $|\psi_{\mathrm{GHZ}}\rangle$, $N(\rho_{\mathrm{GHZ}}) = N_{A-BC} = N_{B-CA} = N_{C-AB}$; thus the sudden death of $N(\rho_{\mathrm{GHZ}})$ indicates the sudden death of N_{A-BC}, N_{B-CA} and N_{C-AB}.

When considering robustness of the initial GHZ state as a quantum channel for teleportation, as can be seen from the dashed curves shown in Fig. 1, $F_{\mathrm{av}}(\rho_{\mathrm{GHZ}})$ decays to the classical limiting value of 2/3 after a critical time τ_T, which decreases with increasing $\bar{n}$, and when $\bar{n} = 0$ one can obtain $\gamma \tau_T = \ln[(3 + \sqrt{5})/2]$. Moreover, one can note that the critical time τ_T is earlier than the death time τ_E of tripartite negativity for any fixed $\bar{n}$. Since a non-zero tripartite negativity signals the entanglement of the system, this phenomenon indicates that not all the three-qubit entangled states generated from the initial GHZ class state are useful for nonclassical teleportation. In fact, a minimum tripartite negativity (at the critical time τ_T) is always necessary for the achievement of nonclassical fidelity for the situations considered here. As displayed evidently in Fig. 2, the critical tripartite negativity $N_c(\rho_{\mathrm{GHZ}})$ after which the teleportation protocol fails to achieve a nonclassical fidelity is increased by increasing the reservoir temperature. This phenomenon is mainly caused by the competition between the increased temperature $\bar{n}$ and the decreased critical time τ_T. Because the increase of $\bar{n}$ always decrease τ_T and $N(\rho_{\mathrm{GHZ}})$, but meanwhile the decrease of τ_T always increase

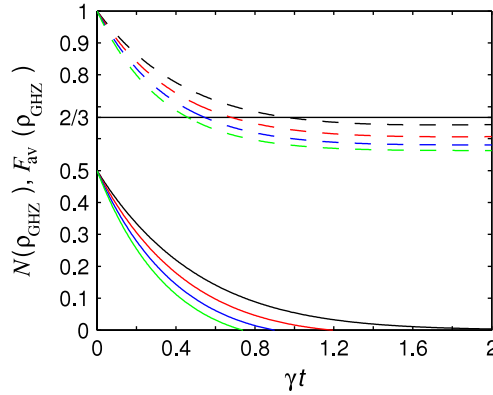**Fig. 1.** (Color online) Tripartite negativity $N(\rho_{GHZ})$ (solid curves) and average fidelity $F_{av}(\rho_{GHZ})$ (dashed curves) versus γt. For every line style, the curves from top to bottom correspond to the cases of $\bar{n} = 0$, $\bar{n} = 0.1$, $\bar{n} = 0.2$ and $\bar{n} = 0.3$.
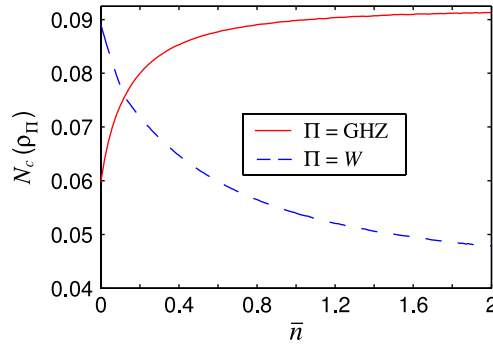

Fig. 2. (Color online) Critical tripartite negativities $N_c(\rho_{GHZ})$ (solid curves) and $N_c(\rho_W)$ (dashed curves) versus $\bar{n}$.

$N(\rho_{GHZ})$. Due to the complexity of Eqs. (8) and (9), it is difficult to obtain an analytic form of $N_c(\rho_{GHZ})$; however, we can make a heuristic analysis for two limiting cases, i.e., $\bar{n} = 0$ and $\bar{n} \to \infty$. For $\bar{n} = 0$, since $\gamma \tau_T = \ln[(3 + \sqrt{5})/2]$, we obtain $N_c(\rho_{GHZ}) = (2 - \sqrt{5} + \sqrt{197 - 88\sqrt{5}})/4 \simeq 0.0598$, while for $\bar{n} \to \infty$, $\gamma \tau_T$ can only be solved numerically from the nonlinear equation $p^4 + 2p^3 = 1$, which gives rise to $p \simeq 0.7166$ and $N_c(\rho_{GHZ}) \simeq 0.0920$. This analysis corroborates our finding presented in Fig. 2.

As mentioned before, Bell-inequality violations may act as an indicator of the usefulness of entanglement. Let us discuss now this issue by quantum states evolving in time. Our aim is to individuate the Svetlichny inequality violation regions characterized by $|\langle \mathcal{S} \rangle_{\rho_{GHZ}(t)}| > 4$ and the WWZB inequality violation regions characterized by $|\langle \mathcal{B}_{PI} \rangle_{\rho_{GHZ}(t)}| > 2(I = 1, 2, 3, 4, 5)$, and to compare their possible relations with disentanglement and average teleportation fidelity. For this purpose, we calculate the expectation values of the Svetlichny operator $\mathcal{S}$ and the WWZB operator $\mathcal{B}_{PI}$. For the initial GHZ state, the measurement operators for the first qubit (i.e., qubit A) are defined as $M_A \equiv \sigma_y$ and $M_A' \equiv \sigma_x$ [19], while the measurement operators for the second and the third qubits are defined with respect to the first one by a rotation

$$\begin{pmatrix} M_K \\ M_K' \end{pmatrix} = \begin{pmatrix} \cos \theta_K & -\sin \theta_K \\ \sin \theta_K & \cos \theta_K \end{pmatrix} \begin{pmatrix} M_A \\ M_A' \end{pmatrix}, \tag{10}$$

where $\theta_K (K = B, C)$ are the rotation angles. Combining this with Eq. (5), one can obtain

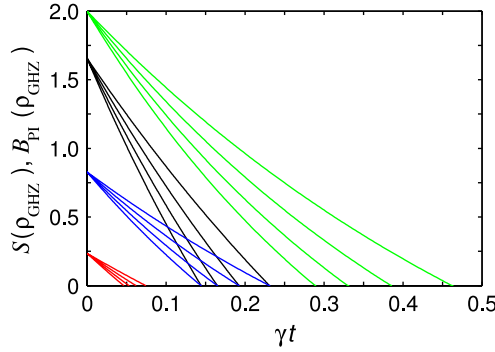$$\langle \mathcal{S} \rangle_{\rho_{GHZ}(t)} = 4p^3(\sin \theta_{BC} - \cos \theta_{BC}), \tag{11}$$

Fig. 3. (Color online) Bell-nonlocality violation of $\rho_{GHZ}(t)$. Here the black, red, blue and green curves show respectively dynamics of $S(\rho_{GHZ})$, $B_{P2}(\rho_{GHZ})$, $B_{P3}(\rho_{GHZ})$ and $B_{P5}(\rho_{GHZ})$. For every line color, the curves from right to left correspond to the cases of $\bar{n} = 0$, $\bar{n} = 0.1$, $\bar{n} = 0.2$ and $\bar{n} = 0.3$.

where $\theta_{BC} = \theta_B + \theta_C$, and the same notation will be used throughout this paper. Recall that if $|\langle \mathcal{S} \rangle_{\rho_{GHZ}(t)}| > 4$, the state $\rho_{GHZ}(t)$ is genuinely tripartite Bell nonlocal [19]. Since $p = e^{-(2\bar{n}+1)\gamma t/2}$ and the modulus of the trigonometric term takes its maximum $\sqrt{2}$ whenever $\theta_{BC} = -\pi/4$ or $3\pi/4$, the maximum expectation value of the Svetlichny operator $\mathcal{S}$ evolves according to $|\langle \mathcal{S} \rangle_{\rho_{GHZ}(t)}| = 4\sqrt{2}e^{-3(2\bar{n}+1)\gamma t/2}$, and approaches the classical threshold value 4 in a finite timescale $\tau_S = \ln 2/[3(2\bar{n}+1)\gamma]$. As can be seen from the dynamics of $S(\rho_{GHZ}) = \max\{|\langle \mathcal{S} \rangle_{\rho_{GHZ}(t)}| - 4, 0\}$ presented in Fig. 3, the temperature of the thermal reservoir can influence $S(\rho_{GHZ})$ to a great extent, and the time regions for genuinely tripartite nonlocality shrink with increasing temperature. Particularly, in the small γt region we find $|\langle \mathcal{S} \rangle_{\rho_{GHZ}(t)}|$ decays up to quadratic terms in time as $|\langle \mathcal{S} \rangle_{\rho_{GHZ}(t)}| = 2\sqrt{2}(a^2 - 2a + 2) + \mathcal{O}(t^3)$, where $a = 3(2\bar{n}+1)\gamma t/2$. The relative difference between the actual and the approximate results, is in practice negligible when both are above the classical threshold $|\langle \mathcal{S} \rangle_{\rho_{GHZ}(\tau_S)}| = 4$.

Next we extend the analysis of Bell-type nonlocal correlations in tripartite states addressed by the violation of the WWZB inequalities. The expectation values for the five distinct classes of the $\mathcal{B}_{PI}$ operators for the state $\rho_{GHZ}(t)$ can also be calculated analytically, which are given by

$$\langle \mathcal{B}_{P1} \rangle_{\rho_{GHZ}(t)} = \langle \mathcal{B}_{P4} \rangle_{\rho_{GHZ}(t)} = 2p^3 \sin \theta_{BC},$$

$$\langle \mathcal{B}_{P2} \rangle_{\rho_{GHZ}(t)} = -p^3 (2 \sin \theta_{BC} + \cos \theta_{BC}),$$

$$\langle \mathcal{B}_{P3} \rangle_{\rho_{GHZ}(t)} = 2p^3 (\sin \theta_{BC} - \cos \theta_{BC}),$$

$$\langle \mathcal{B}_{P5} \rangle_{\rho_{GHZ}(t)} = -4p^3 \cos \theta_{BC}. \tag{12}$$

Recalling that $p = e^{-(2\bar{n}+1)\gamma t/2}$, one sees immediately that the state $\rho_{GHZ}(t)$ does not violate the WWZB-type inequalities with respect to the operators $\mathcal{B}_{P1}$ and $\mathcal{B}_{P4}$ in the full time region. For the remaining three classes, plots of $B_{PI}(\rho_{GHZ}) = \max\{|\langle \mathcal{B}_{PI} \rangle_{\rho_{GHZ}(t)}| - 2, 0\}$ ($I = 2, 3, 5$) versus the rescaled time γt are displayed in Fig. 3 as red, blue and green curves with different $\bar{n}$. Clearly, they show qualitatively similar dynamical behaviors, the most pronounced difference is the time regions during which the corresponding inequalities are violated. For the inequality of form P2, since the trigonometric term is strictly bounded by $\sqrt{5}$, the maximum expectation value of $\mathcal{B}_{P2}$ evolves as $|\langle \mathcal{B}_{P2} \rangle_{\rho_{GHZ}(t)}| = \sqrt{5}e^{-3(2\bar{n}+1)\gamma t/2}$, and approaches the critical value 2 in a short timescale $\tau_{B_{P2}} = \ln(5/4)/[3(2\bar{n}+1)\gamma]$, which decreases gradually with increasing temperature (cf. the red curves shown in Fig. 3). Second, for the inequality of form P3, the trigonometric term is strictly limited by $\sqrt{2}$, and the maximum expectation value of $\mathcal{B}_{P3}$ is $|\langle \mathcal{B}_{P3} \rangle_{\rho_{GHZ}(t)}| = 2\sqrt{2}e^{-3(2\bar{n}+1)\gamma t/2}$, which becomes smaller than 2 in precisely the same timescale as that of the Svetlichny inequality, i.e., $\tau_{B_{P3}} = \tau_S$. Finally, for the inequality of form P5, the maximum $|\langle \mathcal{B}_{P5} \rangle_{\rho_{GHZ}(t)}| = 4e^{-3(2\bar{n}+1)\gamma t/2}$ decays to its critical value 2 in a finite time $\tau_{B_{P5}} = \ln 4/[3(2\bar{n}+1)\gamma]$. Note that after time $\tau_{B_{P5}}$ all the five sets of the
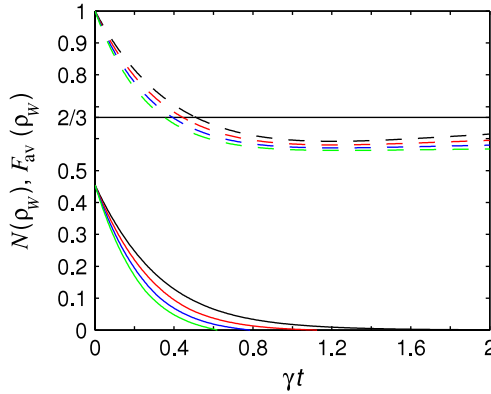
Fig. 4. (Color online) Tripartite negativity $N(\rho_W)$ (solid curves) and average fidelity $F_{\mathrm{av}}(\rho_W)$ (dashed curves) versus γt. For every line style, the curves from top to bottom correspond to the cases of $\bar{n} = 0$, $\bar{n} = 0.1$, $\bar{n} = 0.2$ and $\bar{n} = 0.3$.

WWZB inequalities are satisfied, and the state $\rho_{\mathrm{GHZ}}(t)$ becomes Bell local, i.e., its correlations can be reproduced by a classical local model [1]. This implies that under the influence of thermal reservoir, the phenomenon of BNSD occurs for definitive times for the initial GHZ class state preparation.

Moreover, by comparing Fig. 3 with Fig. 1, one can also note that the lifetime for the tripartite negativity is longer than the time region during which the teleportation protocol outperforms those of classical ones or the lifetime of Bell nonlocality addressed by the violation of the Svetlichny as well as the full set of WWZB inequalities. Since a non-zero tripartite negativity signals the fact that the system is entangled, this phenomenon indicates that for the initial GHZ state preparation, all the Bell-nonlocal (not necessary the genuinely tripartite Bell nonlocal) states are entangled and can be used for nonclassical teleportation, while there are still some tripartite states (i.e., the states locating between the time region τ_{T} and τ_{E}) which are entangled but fail to achieve a nonclassical teleportation fidelity. This conclusion is very similar to that of the two-qubit case [3–5], for which the subset of the entangled X-type states that violate the Bell–CHSH inequality can always be used for nonclassical teleportation.

We now turn our attention to an analysis of disentanglement, Bell nonlocality and teleportation capacity for the initial W state preparation. The density matrix $\rho_W(t)$ can also be obtained directly from Eqs. (2) and (3), from which the complete analytical forms of the tripartite negativity and the average fidelity can be derived exactly; however, we do not list them here explicitly because their expressions are quite involved. Instead, we plot numerically in Fig. 4 the time behaviors of both $N(\rho_W)$ and $F_{\mathrm{av}}(\rho_W)$ for the same values of the system parameters.

The solid curves in Fig. 4 show the numerical results for $N(\rho_W)$ versus γt with different $\bar{n}$, from which one can note a pronounced difference between the dynamical behaviors for the zero and nonzero temperature cases. Similar to that for the initial GHZ state preparation, the tripartite negativity $N(\rho_W)$ decays exponentially and disappears only in the infinite time limit for the zero temperature reservoir; thus no ESD happens for this special case. For the situations of nonzero temperature reservoir, however, the tripartite negativity terminates abruptly in a finite time τ_{E}, which decreases with increasing temperature and is shorter than that for the initial GHZ state preparation (cf. Figs. 1 and 4). Thus one can say that the tripartite negativity of the initial W state is fragile compared with that of the initial GHZ state in the sense that it decays in a faster rate. But it should be noted that this does not mean the fragility of the entanglement for the initial W state, for the disappearance of the tripartite negativity cannot guarantee that the state is always separable. Moreover, we would like to mention here that for the special case of infinite temperature reservoir (i.e., $\bar{n} \rightarrow \infty$), Carvalho et al. [33] have shown that the entanglement measured by the multipartite concurrence for the initial standard W state preparation also dies faster than that for the initial GHZ state preparation.

From the dashed curves shown in Fig. 4 one can see that in the whole temperature region, the teleportation protocol loses its quantum advantage over purely classical communication in a finite
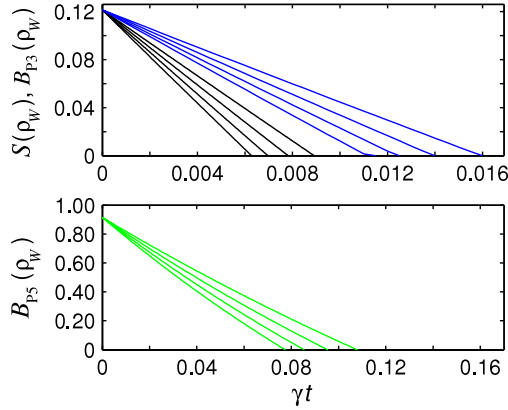
Fig. 5. (Color online) Bell-nonlocality violation of $\rho_W(t)$. Here the black, blue and green curves show respectively dynamics of $S(\rho_W)$, $B_{P3}(\rho_W)$ and $B_{P5}(\rho_W)$. For every line color, the curves from right to left correspond to the cases of $\bar{n} = 0$, $\bar{n} = 0.1$, $\bar{n} = 0.2$ and $\bar{n} = 0.3$.

timescale τ_T, which is significantly shorter than that for the initial GHZ state preparation, and $F_{av}(\rho_W)$ is always smaller than $F_{av}(\rho_{GHZ})$ when $t > 0$. Thus although they both ensure unit fidelity under ideal circumstance [10,11], the quantum channel with the initial GHZ class state preparation outperforms that of the initial W class state preparation in terms of their teleportation capacity. Moreover, from Fig. 4 one can also observe that a non-vanishing tripartite negativity $N_c(\rho_W)$ is always necessary for $F_{av}(\rho_W) > 2/3$; however, as can be seen from the blue dashed curve shown in Fig. 2, $N_c(\rho_W)$ decreases exponentially with increasing $\bar{n}$ and thus behaves in a markedly different way compared to $N_c(\rho_{GHZ})$, which can also be understood from the competitive mechanism between the increased temperature $\bar{n}$ and the decreased critical time τ_T. Although the complexity of $\rho_W(t)$ prevents us to derive an analytic form of $N_c(\rho_W)$ versus $\bar{n}$ for general case, we have $F_{av}(\rho_W) = 5p^4/6 - p^2/2 + 2/3$ for $\bar{n} = 0$, which yields $\gamma\tau_T = \ln(5/3)$ and $N_c(\rho_W) \simeq 0.0891$, while for very large $\bar{n}$ we have $\bar{n}\gamma\tau_T \simeq 0.3257$ and $N_c(\rho_W) \simeq 0.0404$.

Now we discuss Bell-nonlocal behaviors of $\rho_W(t)$. The measurement operators associated with the first qubit of the initial W state are $M_A \equiv \sigma_z$ and $M_A' \equiv \sigma_x$ [18], while for the second and the third qubits they are still defined as those in Eq. (10). Then the expectation value of the Svetlichny operator $\mathscr{S}$ can be readily obtained as

$$\langle\mathscr{S}\rangle_{\rho_W(t)} = (2\rho_W^{11+44+66+77+46+47+67-23-25-35} - 1)(\sin\theta_{BC} + \cos\theta_{BC}). \tag{13}$$

For the zero temperature reservoir (i.e., $\bar{n} = 0$), the maximum value of $\langle\mathscr{S}\rangle_{\rho_W(t)}$ simplifies to $|\langle\mathscr{S}\rangle_{\rho_W(t)}| = (4+5\sqrt{2})p^4 - (2+9\sqrt{2}/2)p^2 + \sqrt{2}$, which decays with time and becomes smaller than 4 in a finite timescale $\gamma\tau_S = \ln[(16+20\sqrt{2})/(4+9\sqrt{2}+\sqrt{274+328\sqrt{2}})] \simeq 0.00891$. For the nonzero temperature reservoirs (i.e., $\bar{n} \neq 0$), since the density matrix elements of $\rho_W(t)$ are so complicated, we define $S(\rho_W) = \max\{|\langle\mathscr{S}\rangle_{\rho_W(t)}| - 4, 0\}$ and present its dynamical behaviors numerically in Fig. 5 as black curves. Clearly, $S(\rho_W)$ decays monotonically with increasing γt and becomes zero in a finite time τ_S, after which the genuinely tripartite Bell-nonlocal correlation disappears. τ_S decreases with increasing temperature, and as can be found obviously from Fig. 5, its magnitude is very small ($\gamma\tau_S < 0.00891$), i.e., the genuinely tripartite Bell nonlocality for the initial W state preparation is fragile and the thermal reservoir destroys it in a very short timescale.

The expectation values for the five WWZB operators $\mathscr{B}_{PI}(I = 1, 2, 3, 4, 5)$ can also be obtained in terms of the density matrix elements of $\rho_W(t)$ as

$$\langle\mathscr{B}_{P1}\rangle_{\rho_W(t)} = (4\rho_W^{11+44+66+77} - 2)\cos\theta_B\cos\theta_C + 4\rho_W^{23-47}\sin\theta_B\sin\theta_C,$$
$$\langle\mathscr{B}_{P2}\rangle_{\rho_W(t)} = \rho_W^{25+35-46-47}\cos\theta_{BC} + \rho_W^{25-35+46-47}\sin(\theta_B - \theta_C)$$
$$+ (\rho_W^{23-67-11-44-66-77} + 1/2)(\cos\theta_{BC} - \sin\theta_{BC}),$$

$$\langle \mathcal{B}_{\text{P3}} \rangle_{\rho_W(t)} = (2\rho_W^{11+44+66+77-35+46} - 1)x_+ + 4\rho_W^{23-47}x_-,$$

$$\langle \mathcal{B}_{\text{P4}} \rangle_{\rho_W(t)} = (2\rho_W^{11+44+66+77} - 1)y_+ + 2\rho_W^{35-46}y_- + 4\rho_W^{23-47}\sin\theta_B\sin\theta_C,$$

$$\langle \mathcal{B}_{\text{P5}} \rangle_{\rho_W(t)} = (2\rho_W^{11+44+66+77+46+47+67-23-25-35} - 1)\sin\theta_{BC} \tag{14}$$

where $x_+ = \sqrt{2}\sin(\theta_B + \pi/4)\cos\theta_C$, $x_- = \sqrt{2}\sin(\theta_B - \pi/4)\sin\theta_C$, $y_\pm = \sqrt{2}\sin(\theta_C \pm \pi/4)\cos\theta_B$, and in deriving the above equations, we have used the hermiticity condition of the density matrix.

We define $B_{\text{PI}}(\rho_W) = \max\{|\langle \mathcal{B}_{\text{PI}} \rangle_{\rho_W(t)}| - 2, 0\}$ (I = 1, 2, 3, 4, 5) to evaluate the extent to which the WWZB-type inequalities are violated. Still due to the fact that the elements of $\rho_W(t)$ are so complicated, it is difficult to express $B_{\text{PI}}(\rho_W)$ compactly with respect to the parameters γt and $\bar{n}$. Thus we resort to numerical calculations. The results show that the inequalities of forms P1, P2 and P4 are satisfied in the whole time region (i.e., $B_{\text{PI}}(\rho_W) \equiv 0$ for I = 1, 2, 4). For the remaining two classes of forms P3 and P5, examples of decays of $B_{\text{P3}}(\rho_W)$ and $B_{\text{P5}}(\rho_W)$ with different $\bar{n}$ are presented graphically in Fig. 5 as blue and green curves, respectively. It is obvious that they decay monotonously with increasing γt and become zero after finite times $\tau_{B_{\text{P3}}}$ and $\tau_{B_{\text{P5}}}$ (when $\bar{n} = 0$ we have $\gamma\tau_{B_{\text{P5}}} = \ln[(20 + 8\sqrt{2})/(9 + 2\sqrt{2} + \sqrt{169 + 68\sqrt{2}})] \simeq 0.10785$), both of which decrease with increasing $\bar{n}$. The time $\tau_{B_{\text{P5}}}$ demonstrates the time of sudden death of all species of Bell-nonlocal correlations for $\rho_W(t)$ under the influence of thermal noise. Particularly, by comparing the present results with those of $\rho_{\text{GHZ}}(t)$, one can note that $\tau_{B_{\text{P5}}}$ here is very small, i.e., the Bell nonlocality for $\rho_W(t)$ is very fragile compared with that of $\rho_{\text{GHZ}}(t)$.

Furthermore, one can note that the death time for Bell nonlocality is much earlier than that for the tripartite negativity or the critical time after which the teleportation fidelity $F_{\text{av}}(\rho_W) < 2/3$. This reveals several common features. First, as mentioned before, the positivity of $N(\rho_W)$ ensures the entanglement of the state $\rho_W(t)$; thus there exist time regions during which $\rho_W(t)$ possesses local correlations even in regard to high values of entanglement. Second, only partial of the tripartite entangled states are useful for nonclassical teleportation. But all the Bell-nonlocal states yield $F_{\text{av}}(\rho_W) > 2/3$.

Finally, we would like to emphasize that even when one or two of the participating qubits can be well preserved, e.g., $\gamma_A = 0$ or $\gamma_A = \gamma_B = 0$, the tripartite negativity and the Bell nonlocality still experience sudden death, although in a comparatively longer time. This shows that the effects of thermal reservoir on entanglement and coherence of a qubit is indeed very different. Since they change in a similar manner as that of $\gamma_A = \gamma_B = \gamma_C = \gamma$ and no other new results can be drawn, we will not have any more discussions about them here.

4. Summary

In summary, we have investigated behaviors of disentanglement and Bell-nonlocality violation for various decaying states and analyzed their relations with the capacity of these states when being used as quantum channels for teleportation. Our system consists of three qubits prepared initially in the GHZ or W class state and coupled to a thermal reservoir, under the influence of which irreversible coherence decay will be unavoidable. Depending on the temperature of the reservoir, the tripartite negativity can reach a value equal to zero asymptotically (if $\bar{n} = 0$) or at a finite time (if $\bar{n} \neq 0$). But the sudden death of tripartite correlations associated with the Svetlichny inequality and nonlocal correlations associated with the WWZB inequalities are irreversible in the whole temperature regions. Moreover, the tripartite negativity and Bell nonlocality for $\rho_{\text{GHZ}}(t)$ are more robust than that for $\rho_W(t)$ in the sense that they survive in a significantly longer times under the influence of thermal reservoir. Particularly, $\rho_{\text{GHZ}}(t)$ gives a wider time region for nonclassical teleportation. Finally, by comparing the survival time for Bell nonlocality with the time region during which $F_{\text{av}}(\rho_\Pi) > 2/3$ ($\Pi = \text{GHZ}$ or W), we showed that all the Bell-nonlocal states considered in this work can be used for quantum teleportation, while there also exist a family of entangled mixed states which do not violate any multipartite Bell-type inequalities, but still yield nonclassical teleportation fidelity.

Acknowledgments

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