International Journal of Control, Automation, and Systems (2010) 8(1):29-35 http://www.springer.com/12555
DOI 10.1007/s12555-010-0104-0

Unified Parametrization for the Solutions to the Polynomial Diophantine Matrix Equation and the Generalized Sylvester Matrix Equation

Bin Zhou, Zhi-Bin Yan, and Guang-Ren Duan

Abstract: The polynomial Diophantine matrix equation and the generalized Sylvester matrix equation are important for controller design in frequency domain linear system theory and time domain linear system theory, respectively. By using the so-called generalized Sylvester mapping, right coprime factorization and Bezout identity associated with certain polynomial matrices, we present in this note a unified parametrization for the solutions to both of these two classes of matrix equations. Moreover, it is shown that solutions to the generalized Sylvester matrix equation can be obtained if solutions to the Diophantine matrix equation are available. The results disclose a relationship between the polynomial Diophantine matrix equation and generalized Sylvester matrix equation that are respectively studied and used in frequency domain linear system theory and time domain linear system theory.

Keywords: Coprime factorization and Bezout identity, Diophantine matrix equation, generalized Sylvester mapping, generalized Sylvester matrix equation, linear system theory, parametrization.

1. INTRODUCTION

The polynomial Diophantine matrix equation in the form of

$$A(s)X(s)+B(s)Y(s)=E(s), \tag{1}$$

where $A(s)$, $B(s)$, and $E(s)$ are some known polynomial matrices of appropriate dimensions and $X(s)$, $Y(s)$ are some polynomial matrices to be solved, plays important role in frequency domain analysis of linear systems, for example, multivariable stochastic optimal control [21], disturbance rejecting [27] and pole placement [28]. See [13,22,25] and [26] for detailed introduction. In the past several decades, a lot of methods has been developed to solve this class of equations (see [15] and the references therein). Some new techniques (for example, geometric method [16]) are also established to solve this old problem in the literature. Very recently, Tzekis has proposed a very interesting method to solve this problem [4]. A more restricted version of equation (1) by imposing symmetry on the coefficient matrices is also studied in [19].

On the other hand, the generalized Sylvester matrix equation

$$\sum_{i=0}^{\phi} A_i X F^i + \sum_{i=0}^{\varphi} B_i Y F^i = \sum_{i=0}^{\omega} E_i R F^i, \tag{2}$$

where $\{A_i\}_{i=0}^{\phi}, \{B_i\}_{i=0}^{\varphi}, \{E_i\}_{i=0}^{\omega}, R \in \mathbf{R}^{h\times p}$, $F \in \mathbf{R}^{p\times p}$ are known matrices and X, Y are matrices to be determined, also plays very important roles in time domain analysis of linear systems. Many control problems in time domain, such as pole/eigenstructure assignment [3], robust pole assignment [10] and fault detection [6] relay on solutions to this class of linear matrix equations. Due to its wide applications, many algorithms have been developed to search for both analytical and numerical solutions to this class of matrix equations. For analytical solutions, the reader may refer to [2,7-9]. For numerical solutions, see [11] and the references therein.

Very recently, we show in [2] that all the solutions to the generalized Sylvester matrix equation (2) can be parameterized by using the right coprime factorization and Bezout identity for certain polynomial matrices. In this paper, we aim to show that both the solutions to the polynomial Diophantine matrix equation (1) and the generalized Sylvester matrix equation (2) can be parameterized in the same manner. In other words, both parametric solutions to the polynomial Diophantine matrix equation (1) and the generalized Sylvester matrix equation (2) can be obtained as soon as certain polynomial matrix pair is obtained. We accomplish this by using the so-called generalized Sylvester mapping which is properly defined and studied in this paper. Upon the proposed results, we can clearly see a relationship between frequency domain linear system theory and time domain linear system theory. We should point out that connection between frequency domain linear system

Manuscript received April 1, 2008; revised December 28, 2008; accepted May 8, 2009. Recommended by Editorial Board member Yoshito Ohta under the direction of Editor Jae Weon Choi. The work of Bin Zhou was partially supported by the National Natural Science Foundation of China under grant number 60904007 and the Development Program for Outstanding Young Teachers at Harbin Institute of Technology under grant number HITQNJS.2009.054. The work of Guang-Ren Duan and Bin Zhou was partially supported by the Major Program of National Natural Science Foundation of China under grant number 60710002.

Bin Zhou and Guang-Ren Duan are with the Center for Control Theory and Guidance Technology, Harbin Institute of Technology, Harbin, Heilongjiang, P. R. China (e-mails: binzhoulee@163.com, g.r.duan@hit.edu.cn).

Zhi-Bin Yan is with the Natural Science Research Center, Harbin Institute of Technology, Harbin, Heilongjiang, P. R. China (e-mail: zbyan@hit.edu.cn).

theory and time domain linear system theory has been well investigated in the past several decades (see, for example, [12]). However, to the best of our knowledge, no result concerning these two classes of linear equations is available in the literature.

We have to emphasize that the aim of this paper is to show the *relationship* between solutions to the Diophantine equation (1) and the generalized Sylvester matrix equation (2), we don't aim to discuss the algorithm and the numerical reliability of the algorithm for the solutions. To be more clear, the explicit solutions to the Diophantine equation (1) used in this paper is standard and can be found in any linear system theory textbook. Indeed, as pointed our by a reviewer, there are plenty of papers by some researchers in the past several decades on this topic and the subject is now mature. There is a Matlab toolbox implementing the better algorithms to solve polynomial equations and control problems, that led to the matlab toolbox POLYX for solving polynomial equations and control problems.

At the end of this section, we would like point out that polynomial matrix and equations have many other applications and research topics in control community. The most important one may be the Youla-Kucera parameterization which has many important applications in robust and H_∞ control (see, for example, [18]) and LQ optimization [20]. Research on polynomial matrix and its relating problems is revived recently. For example, a polynomial approach is proposed to handle input saturation in recently years (see [24] and the references therein) and LMI characterization for robustness conditions [23].

The remainder of this paper is organized as follows. The main results are given in Section 2 which contains two subsections. In Subsection 2, we introduce the generalized Sylvester mapping and some primary results regarding its properties are proposed. In Subsection 2.2, parametrizations of the solutions to both the polynomial Diophantine matrix equation and generalized Sylvester matrix equation are unified by using the generalized Sylvester mapping. Section 4 concludes the paper.

Notations: Throughout this paper, we use $\sigma(A), A^{\mathrm{T}}$ and $\mathrm{rank}(A)$ to denote the eigenvalue set, the transpose and the rank of matrix A, respectively. For two integers m and $n, m \leq n$, we used $\mathbf{I}[m,n]$ to denote the set $\{m, m+1, \cdots, n\}$. The Kronecker product of two matrices A and B is denoted by $A \otimes B$. For a linear mapping S, we use $\ker(S)$ and $\dim(S)$ to denote its kernel space and dimensions, respectively. For two arbitrary integers p and q, we define

$$\mathbf{F}_{(\alpha,\beta)}^{p\times q}[s] = \left\{ \sum_{i=\alpha}^{\beta} T_i s^i \,\middle|\, \begin{array}{c} T_i \in \mathbf{F}^{p\times q}, \alpha \in \mathbf{Z}, \beta \in \mathbf{Z}, \\ -\infty < \alpha \leq \beta < \infty, \end{array} \right\},$$

where $\mathbf{F} = \mathbf{R}$ or $\mathbf{C}$. If $\alpha \geq 0, \beta < \infty$, then $\mathbf{F}_{(\alpha,\beta)}^{p\times q}[s]$ which is the set of all $p \times q$ polynomial matrices over $\mathbf{F}$ of finite degree, will be denoted by $\mathbf{F}^{p\times q}[s]$ for short.

2. MAIN RESULTS

2.1. Generalized Sylvester mapping and its properties

Definition 1: Let $T(s) = \sum_{i=\alpha}^{\beta} T_i s^i \in \mathbf{F}_{(\alpha,\beta)}^{n\times q}[s]$ and $F \in \mathbf{F}^{p\times p}$ be a fixed square matrix.

For any $X \in \mathbf{F}^{q\times p}$, we define the so-called right-Sylvester mapping $S_R^F : X \mapsto T(s) \odot X$, where

$$T(s) \odot X = \sum_{i=\alpha}^{\beta} T_i X F^i. \tag{3}$$

For any $X \in \mathbf{F}^{p\times n}$, we define the so-called left-Sylvester mapping $S_L^F : X \mapsto X * T(s)$, where

$$X * T(s) = \sum_{i=\alpha}^{\beta} F^i X T_i. \tag{4}$$

Obviously, if $\infty < \alpha < 0$, it follows from (3) and (4) that F must be nonsingular which is assumed to be true in that case.

Remark 1: Definition 1 is a generalization of Definition 1 in [1]. Indeed, in Definition 1 of [1], only the right-generalized Sylvester mapping 3 is introduced. Moreover, it is assumed in [1] that $\alpha = -\infty$ and $\beta = +\infty$ which may cause fundamental problem of convergence. Our definitions do not have such problem. Note also that our definitions also generate that in [5].

By definition, the following simple properties of S_R^F and S_L^F can be obtained. Part of the results in this lemma can be found in [1].

Lemma 1: Let $T_1(s), T_2(s), T(s) \in \mathbf{F}_{(\alpha,\beta)}^{n\times q}[s]$, X_R, $Y_R \in \mathbf{F}^{q\times p}$ and $X_L, Y_L \in \mathbf{F}^{p\times n}$.

1. The mappings $S_R^F : X_R \mapsto T(s) \odot X_R$ and $S_L^F : X \mapsto X * T(s)$ are linear mappings, i.e.,

$$\begin{aligned} &T(s) \odot (X_R + Y_R) = T(s) \odot X_R + T(s) \odot Y_R \\ &(X_L + Y_L) * T(s) = X_L * T(s) + Y_L * T(s). \end{aligned}$$

2. The mappings $S_R^F : X_R \mapsto T(s) \odot X_R$ and $S_L^F : X_L \mapsto X_L * T(s)$ satisfy

$$\begin{aligned} &(T_1(s) + T_2(s)) \odot X_R = T_1(s) \odot X_R + T_2(s) \odot X_R \\ &X_L * (T_1(s) + T_2(s)) = X_L * T_1(s) + X_L * T_2(s). \end{aligned}$$

3. Let $f(s) \in \mathbf{F}_{(\alpha,\beta)}[s]$. Then there holds

$$\begin{aligned} &f(s) I \odot X_R = X_R f(F) \\ &X_L * f(s) I = f(F) X_L. \end{aligned}$$

We give some further properties of both the left-Sylvester and right-Sylvester mappings. These results are generalizes of those given in [1]. For detailed proof, see [1].

Lemma 2: Let $T(s)\in\mathbf{F}^{n\times m}_{(\alpha_1,\beta_1)}[s], B(s)\in\mathbf{F}^{m\times q}_{(\alpha_2,\beta_2)}[s]$ and $X_R\in\mathbf{F}^{q\times p}$, $X_L\in\mathbf{F}^{p\times n}$. Then

$$\begin{aligned}(T(s)B(s))\odot X_R&=T(s)\odot(B(s)\odot X_R),\\ X_L*(T(s)B(s))&=(X_L*T(s))*B(s).\end{aligned}\tag{5}$$

Lemma 3: Let $T_{R1}(s)\in\mathbf{F}^{n\times q}_{(\alpha_1,\beta_1)}[s], T_{R2}(s)\in\mathbf{F}^{m\times q}_{(\alpha_2,\beta_2)}$ and $X_R\in\mathbf{F}^{q\times p}$. Then

$$\begin{bmatrix}T_{R1}(s)\\T_{R2}(s)\end{bmatrix}\odot X_R=\begin{bmatrix}T_{R1}(s)\odot X_R\\T_{R2}(s)\odot X_R\end{bmatrix}.\tag{6}$$

Let $T_{L1}(s)\in\mathbf{F}^{q\times n}_{(\alpha_1,\beta_1)}[s]$, $T_{L2}(s)\in\mathbf{F}^{q\times m}_{(\alpha_2,\beta_2)}$ and $X_L\in\mathbf{F}^{p\times q}$. Then

$$X_L*\begin{bmatrix}T_{L1}(s) & T_{L2}(s)\end{bmatrix}=\begin{bmatrix}X_L*T_{L1}(s) & X_L*T_{L2}(s)\end{bmatrix}.$$

Lemma 4: Let $T_{R1}(s)\in\mathbf{F}^{q\times n}_{(\alpha_1,\beta_1)}[s]$, $T_{R2}(s)\in\mathbf{F}^{q\times m}_{(\alpha_2,\beta_2)}$ and $(X_R,Y_R)\in(\mathbf{F}^{n\times p}\times\mathbf{F}^{m\times p})$. Then

$$\begin{bmatrix}T_{R1}(s) & T_{R2}(s)\end{bmatrix}\odot\begin{bmatrix}X_R\\Y_R\end{bmatrix}=T_{R1}(s)\odot X_R+T_{R2}(s)\odot Y_R.$$

Let $T_{L1}(s)\in\mathbf{F}^{n\times q}_{(\alpha_1,\beta_1)}[s]$, $T_{L2}(s)\in\mathbf{F}^{m\times q}_{(\alpha_2,\beta_2)}$ and $(X_L,Y_L)\in(\mathbf{F}^{p\times n}\times\mathbf{F}^{p\times m})$. Then

$$\begin{bmatrix}X_L & Y_L\end{bmatrix}*\begin{bmatrix}T_{L1}(s)\\T_{L2}(s)\end{bmatrix}=X_L*T_{L1}(s)+Y_L*T_{L2}(s).$$

Remark 2: Roughly speaking, Lemmas 1-4 say that we can use the symbols $\odot$ and $*$ as the ordinary matrix product, though they are essentially different from the ordinary matrix product.

We provide the following theorem regarding the properties of the mappings S_R^F and S_L^F. This result generalizes Theorem 1 in [1]. A more simple and elegant proof of this theorem than that proposed in [1] is given in appendix, which can make this subsection more legible.

Theorem 1: Let $T(s)\in\mathbf{F}^{n\times m}[s], X_R\in\mathbf{F}^{m\times p}$ and $X_L\in\mathbf{F}^{p\times n}$.

1) The mapping $S_R^F:X_R\mapsto T(s)\odot X_R$ is surjective (or the mapping $S_L^F:X_L\mapsto X_L*T(s)$ is injective) if and only if

   $\operatorname{rank}(T(\lambda))=n,\forall\lambda\in\sigma(F)$.

2) The mapping $S_R^F:X_R\mapsto T(s)\odot X_R$ is injective (or the mapping $S_L^F:X_L\mapsto X_L*T(s)$ is surjective) if and only if

   $\operatorname{rank}(T(\lambda))=m,\forall\lambda\in\sigma(F)$.

3) The mapping $S_R^F:X_R\mapsto T(s)\odot X_R$ is bijective (or the mapping $S_L^F:X_L\mapsto X_L*T(s)$ is bijective) if and only

   $\operatorname{rank}(T(\lambda))=n=m,\forall\lambda\in\sigma(F)$.

Definition 2: Let $T(s)\in\mathbf{F}^{n\times m}[s]$, $X_R\in\mathbf{F}^{m\times p}$ and $X_L\in\mathbf{F}^{p\times n}$. The mappings $S_R^F:X_R\mapsto T(s)\odot X_R$ and $S_L^F:X_L\mapsto X_L*T(s)$ are said to be universal-surjective, universal-injective and universal-bijective if they are surjective, injective and bijective, respectively, for arbitrary matrix $F\in\mathbf{F}^{p\times p}$.

Definition 3: For a polynomial matrix $T(s)\in\mathbf{F}^{n\times m}[s]$.

1) $T(s)$ is said to be left-coprime if $\operatorname{rank}(T(s))=n$, $\forall s\in\mathbf{C}$.
2) $T(s)$ is said to be right-coprime if $\operatorname{rank}(T(s))=m$, $\forall s\in\mathbf{C}$.
3) $T(s)$ is said to be coprime if it is both left-coprime and right-coprime.

The coprimeness of one polynomial matrix can be extended to a pair of polynomial matrices case. For example, the polynomial matrix pair $(A(s),B(s))\in(\mathbf{F}^{n\times m}[s]\times\mathbf{F}^{n\times p}[s])$ is said to be left-coprime if

$$\operatorname{rank}([A(s)\quad B(s)])=n,\forall s\in\mathbf{C}.$$

Obviously, $T(s)\in\mathbf{F}^{m\times m}[s]$ is coprime if and only if $T(s)$ is a unimodular matrix. By using this fact and the above two definitions, we can obtain the following corollary of Theorem 1.

Corollary 1: Let $T(s)\in\mathbf{F}^{n\times m}[s]$, $X_R\in\mathbf{F}^{m\times p}$ and $X_L\in\mathbf{F}^{p\times n}$.

1) The mapping $S_R^F:X_R\mapsto T(s)\odot X_R$ is universal-surjective (or the mapping $S_L^F:X_L\mapsto X_L*T(s)$ is universal-injective) if and only if $T(s)$ is left-coprime.
2) The mapping $S_R^F:X_R\mapsto T(s)\odot X_R$ is universal-injective (or the mapping $S_L^F:X_L\mapsto X_L*T(s)$ is universal-surjective) if and only if $T(s)$ is right-coprime.
3) The mappings $S_R^F:X_R\mapsto T(s)\odot X_R$ is universal-bijective (or the mappings $S_L^F:X_L\mapsto X_L*T(s)$ is universal-bijective) if and only if $T(s)$ is a unimodular matrix.

2.2. Parametrization of the solutions to the polynomial Diophantine matrix equation and the generalized Sylvester matrix equation

The polynomial Diophantine matrix equation takes the form

$$A(s)X_r(s)+B(s)Y_r(s)=E(s),\tag{7}$$

where $A(s) \in \mathbf{R}^{n\times n}[s]$, $B(s) \in \mathbf{R}^{n\times m}[s]$ and $E(s) \in \mathbf{R}^{n\times h}[s]$ are known polynomial matrices, and $(X_r(s), Y_r(s)) \in (\mathbf{R}^{n\times h}[s] \times \mathbf{R}^{m\times h}[s])$ are polynomial matrices to be determined. Assume that the polynomial matrix pair $(A(s), B(s))$ is left coprime. We recall the following well-known results.

Theorem 2 [13]**:** Assume that $E(s)$ is any known polynomial matrix. Let $(D_r(s), N_r(s)) \in (\mathbf{R}^{m\times m}[s] \times \mathbf{R}^{n\times m}[s])$ be a right coprime pair such that

$$A(s)N_r(s) + B(s)D_r(s) = 0, \tag{8}$$

and $(X_0(s), Y_0(s)) \in (\mathbf{R}^{n\times h}[s] \times \mathbf{R}^{m\times h}[s])$ be any polynomial matrix pair such that

$$A(s)X_{r0}(s) + B(s)Y_{r0}(s) = E(s). \tag{9}$$

Then a global parametrization of the solutions to the polynomial Diophantine matrix equation (7) is given by

$$\begin{cases} X_r(s) = X_{r0}(s) + N_r(s)T_r(s) \\ Y_r(s) = Y_{r0}(s) + D_r(s)T_r(s), \end{cases}$$

where $T_r(s) \in \mathbf{R}^{m\times h}[s]$ is an arbitrary polynomial matrix.

The following corollary can be immediately obtained.

Corollary 2: Let $(D_r(s), N_r(s)) \in (\mathbf{R}^{m\times m}[s] \times \mathbf{R}^{n\times m}[s])$ be a right coprime pair such that (8) holds and $(U_r(s), V_r(s)) \in (\mathbf{R}^{n\times n}[s] \times \mathbf{R}^{m\times n}[s])$ be a polynomial matrix pair satisfying the following Bezout identity

$$A(s)U_r(s) + B(s)V_r(s) = I. \tag{10}$$

Then a global parametrization of the solutions to the polynomial Diophantine matrix equation (7) can be given by

$$\begin{cases} X_r(s) = U_r(s)E(s) + N_r(s)T_r(s) \\ Y_r(s) = V_r(s)E(s) + D_r(s)T_r(s), \end{cases}$$

where $T_r(s) \in \mathbf{R}^{m\times h}[s]$ is an arbitrary polynomial matrix.

The dual form of the polynomial Diophantine matrix equation (7) is

$$X_l(s)A(s) + Y_l(s)C(s) = D(s), \tag{11}$$

where $A(s) \in \mathbf{R}^{n\times n}[s]$, $C(s) \in \mathbf{R}^{q\times n}[s]$ and $D(s) \in \mathbf{R}^{h\times n}[s]$ are known polynomial matrices, and $(X_l(s), Y_l(s)) \in (\mathbf{R}^{h\times n}[s] \times \mathbf{R}^{h\times q}[s])$ are polynomial matrices to be determined.

Corollary 3: Assume that the polynomial matrix pair $(A(s), C(s))$ is right coprime. Let $(D_l(s), N_l(s)) \in (\mathbf{R}^{q\times q}[s] \times \mathbf{R}^{q\times n}[s])$ be a left coprime polynomial matrix pair such that

$$N_l(s)A(s) + D_l(s)C(s) = 0, \tag{12}$$

and $(X_{l0}(s), Y_{l0}(s)) \in (\mathbf{R}^{h\times n}[s] \times \mathbf{R}^{h\times q}[s])$ be any polynomial matrix pair such that

$$X_{l0}(s)A(s) + Y_{l0}(s)B(s) = D(s). \tag{13}$$

Then a global parametrization of the solutions to the polynomial Diophantine matrix equation (11) is given by

$$\begin{cases} X_l(s) = X_{l0}(s) + T_l(s)N_l(s) \\ Y_l(s) = Y_{l0}(s) + T_l(s)D_l(s), \end{cases}$$

where $T_l(s) \in \mathbf{R}^{h\times q}[s]$ is an arbitrary polynomial matrix.

The generalized Sylvester matrix equation takes the form

$$\sum_{i=0}^{\phi} A_i X_r F^i + \sum_{i=0}^{\varphi} B_i Y_r F^i = \sum_{i=0}^{\omega} E_i R F^i, \tag{14}$$

where $A_i \in \mathbf{R}^{n\times n}$, $i \in \mathbf{I}[0,\phi]$, $B_i \in \mathbf{R}^{n\times m}$, $i \in \mathbf{I}[0,\varphi]$, $E_i \in \mathbf{R}^{n\times h}$, $i \in \mathbf{I}[0,\omega]$ and $R \in \mathbf{R}^{h\times p}$ are known matrices and $(X_r, Y_r) \in (\mathbf{R}^{n\times p} \times \mathbf{R}^{m\times p})$ are unknown matrices. This class of equations includes many linear equations, for example, $AX - XF = BY$, $AX - EXF = BY$ and $AX + BY = EXF + R$, as special cases. All these mentioned equations play important role in state-space control system design (see the references cited in Introduction).

By using the so-called right-Sylvester mapping defined before, the generalized Sylvester matrix equation (14) can be rewritten as

$$A(s) \odot X_r + B(s) \odot Y_r = E(s) \odot R, \tag{15}$$

where

$$\begin{cases} A(s) = \sum_{i=0}^{\phi} A_i s^i \in \mathbf{R}^{n\times n}[s] \\ B(s) = \sum_{k=0}^{\varphi} B_k s^k \in \mathbf{R}^{n\times m}[s] \\ E(s) = \sum_{l=0}^{\omega} E_l s^l \in \mathbf{R}^{n\times h}[s]. \end{cases} \tag{16}$$

Theorem 3: Assume that $(A(s), B(s))$ is left coprime. Let $(D_r(s), N_r(s)) \in (\mathbf{R}^{m\times m}[s] \times \mathbf{R}^{n\times m}[s])$ be a right coprime pair such that (8) holds and $(X_{r0}(s), Y_{r0}(s)) \in (\mathbf{R}^{n\times h}[s] \times \mathbf{R}^{m\times h}[s])$ be any polynomial matrix pair such that (9) holds. Then a global parametrization of the solutions to the generalized Sylvester matrix equation (14) for arbitrary $F \in \mathbf{R}^{p\times p}$ is given by

$$\begin{cases} X_r = X_{r0}(s) \odot R + N_r(s) \odot Z_r \\ Y_r = Y_{r0}(s) \odot R + D_r(s) \odot Z_r, \end{cases} \tag{17}$$

where $Z_r \in \mathbf{R}^{m\times p}[s]$ is an arbitrary parameter matrix.

Proof: We first show that (X_r, Y_r) given by (17) satisfies the generalized Sylvester matrix equation (14). Using the properties of right-Sylvester mapping and equations (8)-(9), we can obtain

$$A(s) \odot X_r + B(s) \odot Y_r = E(s) \odot R,$$

which shows that (X_r, Y_r) is indeed a solution to (14).

We next show that all the solutions to (14) can be parameterized as (17). Rewrite the equation (15) as

$$0 = \begin{bmatrix} A(s) & B(s) \end{bmatrix} \odot \begin{bmatrix} X_r \\ Y_r \end{bmatrix} \triangleq T(s) \odot W.$$

Denote $L = \{W : T(s) \odot W = 0\}$, i.e., L is the solution space of the generalized Sylvester matrix equation (14). Since $T(s)$ is left-coprime, it follows from Corollary 1 that $S_R^F : W \mapsto T(s) \odot W$ is universal-surjective. Then we have

$$\dim(L) = \dim\left(\ker\left(S_R^F\right)\right) = (n+m)p - np = rp.$$

parameters denoted by Z in the solution (17) is already rp. Hence, it is sufficient to show that $Z_r \mapsto (X_r, Y_r)$ is universal-injective, or equivalently

$$Z_r \mapsto \begin{bmatrix} N_r(s) \\ D_r(s) \end{bmatrix} \odot Z_r,$$

is universal-injective. This is true as $(N_r(s), D_r(s))$ is right coprime (Corollary 1). This completes the proof.

The following corollary can be obtained in accordance with Corollary 2.

Corollary 4: Assume that $(A(s), B(s))$ is left coprime. Let $(D_r(s), N_r(s)) \in \left(\mathbf{R}^{m\times m}[s] \times \mathbf{R}^{n\times m}[s]\right)$ be a right coprime pair such that (8) holds, and $(U_r(s), V_r(s)) \in \left(\mathbf{R}^{n\times n}[s] \times \mathbf{R}^{m\times n}[s]\right)$ be a polynomial matrix pair such that (10) holds. Then a global parametrization of the solutions to the generalized Sylvester matrix equation (14) for arbitrary $F \in \mathbf{R}^{p\times p}$ is given by

$$\begin{cases} X_r(s) = (U_r(s)E(s)) \odot R + N_r(s) \odot Z_r \\ Y_r(s) = (V_r(s)E(s)) \odot R + D_r(s) \odot Z_r, \end{cases}$$

where $Z_r \in \mathbf{R}^{m\times p}[s]$ is an arbitrary parameter matrix.

The dual form of the generalized Sylvester matrix equation (14) is

$$\sum_{i=0}^{\phi} F^i X_l A_i + \sum_{i=0}^{\varphi} F^i Y_l C_i = \sum_{i=0}^{\omega} F^i R D_i, \tag{18}$$

where X_l and Y_l are matrices to be determined. Equation (18) is known as the generalized Sylvester-observer equation and plays important role in observer design problem. Let $A(s)$ be defined as (16) and $C(s) = \Sigma_{i=0}^{\varphi} C_i s^i$, $D(s) = \Sigma_{i=0}^{\omega} D_i s^i$.

Corollary 5: Assume that $(A(s), C(s))$ is right coprime. Let $(D_l(s), N_l(s)) \in \left(\mathbf{R}^{q\times q}[s] \times \mathbf{R}^{q\times n}[s]\right)$ be a right coprime pair satisfying (12) and $(X_{l0}(s), Y_{l0}(s)) \in \left(\mathbf{R}^{h\times n}[s] \times \mathbf{R}^{h\times q}[s]\right)$ be any polynomial matrix pair satisfying (13). Then a global parametrization of the solutions to the generalized Sylvester-observer matrix equation (18) for arbitrary $F \in \mathbf{R}^{p\times p}$ is given by

$$\begin{cases} X_l = RX_{l0}(s) + Z_l N_l(s) \\ Y_l = RY_{l0}(s) + Z_l D_l(s), \end{cases}$$

where $Z_l \in \mathbf{R}^{p\times q}[s]$ is an arbitrary parameter matrix.

Theorems 2 and 3 (Corollaries 2 and 4) clearly imply a resemblance between the solutions to the polynomial Diophantine matrix equation (7) and solutions to the generalized Sylvester matrix equation (14). To the best of our knowledge, this resemblance has not yet been pointed out before. This result clearly shows an analogy and connection between the control theories developed respectively in frequency and time frameworks. Owning to these comments, one may ask the following question: Does one can immediately get solutions to one of the two equations as soon as solutions to the other one are obtained? The following theorem partly answers this question.

Theorem 4: Let $(X_r(s), Y_r(s))$ be a solution to the polynomial Diophantine matrix equation (7). Then $(X_r(s) \odot R, Y_r(s) \odot R)$ is a solution to the generalized Sylvester matrix equation (14). Let $(X_l(s), Y_l(s))$ be a solution to the polynomial Diophantine matrix equation (11). Then $(R * X_l(s), R * Y_l(s))$ is a solution to the generalized Sylvester-observer matrix equation (14).

Proof: We only prove the first statement. Since $(X_r(s), Y_r(s))$ is a solution to the polynomial Diophantine matrix equation (7), then

$$A(s)X_r(s) + B(s)Y_r(s) = E(s),$$

which in turn implies

$$E(s) \odot R = A(s) \odot (X_r(s) \odot R) + B(s) \odot (Y_r(s) \odot R).$$

That is to say, $(X_r(s) \odot R, Y_r(s) \odot R)$ is a solution to the generalized Sylvester matrix equation (14) in view of (15). The proof is completed.

3. CONCLUSION

In this note, we have considered the problem of parameterizing all solutions to the polynomial Diophantine matrix equation and the generalized Sylvester matrix equation. It is shown that solutions to both of them can be parameterized as soon as two pairs of polynomial matrices satisfying the right coprime factorization and Bezout identity are obtained. Our results show a nice connection between the polynomial Diophantine matrix equation appearing in frequency domain linear system theory and generalized Sylvester matrix equation encountered in time domain linear system theory.

APPENDIX A

A.1. Proof of Theorem 1

For a matrix $A \in \mathbf{R}^{m\times n}$, we define the column stretching function $\mathrm{cs}(Y)$ as follows

$$\mathrm{cs}(A): A \mapsto \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}, A = \begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix}.$$

Then we have the following well-known formulation [17]

$$\mathrm{cs}(AXB) = (B^{\mathrm{T}} \otimes A)\mathrm{cs}(X). \quad (19)$$

Denote $\beta = \deg(T(s))$. Taking cs on both sides of $Y = T(s) \odot X$ and using (19), gives

$$\mathrm{cs}(Y) = \mathrm{cs}\left(\sum_{i=0}^{\beta} T_i X F^i\right) = \left(\sum_{i=0}^{\beta} F^T \otimes T_i\right)\mathrm{cs}(X)$$
$$\triangleq \Pi\mathrm{cs}(X).$$

Since both $Y \mapsto \mathrm{cs}(Y)$ and $X \mapsto \mathrm{cs}(X)$ are bijective mappings, the mapping $X \mapsto T(s) \odot X$ is surjective, injective and bijective if and only if Π is of full row rank, of full column rank and nonsingular, respectively. Let V be a nonsingular matrix such that $F^{\mathrm{T}}V = VJ$ where

$$J = \bigoplus_{i=1}^{r} J_i, J_i = \begin{bmatrix} s_i & 1 & 0 & \cdots & 0 \\ 0 & s_i & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & 1 & 0 \\ \vdots & \ddots & 0 & s_i & 1 \\ 0 & \cdots & 0 & 0 & s_i \end{bmatrix}$$

with $s_i, i \in \mathbf{I}[1, r]$ being the eigenvalues of matrix F. Then we have

$$\Pi = \sum_{i=0}^{\beta} F^{\mathrm{T}} \otimes T_i = \sum_{i=0}^{\beta}\left(VJ^iV^{-1} \otimes T_i\right)$$
$$= \left(V \otimes I_n\right)\left(\sum_{i=0}^{\beta} J^i \otimes T_i\right)\left(V^{-1} \otimes I_m\right)$$
$$= \left(V \otimes I_n\right)\begin{bmatrix} \Pi_1 & * & \cdots & * \\ 0 & \Pi_2 & * & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & \Pi_r \end{bmatrix}\left(V^{-1} \otimes I_m\right),$$

where

$$\Pi_i = \begin{bmatrix} T(s_i) & * & \cdots & * \\ 0 & T(s_i) & * & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & T(s_i) \end{bmatrix}, \ i \in \mathbf{I}[1, r],$$

and the terms denoted by $*$ are not important here. We note that Π is of full row rank, of full column rank and nonsingular, if and only if $\Pi_i, i \in \mathbf{I}[1, r]$ are of full row rank, full column rank and nonsingular, respectively, which are equivalent to that $T(s_i), i \in \mathbf{I}[1, r]$ are of full row rank, of full column rank and nonsingular, respectively. This completes the proof.

REFERENCES

[1] B. Zhou and G. R. Duan, "On the generalized Sylvester mapping and matrix equations," *Systems & Control Letters*, vol. 57, pp. 200-208, 2008.

[2] B. Zhou and G. R. Duan, "A new solution to the generalized Sylvester matrix equation $AV - EVF = BW$," *Systems & Control Letters*, vol. 55, no. 3, pp. 193-198, 2006.

[3] G. R. Duan, "Solutions to matrix equation $AV + BW = VF$ and their application to eigenstructure assignment in linear systems," *IEEE Trans. on Automatic Control*, vol. 38, no. 2, pp. 276-280, 1993.

[4] P. A. Tzekis, "A new algorithm for the solution of a polynomial matrix Diophantine equation," *Appl. Math. Comput,* vol. 193, no. 2, pp. 395-407, 2007.

[5] L. Huang, "To solve matrix equation $\Sigma A^i XB_i = C$ by the smith normal form," *Applied Math. A Journal of Chinese University (Series B),* vol. 17, no. 1, pp. 109-118, 2002.

[6] J. Park and G. Rizzoni, "An eigenstructure assignment algorithm for the design of fault detection filters," *IEEE Trans. on Automatic Control*, vol. 39, no. 7, pp. 1521-1524, 1994.

[7] E. B. Castelan and V. Gomes da Silva, "On the solution of a Sylvester matrix equation appearing in descriptor systems control theory," *Systems & Control Letters,* vol. 54, no. 2, pp. 109-117, 2005.

[8] M. Darouach, "Solution to Sylvester equation associated to linear descriptor systems," *Systems & Control Letters,* vol. 55, no. 10, pp. 835-838, 2006.

[9] C. C. Tsui, "A complete analytical solution to the equation $TA - FT = LC$ and its applications," *IEEE Trans. on Automatic Control*, vol. 32, no. 8, pp. 742-744, 1987.

[10] E. K. Chu and B. N. Datta, "Numerically robust pole assignment for second-order systems," *Int. J.*

Control, vol. 64, no. 4, pp. 1113-1127, 1996.
[11] F. Ding and T. Chen, "Iterative least squares solutions of coupled Sylvester matrix equations," *Systems & Control Letters*, vol. 54, no. 2, pp. 95-107, 2005.
[12] V. Kucera, "A bridge between state-space and transfer-function methods," *Annual Reviews in Control*, vol. 23, pp. 177-184, 1999.
[13] W. A. Wolovich and P. J. Antsaklis, "The canonical Diophantine equations with applications," *SIAM Journal on Control and Optimization*, vol. 22, no. 5, pp. 777-787, 1984.
[14] Y. S. Lai, "An algorithm for solving the matrix polynomial equation $B(s)\,D(s) + A(s)\,N(s) = H(s)$," *IEEE Trans. on Circuits and Systems,* vol. 36, no. 8, pp. 1087-1089, 1989.
[15] C. H. Fang, "A simple approach to solving the Diophantine equation," *IEEE Trans. on Automatic Control,* vol. 37, no. 1, pp. 152-155, 1992.
[16] M. Bonilla Estrada, "Solving the right Diophantine equation in a geometric way," *Proc. of the 34th Conference on Decision & Control*, New Orleans, LA, pp. 309-310, 1995.
[17] J. L. Chen and X. H. Chen, *Special Matrices* (in Chinese), Tsinghua University Press, 2002.
[18] H. Kwakernaak, "Robust control and H_∞-optimization - Tutorial paper," *Automatica*, vol. 29, no. 2, pp. 255-273, 1993.
[19] D. Henrion and M. Sebek, "Symmetric matrix polynomial equation: interpolation results," *Automatica*, vol. 34, no.7, pp. 811-824, 1998.
[20] K. J. Hunt and M. Sebek, "Polynomial LQ optimization for the standard control structure: scalar solution," *Automatica*, vol. 29, no. 2, pp. 507-509, 1993.
[21] K. J. Hunt and M. Sebek, "Implied polynomial matrix equations in multivariable stochastic optimal control," *Automatica*, vol. 27, no. 2, pp. 395-398, 1991.
[22] M. Sebek, "Two-sided equations and skew primeness for n-D polynomial matrices," *Systems & Control Letters*, vol. 12, no. 4, pp. 331-337, 1989.
[23] D. Henrion, D. Arzelier, and D. Peaucelle, "Positive polynomial matrices and improved LMI robustness conditions," *Automatica*, vol. 39, no. 8, pp. 1479-1485, 2003.
[24] D. Henrion, S. Tarbouriech, and V. Kucera, "Control of linear systems subject to input constraints: a polynomial approach," *Automatica*, vol. 37, no. 4, pp. 597-604, 2001.
[25] V. Kucera, *Analysis and Design of Discrete Linear Control Systems*, Academia and Prentice Hall, Prague and London, 1991.
[26] V. Kucera, "Diophantine equations in control-a survey," *Automatica,* vol. 29, no. 6, pp. 1361-1375, 1993.
[27] R. Lopezlena Estrada and J. C. Martinez-Garcia, "Fixed poles and disturbance rejecting feedback synthesis," *Automatica*, vol. 35, pp. 1737-1740, 1999.
[28] T. Koussiouris and K. Tzierakis, "Rejection of measurable disturbances with decoupling or pole placement," *Proc. of the third European Control Conference*, Rome, Italy, pp. 2257-2262, 1995.

Bin Zhou was born in Hubei Province, China, on July 28, 1981. He received the Bachelor's degree from the Department of Control Science and Engineering at Harbin Institute of Technology, Harbin, China, in 2004. He was a research associate in the Department of Mechanical Engineering, University of Hong Kong, Hong Kong, from December 2007 to March 2008. He is now working toward a Ph.D. degree in the Centre for Control Theory and Guidance Technology at the same institute. His current research interests include constrained control systems and signal processing and he has published over 30 journal papers within these areas.

Zhi-Bin Yan was born in Hanchuan, Hubei province, China, in 1967. He received the B.S. degree from Nankai University in 1988, the M.S. and Ph.D. degrees from Harbin Institute of Technology in 1991 and 2002 respectively, all in Mathematics. Since 1991, he has been with Harbin Institute of Technology, where he is presently a professor in the Natural Science Research Center. His current research interests include differential-algebraic equation, control under communication constraint.

Guang-Ren Duan was born in Heilongjiang Province, P.R. China, on April 5, 1962. He received the B.Sc. degree in Applied Mathematics, and both the M.Sc. and Ph.D. degrees in Control Systems Theory. From 1989 to 1991, he was a post-doctoral researcher at Harbin Institute of Technology, where he became a professor of control systems theory in 1991. He is currently the Director of the Centre for Control Systems and Guidance Technology at Harbin Institute of Technology. He is the author and co-author of over 300 publications. His main research interests include robust control, eigenstructure assignment, descriptor systems, missile autopilot control and magnetic bearing control.