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Adjacent vertex distinguishing total colorings of outerplanar graphs

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Abstract An adjacent vertex distinguishing total coloring of a graph G is a proper total coloring of G such that any pair of adjacent vertices are incident to distinct sets of colors. The minimum number of colors required for an adjacent vertex distinguishing total coloring of G is denoted by $\chi''_a(G)$. In this paper, we characterize completely the adjacent vertex distinguishing total chromatic number of outerplanar graphs.

Keywords Adjacent vertex distinguishing total coloring · Outerplanar graph · Maximum degree

1 Introduction

We only consider simple graphs, i.e., graphs without self loops or multiple edges, throughout this paper. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. A *proper total k-coloring* is a mapping $\phi : V(G) \cup E(G) \to \{1, 2, \ldots, k\}$ such that any two adjacent or incident elements in $V(G) \cup E(G)$ receive different colors. The *total chromatic number* $\chi''(G)$ of G is the smallest integer k such that G has a total k-coloring. Let $C_\phi(v) = \{\phi(v)\} \cup \{\phi(xv) \mid xv \in E(G)\}$ denote the set of colors assigned to a vertex v and those edges incident to v. A proper total k-coloring ϕ of G is *adjacent vertex distinguishing*, or a *total-k-avd-coloring*, if $C_\phi(u) \neq C_\phi(v)$ whenever $uv \in E(G)$. The *adjacent vertex distinguishing total chromatic number* $\chi''_a(G)$ is the smallest integer k such that G has a total-k-avd-coloring.

Let $\Delta(G)$ and $\delta(G)$ denote the maximum degree and the minimum degree of a graph G, respectively. By definition, it is evident that $\chi''_a(G) \geq \chi''(G) \geq \Delta(G) + 1$

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for any graph G. Zhang et al. (2005) first investigated the adjacent vertex distinguishing total coloring of graphs. They determined the adjacent vertex distinguishing total chromatic numbers for paths, cycles, fans, wheels, trees, complete graphs, and complete bipartite graphs. The well-known Total Coloring Conjecture, made independently by Behzad (1965) and Vizing (1968), says that every simple graph G has $\chi''(G) \le \Delta(G) + 2$. This conjecture still remains open. Zhang et al. (2005) put forward the following conjecture:

Conjecture 1 *If G is a graph with at least two vertices, then $\chi''_a(G) \le \Delta(G) + 3$.*

Note that $\chi''_a(K_{2n+1}) = \Delta(K_{2n+1}) + 3 = 2n + 3$ for any $n \ge 1$. This example shows that the upper bound $\Delta(G) + 3$ for $\chi''_a(G)$ is tight if Conjecture 1 is true. More recently, Chen (2007) and Wang (2007), independently, confirmed Conjecture 1 for graphs G with $\Delta(G) \le 3$.

Let $\chi(G)$ and $\chi'(G)$ denote the (vertex) chromatic number and the edge chromatic number of a graph G, respectively. Vizing Theorem (Vizing 1964) asserts that every simple graph G satisfies $\Delta(G) \le \chi'(G) \le \Delta(G) + 1$. The graph G is of *Class 1* if $\chi'(G) = \Delta(G)$, and *Class 2* if $\chi'(G) = \Delta(G) + 1$. As a direct consequence of definitions, we have the following relation:

Proposition 1 *For any graph G, $\chi''_a(G) \le \chi(G) + \chi'(G)$.*

Some upper bounds for $\chi''_a(G)$ can be easily derived from Proposition 1 as follows:

(i) If G is a planar graph, then $\chi(G) \le 4$ by the Four-Color Theorem (Appel and Haken 1976), thus $\chi''_a(G) \le 4 + \Delta(G) + 1 = \Delta(G) + 5$.
(ii) If G is a bipartite graph, then $\chi(G) \le 2$ and $\chi'(G) = \Delta(G)$, thus $\chi''_a(G) \le \Delta(G) + 2$.
(iii) If G is a Class 1 graph and $\chi(G) \le 3$, then $\chi''_a(G) \le \Delta(G) + 3$.

A planar graph is called *outerplanar* if there is an embedding of G into the Euclidean plane such that all the vertices are incident to the unbounded face. An *outerplane* graph is a particular embedding of an outerplanar graph. Obviously, all trees and graphs of maximum degree at most 2 are outerplanar graphs.

The following result first appeared in Zhang et al. (2005):

Lemma 2 *Let G be a graph with $\Delta(G) \le 2$. Then $\chi''_a(G) \le 5$. Moreover, $\chi''_a(G) = 5$ if and only if G is a 3-cycle.*

It is easy to show that every outerplanar graph G has $\chi(G) \le 3$. It is known in Fiorini (1975) that an outerplanar graph G is of Class 1 if and only if G is not an odd cycle. These facts, Lemma 2 and Proposition 1 give the following:

(iv) If G is an outerplanar graph, then $\chi''_a(G) \le \Delta(G) + 3$.

Another easy observation was made in Zhang et al. (2005):

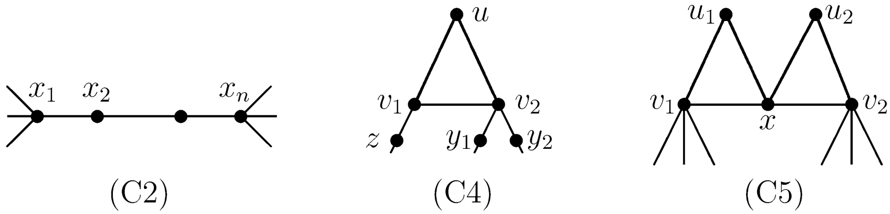

Fig. 1 Configurations (C2), (C4) and (C5)

Proposition 3 *If G is a graph with two adjacent vertices of maximum degree, then* $\chi''_a(G) \geq \Delta(G) + 2$.

Several authors, e.g., Chen and Zhang (2006) and Zhang et al. (2005), considered the adjacent vertex distinguishing total coloring of 2-connected outerplane graphs in the past years. In this paper, we extend their results by characterizing completely the adjacent vertex distinguishing total chromatic number of outerplane graphs. More precisely, we prove the following:

Main Theorem *Let G be an outerplane graph with $\Delta(G) \geq 3$. Then $\Delta(G) + 1 \leq \chi''_a(G) \leq \Delta(G) + 2$; and $\chi''_a(G) = \Delta(G) + 2$ if and only if G contains two adjacent vertices of maximum degree.*

2 Structural lemmas

Let G be an outerplane graph. Let $F(G)$ denote the set of faces in G. For $f \in F(G)$, we use $b(f)$ to denote the boundary walk of f and write $f = [u_1u_2 \cdots u_n]$ if $u_1, u_2, \ldots, u_n$ are all the vertices of $b(f)$ traversed once in cyclic order. A *k-vertex* is a vertex of degree k. A 1-vertex is also said to be a *leaf*.

We define some configurations as follows:

(C1) A vertex v of degree at most 3 is adjacent to a leaf.
(C2) A path $x_1x_2 \cdots x_n$, $n \geq 4$, with $d_G(x_1) \neq 2$, $d_G(x_n) \neq 2$, and $d_G(x_i) = 2$ for all $i = 2, 3, \ldots, n-1$.
(C3) A k-vertex v, $k \geq 4$, is adjacent to a leaf and $k-3$ vertices of degree ≤ 2.
(C4) A 3-face $[uv_1v_2]$ satisfies $d_G(u) = 2$ and $d_G(v_1) = 3$.
(C5) Two 3-faces $[u_1v_1x]$ and $[u_2v_2x]$ satisfy $d_G(x) = 4$ and $d_G(u_1) = d_G(u_2) = 2$.

Lemma 4 (Wang and Zhang 1999) *Every outerplane graph G with $\delta(G) = 2$ contains one of the configurations* (C2), (C4) *and* (C5) (*see Fig.* 1).

Lemma 5 *Every connected outerplane graph G with at least two vertices contains one of the configurations* (C1) *to* (C5).

Proof Assume to the contrary that G contains none of (C1)–(C5). Since G has no (C1), there is no vertex of degree at most 3 adjacent to a leaf. Since G contains no

(C3), every vertex v of degree at least 4 is adjacent to at most $d_G(v) - 3$ leaves; that is, it has at least three neighbors that are not leaves.

Let H be the graph obtained by removing all leaves of G. Then H is a connected outerplane graph. It follows from the previous argument that, for every $v \in V(H)$, $d_H(v) \geq 2$ and $d_H(v) = d_G(v)$ if $2 \leq d_G(v) \leq 3$. By Lemma 4, H contains one of (C2), (C4) and (C5), see Fig. 1. If H contains (C2) or (C4), then (C2) or (C4) must be a configuration of G because $d_H(t) = d_G(t)$ for all $t \in \{x_2, \ldots, x_{n-1}, u, v_1\}$. If H contains (C5), then since $d_H(u_i) = d_G(u_i)$ for $i = 1, 2$, and x cannot be adjacent to any leaf in G by the excluding of (C3) from G, (C5) also is a configuration of G. This contradicts the assumption on G. □

Lemma 6 *Every connected outerplane graph G with $\Delta(G) \leq 3$ contains one of the following configurations*:

- (B1) *A vertex v adjacent to at most one vertex that is not a leaf.*
- (B2) *A path $x_1x_2x_3x_4$ such that each of x_2 and x_3 is either a 2-vertex, or a 3-vertex that is adjacent to a leaf.*
- (B3) *A 3-face $[uxy]$ such that either $d_G(u) = 2$, or $d_G(u) = 3$ and u is adjacent to a leaf.*

Proof Assume to the contrary that G contains none of (B1)–(B3). Since G has no (B1), there is no 2-vertex adjacent to a leaf, and there is no 3-vertex adjacent to two or more leaves. Equivalently, every 3-vertex is adjacent to at most one leaf.

Let H be the graph obtained by removing all leaves of G. Then H is a connected outerplane graph. It is easy to inspect that $\delta(H) = 2$. If H is a cycle, then G contains (B2) or (B3). Thus, suppose that H is not a cycle, so $\Delta(H) = 3$. If H is 2-connected, then there is an end face $f = [u_1u_2 \cdots u_n]$, $n \geq 3$, such that $d_H(u_1) = d_H(u_n) = 3$ and $d_H(u_i) = 2$ for all $i = 2, 3, \ldots, n-1$. Since G has no (B2), we derive that $n = 3$ and hence (B3) is contained in G. If H is not 2-connected, then there is an end block M which is a cycle C in H with a cut vertex of degree 3 in G. Again, since (B2) does not appear in G, C is a 3-cycle. Thus, (B3) holds. We always obtain a contradiction. □

Lemma 7 *Every connected outerplane graph G with $\Delta(G) = 4$ and without adjacent 4-vertices contains one of the following configurations*:

- (A1) *A vertex v with $d_G(v) \neq 3$ is adjacent to a leaf.*
- (A2) *A 3-vertex is adjacent to at least two leaves.*
- (A3) *A path $x_1x_2x_3x_4$ such that each of x_2 and x_3 is either a 2-vertex, or a 3-vertex that is adjacent to a leaf.*
- (A4) *A 3-face $[uxy]$ with $d_G(x) = 3$ such that either $d_G(u) = 2$, or $d_G(u) = 3$ and u is adjacent to a leaf.*

Proof Assume to the contrary that G contains none of (A1)–(A4). Since G has no (A1), there does not exist a vertex of degree 1, 2 or 4 adjacent to a leaf. Since G has no (A2), every 3-vertex is adjacent to at most one leaf. Thus, each leaf of G must be adjacent to a 3-vertex.

Let H be the graph obtained by removing all leaves of G. Then H is a connected outerplane graph. It is easy to derive that $\delta(H) = 2$. By Lemma 4 and noting the fact that there are no adjacent 4-vertices, H contains (C2) or (C4). If H contains (C2), then G will contain (A3). If H contains (C4), G will contain (A4). We always get a contradiction. □

Lemma 8 *Every connected outerplane graph G with $\Delta(G) = 3$ and without adjacent 3-vertices contains one of the following configurations*:

(D1) *A leaf.*
(D2) *A cycle $C = x_1x_2 \cdots x_n$, with $n \geq 3$, such that $d_G(x_1) = 3$ and $d_G(x_i) = 2$ for all $i = 2, 3, \ldots, n$.*

Proof Suppose that G contains no (D1), i.e., $\delta(G) = 2$. Let M be an end block of G. Then M is a cycle C since G contains no adjacent 3-vertices. However, since $\Delta(G) = 3$, there is a vertex $v \in V(C)$ such that $d_G(v) = 3$. Thus, G contains (D2). □

Given an outerplane graph G, we write $|T(G)| = |V(G)| + |E(G)|$. Suppose that ϕ is a total-k-avd-coloring of G with a color set $C = \{1, 2, \ldots, k\}$, where $k \geq 5$. Assume that $v \in V(G)$ with $d_G(v) \leq 2$ is not adjacent to any vertex of the same degree as itself. Since v has at most two adjacent vertices and two incident edges and $|C| \geq 5$, we always can color v in the last stage when all its incident or adjacent elements have been colored. In other words, we may omit the coloring for such 1-vertices and 2-vertices in the following proofs of several theorems.

The proof of the Main Theorem is divided into two cases: $\Delta(G) = 3$ and $\Delta(G) \geq 4$.

3 $\Delta(G) = 3$

Theorem 9 *If G is an outerplane graph with $\Delta(G) \leq 3$, then $\chi''_a(G) \leqslant 5$.*

Proof The proof proceeds by induction on $|T(G)|$. If $|T(G)| \leq 5$, the theorem holds trivially. Suppose that G is an outerplane graph with $\Delta(G) \leq 3$ and $|T(G)| \geq 6$. We may assume that G is connected since $\chi''_a(G) = \max\{\chi''_a(G_i)\}$ and $\Delta(G) = \max\{\Delta(G_i)\}$, where both maxima are taken over all components G_i of G. By the induction assumption, any outerplane graph H with $\Delta(H) \leq 3$ and $|T(H)| < |T(G)|$ has a total-5-avd-coloring ϕ.

By Lemma 6, G contains one of the configurations (B1)–(B3). To complete the proof, we need to handle separately every possible case. In the subsequent proofs, we routinely construct appropriate proper total colorings without verifying in detail that they are adjacent vertex distinguishing because that usually can be supplied in a straightforward manner.

(B1) G contains a vertex v adjacent to at most one vertex that is not a leaf.

Let $v_1, \ldots, v_n$ be all the neighbors of v with $d_G(v_1) = \cdots = d_G(v_{n-1}) = 1$ and $d_G(v_n) \geq 1$. Clearly, $2 \leq n \leq 3$. Let $H = G - \{v_1, \ldots, v_{n-1}\}$. Then, H is an outerplane graph with $\Delta(H) \leq 3$ and $|T(H)| < |T(G)|$, hence it has a total-5-avd-coloring ϕ with the color set $C = \{1, 2, \ldots, 5\}$. We color $vv_1, \ldots, vv_{n-1}$ with different colors in $C \setminus \{\phi(v), \phi(v_n), \phi(vv_n)\}$. Since $n-1 \leq 2$ and $|C \setminus \{\phi(v), \phi(v_n), \phi(vv_n)\}| \geq 2$, the extended coloring is a total-5-avd-coloring of G.

(B2) G contains a path $x_1x_2x_3x_4$ such that each of x_2 and x_3 is either a 2-vertex, or a 3-vertex that is adjacent to a leaf.

For $i \in \{2, 3\}$, let x_i' be a leaf adjacent to x_i provided x_i is a 3-vertex.

If both x_2 and x_3 are 2-vertices, then the proof can be given with a similar argument as in the case (C2) of the following Theorem 11.

If $d_G(x_2) = d_G(x_3) = 3$, let H denote the graph obtained from G by identifying x_2' and x_3'. Then, H is an outerplane graph with $\Delta(H) \leq 3$ and $|T(H)| = |T(G)| - 1$. Obviously, any total-5-avd-coloring of H can induce a total-5-avd-coloring of G.

If $d_G(x_2) = 3$ and $d_G(x_3) = 2$, say, let $H = G - x_2'$. By the induction assumption, H has a total-5-avd-coloring ϕ with the color set $C = \{1, 2, \ldots, 5\}$. If $d_G(x_1) \neq 3$, we only need to color properly x_2x_2'. Assume that $d_G(x_1) = 3$, and further suppose $C_\phi(x_1) = \{1, 2, 3, 4\}$. If $5 \in \{\phi(x_2), \phi(x_2x_3)\}$, we properly color x_2x_2'. Otherwise, we color x_2x_2' with 5.

(B3) G contains a 3-face $[uxy]$ such that either $d_G(u) = 2$, or $d_G(u) = 3$ and u is adjacent to a leaf u'.

Based on the proof of (B2), we may assume that $d_G(x) = d_G(y) = 3$. Let $x' \neq u, y$ be the third neighbor of x, and $y' \neq u, x$ be the third neighbor of y. Let f' denote the face adjacent to $[uxy]$ with xy as a common edge. We need to consider some subcases, depending on the size of f'.

(B3.1) $d_G(f') = 3$, i.e., x' is identical to y'. By the induction assumption, $G - \{u, xy\}$ has a total-5-avd-coloring ϕ with the color set $C = \{1, 2, \ldots, 5\}$. If $d_G(x') = 2$, i.e., G is a graph of order 4 obtained from K_4 by removing an edge, then the theorem holds obviously. Assume that $d_G(x') = 3$ and let $t \neq x, y$ be the third neighbor of x'. Let $\phi(x't) = 1$, $\phi(x') = 2$, $\phi(x'x) = 3$, and $\phi(x'y) = 4$. We color uy with 1, ux with 2, $\{y, uu'\}$ with 3 (if u' exists), x with 4, and $\{u, xy\}$ with 5.

(B3.2) $d_G(f') = 4$, i.e., x' is adjacent to y'. Without loss of generality, we assume that both x' and y' are 3-vertices (otherwise, we have an easier proof). Let $x'' \neq x, y'$ denote the third neighbor of x', and $y'' \neq y, x'$ be the third neighbor of y'. By the induction assumption, $G - \{u, xy\}$ has a total-5-avd-coloring ϕ with the color set $C = \{1, 2, \ldots, 5\}$. Let $\phi(x') = 1$, $\phi(x'y') = 2$, $\phi(x'x'') = 3$, $\phi(x'x) = 4$, $\phi(y') = a$, $\phi(yy') = b$, and $\phi(y'y'') = c$. Since x' is adjacent to y', we see that $5 \in \{a, b, c\}$.

(3.2.1) $a = 5$. Then $b \in \{1, 3, 4\}$. If $b = 1$, we color or recolor u with 1, $\{y, ux\}$ with 2, xy with 3, uy with 4, and $\{x, uu'\}$ with 5 (if u' exists). If $b = 3$, we color or recolor xy with 1, $\{y, ux\}$ with 2, u with 3, uy with 4, and $\{x, uu'\}$ with 5 (if u' exists). If $b = 4$, we color or recolor uy with 1, $\{y, ux\}$ with 2, $\{u, xy\}$ with 3, and $\{x, uu'\}$ with 5 (if u' exists).

(3.2.2) $b = 5$. We color or recolor $\{y, uu'\}$ with 1 (if u' exists), ux with 2, $\{u, xy\}$ with 3, uy with 4, and x with 5 (if u' exists).

(3.2.3) $c = 5$. Then $b \in \{1, 3, 4\}$. We first color or recolor $\{y, ux\}$ with 2 and $\{x, uu'\}$ with 5 (if u' exists). If $b = 1$, we further color u with 1, xy with 3, uy with 4. If $b = 3$, we color xy with 1, u with 3, uy with 4. If $b = 4$, we color xy with 1, uy with 3, and u with 4.

(B3.3) $d_G(f') \geq 5$, i.e., x' is not adjacent to y'. Let $H = G - \{u, x, y\} + x'y'$. By the induction assumption, H has a total-5-avd-coloring ϕ with the color set $C = \{1, 2, \ldots, 5\}$. Suppose that $\phi(x') = 1$, $\phi(x'y') = 2$, and $\phi(y') = 3$. In G, we color $\{y, uu'\}$ with 1 (if u' exists), $\{u, xx', yy'\}$ with 2, ux with 3, $\{x, uy\}$ with 4, and xy with 5. □

Theorem 10 *If G is an outerplane graph with $\Delta(G) = 3$ and without adjacent 3-vertices, then $\chi''_a(G) = 4$.*

Proof The lower bound that $\chi''_a(G) \geq 4$ is trivial. We prove the upper bound $\chi''_a(G) \leq 4$ by induction on the vertex number $|V(G)|$. If $|V(G)| = 4$, then G is either $K_{1,3}$, or a graph obtained from $K_{1,3}$ by joining a pair of leaves. It is easy to verify that $\chi''_a(G) = 4$ for both these cases. Let G be a connected outerplane graph with $\Delta(G) = 3$ and $|V(G)| \geq 5$ and having no adjacent 3-vertices. By Lemma 8, G contains (D1) or (D2).

If G contains (D1), i.e., a leaf v adjacent to a vertex u, let $H = G - v$. Then, H is a connected outerplane graph with $\Delta(H) \leq 3$ and $|V(H)| \geq 4$ and without adjacent 3-vertices. This means that H cannot be a 3-cycle. By the induction assumption or Lemma 2 in Sect. 1, H has a total-4-avd-coloring ϕ with the color set $C = \{1, 2, 3, 4\}$.

If $d_G(u) = 2$, let $x \neq v$ be the second neighbor of u. We color uv with a color different from the colors of u, x, xu, and v with a color different from the colors of u, uv.

If $d_G(u) = 3$, let $u_1, u_2 \neq v$ be the other neighbors of u. Since G contains no adjacent 3-vertices, $d_G(u_i) \leq 2$ for all $i = 1, 2$. We color uv with a color different from the colors of u, uu_1, uu_2, and v with a color different from the colors of u, uv.

If G contains (D2), i.e., a cycle $C = x_1x_2 \cdots x_n$, with $n \geq 3$, such that $d_G(x_1) = 3$ and $d_G(x_i) = 2$ for all $i = 2, 3, \ldots, n$, let $y \neq x_2, x_n$ be the third neighbor of x_1. We see that y is not a 3-vertex. Let $H = G - x_2$. Then H is a connected outerplane graph with $\Delta(H) \leq 3$ and $|V(H)| \geq 4$ and without adjacent 3-vertices. We note that H is not a 3-cycle. By the induction assumption or Lemma 2, H has a total-4-avd-coloring ϕ with the color set $C = \{1, 2, 3, 4\}$. Assume that $\phi(y) = 1$, $\phi(x_1y) = 2$, and $\phi(x_1) = 3$. Erase the colors of all edges and all vertices other than x_1 in C.

If $n = 3$, we color $\{x_3, x_1x_2\}$ with 1, x_2 with 2, x_2x_3 with 3, and x_1x_3 with 4.

If $n \geq 4$, we first color x_1x_2 with 1 and x_1x_n with 4, then extend the current coloring to the other vertices and edges of C, with a similar method in the proof of Theorem 2.1 in Zhang et al. (2005). □

Combining Proposition 3, Theorems 9 and 10, we complete the proof of Main Theorem for the case $\Delta(G) = 3$.

4 $\Delta \geq 4$

Theorem 11 *If G is an outerplane graph with $\Delta(G) \geq 4$, then $\chi''_a(G) \leq \Delta(G) + 2$.*

Proof We prove the theorem by induction on $|T(G)|$. If $|T(G)| \le 5$, the theorem holds clearly. Suppose that G is a connected outerplane graph with $\Delta(G) \ge 4$ and $|T(G)| \ge 6$. By the induction assumption or Theorem 9, every outerplane graph H with $\Delta(H) \le \Delta(G)$ and $|T(H)| < |T(G)|$ has $\chi''_a(H) \le \Delta(H) + 2 \le \Delta(G) + 2$.

By Lemma 5, G contains one of the configurations (C1)–(C5). Since $\Delta(G) \ge 4$, the number of colors used is $\Delta(G) + 2 \ge 6$.

(C1) G contains a vertex v with $d_G(v) \le 3$ which is adjacent to a leaf.

Without loss of generality, we may assume that $d_G(v) = 3$ and u_1, u_2, u_3 are neighbors of v with $d_G(u_1) = 1$. Let $H = G - u_1$. Then H is a connected outerplane graph with $|T(H)| < |T(G)|$. By the induction assumption or Theorem 9, H has a total-$(\Delta(G) + 2)$-avd-coloring ϕ with the color set $C = \{1, 2, \dots, \Delta(G) + 2\}$. Suppose that $\phi(v) = 1$, $\phi(vu_2) = 2$, and $\phi(vu_3) = 3$.

If $|\{4, 5, 6\} \cap C_\phi(u_i)| \ge 2$ for all $i = 2, 3$, we color vu_1 with 4. If $|\{4, 5, 6\} \cap C_\phi(u_i)| \le 1$ for all $i = 2, 3$, we color uv_1 with a color in $\{4, 5, 6\} \backslash (C_\phi(u_2) \cup C_\phi(u_3))$. If $|\{4, 5, 6\} \cap C_\phi(u_2)| \ge 2$ and $|\{4, 5, 6\} \cap C_\phi(u_3)| \le 1$, say, we color vu_1 with a color in $\{4, 5, 6\} \backslash C_\phi(u_3)$.

(C2) G contains a path $x_1 x_2 \cdots x_n$ with $d_G(x_1) \ne 2$, $d_G(x_n) \ne 2$, and $d_G(x_i) = 2$ for all $i = 2, 3, \dots, n-1$, where $n \ge 4$.

By the induction assumption or Theorem 9, $G - x_2x_3$ has a total-$(\Delta(G)+2)$-avd-coloring ϕ with the color set $C = \{1, 2, \dots, \Delta(G) + 2\}$.

If $n = 4$, we recolor x_2 with a color $a \in C \setminus \{\phi(x_1), \phi(x_3), \phi(x_1x_2), \phi(x_3x_4)\}$, and color x_2x_3 with a color in $C \setminus \{a, \phi(x_3), \phi(x_1x_2), \phi(x_3x_4)\}$.

If $n \ge 5$, we recolor x_3x_4 with $a \in C \setminus \{\phi(x_2), \phi(x_4), \phi(x_5), \phi(x_4x_5)\}$, x_3 with $b \in C \setminus \{a, \phi(x_2), \phi(x_4), \phi(x_4x_5)\}$, and color x_2x_3 with a color in $C \setminus \{a, b, \phi(x_2), \phi(x_1x_2)\}$.

(C3) G contains a vertex v with neighbors $v_1, v_2, \dots, v_k$, $k \ge 4$, such that $d_G(v_1) = 1$ and $d_G(v_i) \le 2$ for all $i = 2, 4, \dots, k-2$.

For $2 \le i \le k-2$, if v_i is a 2-vertex, we denote by $u_i \ne v$ the second neighbor of v_i. It follows from (C2) that $d_G(u_i) \ge 3$. By the induction assumption or Theorem 9, $G - v_1$ has a total-$(\Delta(G) + 2)$-avd-coloring ϕ with the color set $C = \{1, 2, \dots, \Delta(G) + 2\}$. We may assume that $\phi(v) = 1$, $\phi(vv_i) = i$ for $i = 2, 3, \dots, k$. Since $\Delta(G) \ge d_G(v) = k$, $|C| \ge \Delta(G) + 2 \ge k + 2$. Thus, $k + 1, k + 2 \in C$.

If $k + 1 \in C_\phi(v_{k-1}) \cap C_\phi(v_k)$, we color vv_1 with $k + 2$. If $k + 1 \notin C_\phi(v_{k-1}) \cup C_\phi(v_k)$, we color vv_1 with $k + 1$. The similar argument works for the color $k + 2$. If $\{k + 1, k + 2\} \subseteq C_\phi(v_{k-1}) \backslash C_\phi(v_k)$ or $\{k + 1, k + 2\} \subseteq C_\phi(v_k) \backslash C_\phi(v_{k-1})$, we color vv_1 with $k + 1$.

Now suppose that $k + 1 \in C_\phi(v_{k-1}) \backslash C_\phi(v_k)$ and $k + 2 \in C_\phi(v_k) \backslash C_\phi(v_{k-1})$, say. If $d_G(v_2) = 1$, we recolor (or color) vv_2 with $k+1$ and vv_1 with $k+2$. If $d_G(v_2) = 2$, we recolor (or color) vv_2 with a color $a \in \{k + 1, k + 2\} \backslash \{\phi(v_2u_2)\}$, vv_1 with a color in $\{k + 1, k + 2\} \backslash \{a\}$, and v_2 with a color different from $1, a, \phi(u_2), \phi(u_2v_2)$.

(C4) G contains a 3-face $[uv_1v_2]$ with $d_G(u) = 2$ and $d_G(v_1) = 3$.

Let $z \ne u, v_2$ be the third neighbor of v_1. Let $y_1, \dots, y_m$ be the neighbors of v_2 different from u and v_1, where $m \ge 1$. By the induction assumption or Theorem 9, $G - uv_1$ has a total-$(\Delta(G) + 2)$-avd-coloring ϕ with $C = \{1, 2, \dots, \Delta(G) + 2\}$.

If $m = 1$, the proof is similar to the case (B3) in Theorem 9.

Assume that $m \geq 2$. If $d_G(z) \neq 3$, we color properly uv_1. Assume that $d_G(z) = 3$. If $\phi(z) \neq \phi(v_1v_2)$, we color uv_1 with a color different from those of $z, v_1, v_1v_2, zv_1, uv_2$. Otherwise, we recolor v_1 with a color $a \in C \setminus (C_\phi(z) \cup \{\phi(v_2)\})$, and then color properly uv_1. Since $|C_\phi(z) \cup \{\phi(v_2)\}| \leq 4 + 1 = 5$ and $|C| \geq 6$, the extended coloring is feasible.

(C5) G contains two 3-faces $[u_1v_1x]$ and $[u_2v_2x]$ such that $d_G(x) = 4$ and $d_G(u_1) = d_G(u_2) = 2$.

Based on the proofs of (C2) and (C4), we may assume that $d_G(v_i) \geq 4$ for $i = 1, 2$. Let $z_1, z_2, \ldots, z_m$ be the neighbors of v_1 different from x and u_1. Let $y_1, y_2, \ldots, y_n$ be the neighbors of v_2 different from x and u_2. Then, $m \geq 2$ and $n \geq 2$.

If $m, n \geq 3$, then any total-$(\Delta(G) + 2)$-avd-coloring of $G - xu_1$ can be easily extended to the whole graph G. Otherwise, assume that $n \geq 2$ and $m = 2$ by symmetry. By the induction assumption or Theorem 9, $G - xu_1$ has a total-$(\Delta(G) + 2)$-avd-coloring ϕ with $C = \{1, 2, \ldots, \Delta(G) + 2\}$. Let $\phi(v_1) = 1$, $\phi(v_1z_1) = 2$, $\phi(v_1z_2) = 3$, $\phi(v_1x) = 4$, and $\phi(v_1u_1) = 5$. This implies that $6 \notin C_\phi(v_1)$. We need to consider two subcases as follows:

(C5.1) $n \geq 3$. If $6 \notin \{\phi(x), \phi(xu_2), \phi(xv_2)\}$, we color xu_1 with 6. Otherwise, we properly color xu_1.

(C5.2) $n = 2$. Since $|C_\phi(v_2)| = 5$ and $|C| \geq 6$, there is some color $a \in C \setminus C_\phi(v_2)$.

(i) $a \geq 6$. If $a \notin C_\phi(x)$, we color xu_1 with a; Otherwise, we color properly xu_1.

(ii) $1 \leq a \leq 5$. First, assume that $a \in C_\phi(x)$; especially, this is true for $a = 4$. If $6 \notin C_\phi(x)$, we color xu_1 with 6; otherwise, we color properly xu_1. Next, assume that $a \notin C_\phi(x)$. We recolor xu_2 with a and then reduce the proof to the previous case.

□

Theorem 12 *If G is an outerplane graph with $\Delta(G) \geq 4$ and without adjacent vertices of maximum degree, then $\chi''_a(G) = \Delta(G) + 1$.*

Proof It is evident that $\chi''_a(G) \geq \Delta(G) + 1$. It suffices to show that $\chi''_a(G) \leq \Delta(G) + 1$ by induction on $|T(G)|$. If $|T(G)| \leq 5$, the theorem holds clearly. Suppose that G is a connected outerplane graph with $\Delta(G) \geq 4$ and $|T(G)| \geq 6$ and without adjacent vertices of maximum degree. We divide the proof into the following two parts:

Part 1 $\Delta(G) \geq 5$.

By Lemma 5, G contains one of the configurations (C1)–(C5). We know that $|C| = \Delta(G) + 1 \geq 6$ in this case.

If G contains (C1), (C2), (C4) or (C5), the proof is analogous to the corresponding case of Theorem 11.

Assume that G contains (C3), i.e., a vertex v with neighbors $v_1, v_2, \ldots, v_k$, $k \geqslant 4$, such that $d_G(v_1) = 1$ and $d_G(v_i) \leq 2$ for all $i = 2, 4, \ldots, k - 2$. By the induction assumption or Theorem 11, $G - v_1$ has a total-$(\Delta(G) + 1)$-avd-coloring ϕ with the color set $C = \{1, 2, \ldots, \Delta(G) + 1\}$. If $k = d_G(v) < \Delta(G)$, then $|C| = \Delta(G) + 1 \geq$

$k+2$, and hence we can give a similar proof as in the case (C3) of Theorem 11. If $k = d_G(v) = \Delta(G)$, then $d_G(v_i) < \Delta(G)$ for $i = k-1, k$ by the assumption, we only need to color properly the edge vv_1.

Part 2 $\Delta(G) = 4$.

In this case, $|C| = 5$. By Lemma 7, G contains one of the configurations (A1)–(A4).

If G contains (A2), then the proof is similar to the Case (B1) in Theorem 9.

If G contains (A3), the proof is similar to the case (C2) in Theorem 11.

Assume that G contains (A1), i.e., a vertex v with $d_G(v) \neq 3$ adjacent to a leaf u. If $d_G(v) \leq 2$, the proof is similar to the case (C1) in Theorem 11. If $d_G(v) = 4$, then since every neighbor of v is not a 4-vertex, any total-5-avd-coloring of $G - uv$ can be extended to the whole graph G.

If G contains (A4), i.e., a 3-face $[uxy]$ with $d_G(x) = 3$ such that either $d_G(u) = 2$, or $d_G(u) = 3$ and u is adjacent to a leaf u', let $x' \neq u, y$ be the third neighbor of x. By (A3), we may assume that $d_G(y) \geq 3$. We consider the following two cases, depending on the value of $d_G(y)$.

Case 1. $d_G(y) = 4$

If $d_G(u) = 3$, we let $H = G - u'$. Then H is an outerplane graph with $\Delta(H) = 4$ and without adjacent 4-vertices. By the induction assumption, H has a total-5-avd-coloring ϕ with the color set $C = \{1, 2, \ldots, 5\}$. Suppose that $\phi(x) = 1$, $\phi(xy) = 2$, $\phi(xu) = 3$, and $\phi(xx') = 4$. If $5 \in \{\phi(u), \phi(uy)\}$, we properly color uu'; otherwise, we color uu' with 5.

If $d_G(u) = 2$, we let $H = G - xu$ and let ϕ be a total-5-avd-coloring of H with the color set $C = \{1, 2, \ldots, 5\}$. If $d_G(x') \neq 3$, we properly color xu. Assume that $d_G(x') = 3$ and let $C_\phi(x') = \{1, 2, 3, 4\}$. If $5 \notin \{\phi(x), \phi(xy), \phi(uy)\}$, we color xu with 5. If $5 \in \{\phi(x), \phi(xy)\}$, we properly color xu. If $5 \notin \{\phi(x), \phi(xy)\}$ and $\phi(uy) = 5$, we exchange the colors of uy and xy, then properly color xu.

Case 2. $d_G(y) = 3$

Let $y' \neq u, x$ be the third neighbor of y. By the induction assumption, $G - \{u, xy\}$ has a total-5-avd-coloring ϕ with the color set $C = \{1, 2, \ldots, 5\}$. If $d_G(x') = d_G(y') = 3$, the proof is similar to the case (B3) in Theorem 9. Thus, without loss of generality, assume that $d_G(y') \neq 3$, and let $\phi(x') = 1$, $\phi(xx') = 2$, $\phi(y') = a$, and $\phi(yy') = b$. Moreover, if $d_G(x') = 3$, we suppose that $C_\phi(x') = \{1, 2, 3, 4\}$. By symmetry, it suffices to consider the following subcases, depending on the values of a and b:

(2.1) $a = 1$. If $b = 2$, we color xu with 1, $\{y, uu'\}$ with 3, $\{x, uy\}$ with 4, and $\{xy, u\}$ with 5. If $b = 3$, we exchange the colors of y and yy' in the previous case. If $b = 5$, we color uy with 1, $\{y, uu'\}$ with 2, $\{u, xy\}$ with 3, xu with 4, and x with 5.

(2.2) $a = 2$. If $b = 1$, we color u with 1, uu' with 2, $\{y, xu\}$ with 3, $\{x, uy\}$ with 4, and xy with 5. If $b = 3$, we color $\{y, xu\}$ with 1, u with 2, uu' with 3, $\{x, uy\}$ with 4, and xy with 5. If $b = 5$, we color $\{u, xy\}$ with 1, uy with 2, xu with 3, $\{y, uu'\}$ with 4, and x with 5.

(2.3) $a = 3$. If $b = 1$, we color uu' with 1, y with 2, xu with 3, $\{x, uy\}$ with 4, and $\{u, xy\}$ with 5. If $b = 2$, we exchange the colors of y and yy' in the previous case. If $b = 4$, we color $\{y, uu'\}$ with 1, uy with 2, xu with 3, x with 4, and $\{u, xy\}$ with 5. If $b = 5$, we color $\{u, xy\}$ with 1, uy with 2, xu with 3, $\{y, uu'\}$ with 4, and x with 5.

(2.4) $a = 5$. If $b = 1$, we color u with 1, y with 2, $\{xy, uu'\}$ with 3, $\{x, uy\}$ with 4, and xu with 5. If $b = 2$, we color uy with 1, u with 2, xy with 3, $\{y, xu\}$ with 4, and $\{x, uu'\}$ with 5. If $b = 3$, we color xy with 1, u with 2, uu' with 3, $\{y, xu\}$ with 4, and $\{x, uy\}$ with 5.

□

By Proposition 3, Theorems 11 and 12, we complete the proof of Main Theorem for the case $\Delta(G) \geq 4$. □

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