Deriving the gravitational field equation and horizon entropy for arbitrary diffeomorphism-invariant gravity from spacetime solid

Shao-Feng Wu[1*], Bin Wang[2], Xian-Hui Ge[1], and Guo-Hong Yang[1]

[1] *Department of Physics, Shanghai University, Shanghai 200444, China*

[2] *Department of Physics, Fudan University, Shanghai 200433, China*

Abstract

Motivated by the analogy between the spacetime and the solid with inhomogeneous elasticity modulus, we present an alternative method to obtain the field equation of any diffeomorphism-invariant gravity, by extremizing the constructed entropy function of the displacement vector field of spacetime solid. In general stationary spacetimes, we show that the Wald entropy of horizon arises from the on-shell entropy function of spacetime solid.

PACS numbers: 04.70.Dy; 04.20.Fy; 04.50.-h

[*] Corresponding author. Email: sfwu@shu.edu.cn; Phone: +86-021-66136202.

I. INTRODUCTION

There has been growing interest recently in studying the relation between gravitational field equations describing bulk spacetime dynamics and horizon thermodynamics. A pioneer work on this topic was done in [1] where Einstein's equation emerges as an equation of state from the basic thermodynamic relation in the Rindler spacetime. A lot of effort has been spent to quantify this relation including studies in the so called $f(\mathbf{R})$ gravity [2] and the scalar tensor gravity [3]. Moreover, it has been found that the relation between gravity and thermodynamics exists also in other spacetimes, including a general static spherically symmetric spacetime [4], the dynamical Vaidya spacetime [5] and cosmological spacetime [6–9], where more modified gravity theories like Lovelock gravity [4] and braneworld gravity [10–12] have been described by the first law of thermodynamics on the black hole horizon or cosmological apparent horizon. Recently, it was further found that for arbitrary diffeomorphism-invariant gravity theories, the field equation can be obtained as a state equation of Rindler horizon thermodynamics [13–15]. The disclosed relation between horizon thermodynamics and bulk gravitational field equation can shed the light on holography [16] and even may change the understanding of gravity. In fact, the puzzling thermodynamic feature of gravity and/or spacetime is one important motivation of the proposal that gravity might not be a fundamental interaction but rather an emergent large scale/numbers phenomenon [17].

In thermodynamics, the equations governing the equilibrium state of the system can be obtained by the extremization of the entropy function. In terms of the connection between gravity and thermodynamics, it seems natural that in the gravity context one could also define an entropy function(al) for spacetime and derive the gravitational field equation from it to describe the spacetime. A novel approach to realize this idea was presented in [18, 19], where they made an analogy between the spacetime and an elastic solid. The idea of spacetime-solid analogy is not new but has a long history, since Sakharov's induced gravity, see [20]. At the microscopic level of the analogous picture, its associated degrees of freedom, the 'spacetime atoms', are still elusive, but its existence is crucial to interpret the thermal phenomenon associated with spacetime. At the macroscopic level, what may be really interesting is that, one can expect to develop a theory of elasticity phenomenologically to describe the spacetime solid, envisaging gravity as analogous to the elasticity. Obviously, the fundamental dynamical variable of spacetime solid is the displacement vector field which describes the elastic deformation of the solid, while not the metric, which may be thought of as a coarse grained description of the spacetime at macroscopic scales, somewhat like the density of a solid. Although the analogy is not strong, Padmanabhan et al. [18, 19] successfully constructed an entropy function of a displacement vector field and obtained the correct field equations of Einstein and Lovelock gravity theories by a special variational principle on the entropy function. Moreover, when evaluated on-shell for a solution admitting a horizon, the extreme value of the entropy agrees with standard Wald entropy [21, 22]. This justifies that the constructed elastic entropy has the significance of gravity entropy. Beside further supporting the deep connection between gravity and thermodynamics, this approach is attractive since it has the following two ingredients: (a) the metric is not the dynamical variable; (b) the field equations remain invariant under the shift $T_{ab} \to T_{ab} + \bar{\lambda} g_{ab}$ of the matter energy-momentum tensor T_{ab}. It was pointed out that these ingredients may be important to solve the cosmological constant problem [19].

The spacetime-solid analogy formalism proposed in [18, 19] leads to the gravitational field equations restricted in Einstein theory and Lovelock theory. The standard Wald entropy in [18] is obtained only on the local Rindler frame. Considering that the thermodynamic derivation of field equation can be applicable to arbitrary diffeomorphism-invariant gravity theories, it is natural to ask: Does the proposed formalism hold in all spacetimes beyond Einstein gravity and Lovelock theory? Does it imply something in deep? In this work we are going to address these questions. We will analogize the spacetime as an inhomogeneous elastic solid, and develop a general formalism to obtain the theory for arbitrary diffeomorphism-invariant gravity. Taking the homogeneous limit, our formalism can reduce to obtain the Einstein and Lovelock gravity theories got in [18].

The organization of the paper is the following. In section II, we define the entropy function of a displacement vector field in the spacetime solid for arbitrary diffeomorphism-invariant gravity. Then in section III, we extremize the entropy function to derive the equilibrium equation. In section IV, we compute the extreme of entropy function on general static and non-static spacetimes and show that under appropriate circumstances it is identical to the expression for the Wald horizon entropy. We conclude our results in the last section.

II. ENTROPY FUNCTION FOR ANY DIFFEOMORPHISM-INVARIANT GRAVITY

Now we will describe the spacetime as a deformed solid at the macroscopic level. Usually, to describe a general deformable solid with the theory of elasticity, one can define the thermodynamical function, like the entropy (or free energy, internal energy etc.) with the displacement vector field $\xi^a(x)$ which describes the elastic displacement of the solid through the equation $x^a \rightarrow x^a + \xi^a(x)$, to capture the relevant dynamics in the long-wavelength limit. Varying the thermodynamical function with respect to ξ^a, one can obtain the equilibrium equation of ξ^a (see the standard book of elastic mechanics by Landau and Lifshitz [23]). Analogizing spacetime to solid, the equilibrium equation of spacetime solid can be expected. However, what is remarkable is that the field equations will appear from the equilibrium equation [18]. Before achieving this, the key task is how to specify the thermodynamical function of spacetime.

There are several restrictions for the form of thermodynamical function. At first, the thermodynamical function of the spacetime solid should be a scalar, preserving the covariance of equilibrium equation. Second, in the theory of elasticity, the thermodynamical function can be written as an integral over a quadratic function of small strain tensor, preserving the function density to be translationally invariant. In [18] the entropy function of spacetime solid was proposed as

$$S_g[\xi] \sim \int_V d^D x \sqrt{-g} \left(P^{abcd} \nabla_c \xi_b \nabla_d \xi_b\right), \tag{1}$$

by treating $\nabla_a \xi_b$ as the strain tensor and P^{abcd} as the elasticity modulus. However, in the presence of non-gravitational matter distribution in spacetime, one can not demand the translational invariance of entropy density. Hence, the entropy density can have quadratic terms in both the derivatives $\nabla_a \xi_b$ as well as ξ_b itself. So it was proposed that the total entropy function should also include the matter entropy function [18]

$$S_m[\xi] \sim \int_V d^D x \sqrt{-g} T^{ab} \xi_a \xi_b.$$

Assuming the "constancy" conditions for elasticity modulus $\nabla_d P^{abcd} = 0$ analogous to a homogeneous (and isotropy) solid, it was argued that the total entropy function can be used to describe the Einstein and Lovelock gravity [18].

In our study here we are going to release the constraint $\nabla_d P^{abcd} = 0$, which is analogous to treat the spacetime as an inhomogeneous solid. By this nontrivial extension we will achieve to obtain gravitational field equations for more general gravity theories. Before constructing the concrete form of entropy function, it is natural to expect to get some hints from the inhomogeneous elastic theory. In mechanics of materials, the single and homogeneous materials can be fabricated as piezocomposite materials. The interface between materials produces an uneven distribution of stresses which reduces the electric-field-induced displacement characteristics. It is interesting to see that these materials, the called Functionally Graded Materials [24], have a total free energy:

$$F = \int_V d^3x \left(\frac{1}{2} P^{ijkl} \varepsilon_{ij} \varepsilon_{kl} - \varepsilon^{ijk} \varepsilon_{ij} E_k - \frac{1}{2} K^{ij} E_i E_j \right), \tag{2}$$

where ε_{ij} denotes the components of the strain tensor, E_i is the component of the electric field vector related with the displacement vector by the constitutive equations. The quantities P^{ijkl} and ε_{ijk} represent the elastic and the piezoelectric modulus, respectively, and K^{ij} is the electric permittivity. Motivated by this free energy, we present that a general entropy function of spacetime solid should be

$$S_g\left[\xi\right] \sim \int_V d^D x \sqrt{-g} \left(P^{abcd} \nabla_c \xi_a \nabla_d \xi_b + \nabla_d P^{abcd} \nabla_c \xi_a \xi_b + \nabla_c \nabla_d P^{acbd} \xi_a \xi_b \right). \tag{3}$$

Thus roughly one may make the analogy of the last two terms in (3) as the piezoelectric effect of inhomogeneous spacetime solid.

We need to determine the elasticity modulus P^{abcd}, especially its symmetry which, in the usual elastic solid, will characterize symmetry of structures of crystals, such as monoclinic and tetragonal, etc. However, the concrete form of P^{abcd} only can be determined in a complete theory by the long wavelength limit of the microscopic theory just as the elastic constants can in principle be determined from the microscopic theory of the lattice. In macroscopic level, one can only know that the structure of the gravitational sector is encoded in the form of P^{abcd}, so the object P^{abcd} should be built out of metric and other geometric quantities. Such a tensor can be constructed as a series in the powers of the derivatives of the metric. One may expect that the lowest order term leads to Einstein's theory while higher order terms come from the quantum corrections of underlying microscopic theory. In [18], assuming P^{abcd} has symmetry with Riemann tensor R^{abcd} and impose $\nabla_d P^{abcd} = 0$, the lowest and second order correspond to Einstein's theory and Gauss-Bonnet theory (then Lovelock theory), respectively. However, if we release the condition $\nabla_d P^{abcd} = 0$, more general tensor are possible. The general construction of P^{abcd} should be universal for arbitrary diffeomorphism-invariant gravity, including the tensor for Einstein gravity and Lovelock gravity as our special cases. Assuming the symmetry of P^{abcd} in identical with R^{abcd}, a simplest candidate is

$$P^{abcd} = \frac{\partial L}{\partial R_{abcd}}, \tag{4}$$

where L is the Lagrangian of gravity theories.

Finally, the total entropy can be written as

$$\begin{aligned} S[\xi] &= S_g\left[\xi\right] + S_m\left[\xi\right] \qquad (5)\\ &= 4\int_V d^D x\sqrt{-g}\left(P^{abcd}\nabla_c\xi_a\nabla_d\xi_b + \nabla_d P^{abcd}\nabla_c\xi_a\xi_b + \nabla_c\nabla_d P^{acbd}\xi_a\xi_b - \frac{1}{4}T^{ab}\xi_a\xi_b\right), \end{aligned}$$

where some proportional constants are chosen with hindsight. It should be pointed out that the entropy function Eq. (1) for Einstein and Lovelock gravity and their generalization Eq. (5) for general gravity can not be fixed completely by the analogy with elastic theory. One can understand that the entropy functions Eqs. (1) and (5) are constructed phenomenologically. One of the desired phenomenon is to obtain the field equation from the equilibrium equation. The other is to justify the entropy function with the significance of gravity entropy, which is implemented by identifying the on-shell entropy function with the Wald horizon entropy. We will show both of them in sections below.

III. FIELD EQUATIONS FROM EXTREMIZING THE ENTROPY

In this section, we will derive the equilibrium equation of spacetime solid. Although the expression (5) is well defined for any displacement vector field, one can only obtain significant results for suitably chosen vector fields. The vector is required to characterize the special property of spacetime solid. The most nontrivial property of spacetime is the existence of the horizons which act as one-way membranes which block information for a specific class of observers. The existence of horizons, which are null hypersurfaces, is a feature of any geometrical theory of gravity and is reasonably independent of the field equations. We hence assume that the spacetime solid is deformed induced by the change of horizon. As the simplest case, one can consider that the matter is freely falling into the horizon along the transverse invariant ingoing geodesics. Alternatively, one can also consider a virtual displacement of horizon radially normal to itself engulfing the matter. Obviously in this case the total displacement of spacetime solid is induced by the change of horizon. On the true horizon, this change, i.e. the displacement vector ξ_a, should be characterized by its outward null normal vectors. Actually, we will derive the correct field equations when we impose ξ_a as null vectors and obtain the entropy of horizon when we use the outward unit normal of a near horizon surface to approach the null normal of true horizon.

Varying the vector field ξ_a after adding a Lagrangian multiplier $\lambda(x)$ for imposing ξ_a with constant null norm $\delta\left(\xi_a\xi^a\right)=0$, we find:

$$\begin{aligned}
\delta S[\xi] = 4\int_V d^Dx\sqrt{-g}[&2P^{abcd}\nabla_c\xi_a\nabla_d\delta\xi_b + \nabla_d P^{abcd}\nabla_c\xi_a\delta\xi_b + \nabla_d P^{abcd}\nabla_c\delta\xi_a\xi_b \\
&+\nabla_c\nabla_d P^{acbd}(\delta\xi_a\xi_b+\xi_a\delta\xi_b) - \frac{1}{2}\left(T^{ab}+\lambda g^{ab}\right)\xi_a\delta\xi_b] \\
= 4\int_V d^Dx\sqrt{-g}[&2\nabla_d(P^{abcd}\nabla_c\xi_a\delta\xi_b) - 2P^{abcd}\nabla_d\nabla_c\xi_a\delta\xi_b - 2\nabla_d P^{abcd}\nabla_c\xi_a\delta\xi_b + \nabla_d P^{abcd}\nabla_c\xi_a\delta\xi_b \\
&+\nabla_c(\nabla_d P^{abcd}\delta\xi_a\xi_b) - \nabla_c\nabla_d P^{abcd}\delta\xi_a\xi_b - \nabla_d P^{abcd}\delta\xi_a\nabla_c\xi_b - \nabla_c\nabla_d\left(2P^{acdb}+P^{abcd}\right)\xi_b\delta\xi_a \\
&-\frac{1}{2}\left(T^{ab}+\lambda g^{ab}\right)\xi_a\delta\xi_b].
\end{aligned}$$

where we have used the symmetry of Riemann tensor $P^{abcd}=P^{[ab][cd]}$, $P^{abcd}=P^{cdab}$. And we get the second to the last term above following

$$\nabla_c\nabla_d P^{acbd}(\delta\xi_a\xi_b+\xi_a\delta\xi_b) = -\nabla_c\nabla_d\left(P^{cabd}+P^{cbad}\right)\xi_a\delta\xi_b = -\nabla_c\nabla_d\left(2P^{acdb}+P^{abcd}\right)\xi_a\delta\xi_b, \tag{6}$$

by using rotational symmetry of $P^{a[bcd]}=0$. Then $\delta S[\xi]$ can be simplified as

$$\begin{aligned}
\delta S[\xi] = 4\int_V d^Dx\sqrt{-g}[&2\nabla_d(P^{abcd}\nabla_c\xi_a\delta\xi_b) + \nabla_c(\nabla_d P^{abcd}\delta\xi_a\xi_b) - 2P^{abcd}\nabla_d\nabla_c\xi_a\delta\xi_b \\
&-2\nabla_c\nabla_d P^{acdb}\xi_a\delta\xi_b - \frac{1}{2}\left(T^{ab}+\lambda g^{ab}\right)\xi_a\delta\xi_b],
\end{aligned}$$

which leads, under the stokes theorem,

$$\begin{aligned}
\delta S[\xi] = &4\int_{\partial V} d^{D-1}x\sqrt{h}[2n_d(P^{abcd}\nabla_c\xi_a\delta\xi_b) + n_c(\nabla_d P^{abcd}\delta\xi_a\xi_b)] \\
&-8\int_V d^Dx\sqrt{-g}[P^{abcd}\nabla_d\nabla_c\xi_a\delta\xi_b + \nabla_c\nabla_d P^{acdb}\xi_a\delta\xi_b + \frac{1}{4}\left(T^{ab}+\lambda g^{ab}\right)\xi_a\delta\xi_b],
\end{aligned}$$

where n_a is the vector outward normal to boundary ∂V and h is the determinant of the intrinsic metric on ∂V. As usual, we set the variation $\delta\xi_a$ to zero at boundary (Even though the ξ_a is not fixed on the boundary, we still do not care about the boundary term since it only gives the boundary condition of ξ_a, which will not affect the bulk equilibrium equation which is independent of ξ_a as we will show.). Therefore the first integration in the above equation vanishes, and the condition that $S[\xi]$ be an extremum for arbitrary variations of ξ_a leads

$$\begin{aligned}\delta S[\xi] &= -8\int_V d^Dx\sqrt{-g}\left[P^{abcd}\nabla_d\nabla_c\xi_a\delta\xi_b + \nabla_c\nabla_d P^{acdb}\xi_a\delta\xi_b + \frac{1}{4}\left(T^{ab}+\lambda g^{ab}\right)\xi_a\delta\xi_b\right] \\ &= 4\int_V d^Dx\sqrt{-g}\left[P^{bedc}R^a_{\ edc} - 2\nabla_c\nabla_d P^{acdb} - \frac{1}{2}\left(T^{ab}+\lambda g^{ab}\right)\right]\xi_a\delta\xi_b, \end{aligned} \tag{7}$$

where we have used the definition of the Riemann tensor in terms of commutator of covariant derivatives, as well as some alteration of index. One can find that the equilibrium equation that follows from our variation principle of entropy function is

$$\left(P^{bedc}R^a_{\ edc} - 2\nabla_c\nabla_d P^{acdb} - \frac{1}{2}T^{ab} - \frac{1}{2}\lambda g^{ab}\right)\xi_a = 0. \tag{8}$$

Requiring the condition (8) to hold for arbitrary null vector field ξ_b (It is interesting to note that this requirement is also invoked in the derivation of field equation as a state equation in [1]), one finds that the equilibrium equation from extremizing the total entropy is reduced to

$$P^{bedc}R^a_{\ edc} - 2\nabla_c\nabla_d P^{acdb} - \frac{1}{2}T^{ab} - \frac{1}{2}\lambda g^{ab} = 0. \tag{9}$$

This is a remarkable result that we have obtained a dynamical equation governing the background instead of the usual equilibrium equation determine the displacement vector field. This uninstinctive situation happens because the symmetry of tensor P^{abcd} and the entropy function are so special that Eq. (8) does not contain derivatives with respect to ξ^a.

By demanding conservation of the stress tensor and using the Bianchi identities, one can find that the Lagrangian multiplier

$$\lambda = L - \frac{\Lambda}{8\pi G}, \tag{10}$$

where Λ is an integration constant and G is the Newton gravitational constant. One can immediately see that the equation (9) is just the exact field equation derived from ordinary variational principle from the action of arbitrary diffeomorphism invariant theories of gravity [22] (involving no more than the second derivatives of the spacetime metric g_{ab} and the first derivatives of the matter fields Ψ_m)

$$I = \int_V d^Dx\sqrt{-g}L(g_{ab}, R_{abcd}, \Psi_m, \nabla_a\Psi_m),$$

supplemented by appropriate generalizations of Gibbons-Hawking-like boundary terms. One can find two crucial features of our derivation that, the variational principle is based on a vector with constant norm in spacetime instead of the usual metric field g_{ab}, and the equilibrium equation is invariant under the shift $T_{ab} \to T_{ab} + \bar{\lambda} g_{ab}$ since $\bar{\lambda} g_{ab} \xi^a \xi^b = \bar{\lambda}\varepsilon$ is not varied when ξ^a is varied, regardless of whether $\nabla_d P^{abcd} = 0$.

IV. ON-SHELL ENTROPY FUNCTION

The result in the previous section provides an alternative variational principle in deriving field equations of arbitrary diffeomorphism invariant theories of gravity. In the following, we will show that the boundary term of entropy function $S[\xi]$ will lead to the standard Wald entropy. The specific case on this topic was discussed in [18].

Manipulating the covariant derivatives of Eq. (5), the entropy function can be rewritten as

$$\begin{aligned}
S[\xi] &= 4\int_V d^D x\sqrt{-g}(P^{abcd}\nabla_c\xi_a\nabla_d\xi_b + \nabla_d P^{abcd}\nabla_c\xi_a\xi_b + \nabla_c\nabla_d P^{acbd}\xi_a\xi_b - \frac{1}{4}T^{ab}\xi_a\xi_b) \\
&= 4\int_V d^D x\sqrt{-g}\nabla_d\left(P^{abcd}\nabla_c\xi_a\xi_b\right) - 4\int_V d^D x\sqrt{-g}(P^{abcd}\nabla_d\nabla_c\xi_a\xi_b + \nabla_c\nabla_d P^{acbd}\xi_a\xi_b - \frac{1}{4}T^{ab}\xi_a\xi_b) \\
&= 4\int_{\partial V} d^{D-1}x n_d\left(P^{abcd}\nabla_c\xi_a\xi_b\right) + 2\int_V d^D x\sqrt{-g}(P^{bedc}R^a{}_{edc} - 2\nabla_c\nabla_d P^{acdb} - \frac{1}{2}T^{ab})\xi_b\xi_a \\
&= 4\int_{\partial V} d^{D-1}x n_d\left(P^{abcd}\nabla_c\xi_a\xi_b\right) + \int_V d^D x\sqrt{-g}\lambda\varepsilon, \qquad (11)
\end{aligned}$$

where similar derivations of Eq. (6) has been used in the second equality, and the field equation (9) has been used in getting the last line. This result shows that, as mentioned in [15], the second term of third line can be thought of as the entropy in any diffeomorphism-invariant gravity theory, except a total divergence. Then one can recover the entropy function Eq. (5) by reversing the derivation in Eq. (11).

One can find that the final result of Eq. (11) is not affected by $\nabla_d P^{abcd} \neq 0$. The second term of the last line vanishes since $\varepsilon = 0$ (Even if ε is a nonvanishing constant, it is not a surface term [18, 19]). Therefore, we will concentrate on the first term, which will be interpreted as the surface entropy of horizon.

At this stage, we have not put any restriction of the boundary ∂V. In general, the boundary is $(D-1)$-dimensional. Hence the entropy function should not be the desired entropy of some $(D-2)$-dimensional section of horizons. However, it was pointed out in [18] that, when part of the boundary ∂V is null, it is intrinsically $(D-2)$-dimensional. This case needs to be handled by a limiting procedure from near horizon to true horizon.

In [18], the standard Wald entropy of Einstein and Lovelock gravities are obtained from the boundary term on the local Rindler frame through a limiting process to the null Rindler horizon. They also mentioned that the same result can be recovered for any static spherically symmetric spacetime. Here we will extend their discussion to the general static spacetime without spherical symmetry and even to the stationary but non-static spacetime.

We will briefly introduce the coordinate system which is suited for the discussion of the general static spacetime (see the details in [25]). Given the general static spacetime, one can decompose the metric into a block-diagonal form as follows

$$ds^2 = -N^2 dt^2 + g_{\mu\nu} dx^\mu dx^\nu, \ \mu, \nu = 1, 2, \cdots$$

We can arbitrarily choose a particular $(D-2)$-surface in the constant-time slice and utilize Gaussian normal coordinates in the surrounding region,

$$g_{\mu\nu} dx^\mu dx^\nu = dn^2 + g_{AB} dy^A dy^B, \ A, B = 2, 3, \cdots$$

where n represents the spatial direction normal to the specified $(D-2)$-surface. The Killing horizon H, generated by the timelike Killing vector field $\chi = \partial_t$ is approached as $N^2 \to 0$. One can verify that $\kappa \equiv \partial_n N|_{n\to 0}$ complies with the standard version of the surface gravity. This enables us to write a near horizon Taylor expansion for the lapse $N(n, y) = \kappa n + O(n^2)$. Since we want the horizon to be regular and not possess a curvature singularity, some curvature invariants must remain finite in the horizon limit, which enables us to refine the expansion for the lapse as

$$N(n, y) = \kappa n + \frac{\kappa_2(y)}{3!} n^3 + O(n^4), \tag{12}$$

and write

$$g_{AB}(n, y) = [g_H]_{AB}(y) + \frac{[g_2]_{AB}(y)}{2!} n^2 + O(n^3). \tag{13}$$

One can find that the metric (12) and (13) are more general than the metric of local Rindler frame in the presence of κ_2 and $[g_2]_{ab}$. The later directly leads to the nonvanishing component of $\nabla_b \xi_d$, not only $\nabla_t \xi_t$, which makes the identification in [18] become invalid between the entropy function and Wald entropy through introducing binormal to the cross section of H.

We shall evaluate the surface integral for $S[\xi]_{on-shell}$ near Killing horizon H on a timelike surface Σ, which is denoted as n =constant and the neighboring spacetime is described by metric (12) and (13). This timelike surface Σ can be called as the stretched horizon [26, 27], which has a non-singular induced metric and then provides a more tractable boundary than the true horizon. A rigorous one-to-one correspondence between points on the true and stretched horizons can be realized by using, for example, the null rays that pierce both surfaces. Roughly, we can expect that the on-shell entropy will match the Wald entropy under the limit $n \to 0$ in the end of the calculation. Take $\xi^a = n^a$ as the spacelike normal to these surfaces Σ, with components

$$\xi^a = n^a = (0, 1, 0, 0, \ldots) \tag{14}$$

and unit norm. In the limiting process the spacelike unit vector will be a null vector, which denotes that we are considering the null surface as a limit of a sequence of timelike surfaces. The metric determinant h of these surfaces Σ can be decomposed as $\sqrt{h} = N\sqrt{\sigma}$, where σ is the metric determinant on the transverse spatial surfaces, having the limit on the true horizon $\sqrt{\sigma} \to \sqrt{g_H}$.

Now we will check whether the surface term

$$S_{Pad} = 4 \int_\Sigma d^{D-1}x\sqrt{h} n_d \left(P^{abcd} \nabla_c \xi_a \xi_b\right) \tag{15}$$

will be reduced to Wald entropy in the limiting process. At first, we will introduce the Wald entropy [21, 22] based on a simplified version of the formalism [28]. Consider a generally covariant Lagrangian L, that involves no more than quadratic derivatives of the spacetime metric g_{ab}. Under the diffeomorphism $x^a \to x^a + \chi^a$ the metric changes via $\delta g_{ab} = -\nabla_a \chi_b - \nabla_a \chi_b$. By diffeomorphism-invariance, the change in the action, when evaluated on-shell, is given only by a surface term. This leads to a conservation law, $\nabla_a J^a = 0$, for which we can write $J^a = \nabla_b J^{ab}$. Here J^{ab} defines (not uniquely) the antisymmetric Noether potential associated with the diffeomorphism χ^a, which can be formulated as

$$J^{ab} = -32\pi \left(P^{abcd} \nabla_c \chi_d - 2\chi_d \nabla_c P^{abcd}\right).$$

Associated with a rigid diffeomorphism χ^a, there is the Noether charge defined by integrating the Noether potential over any closed spacelike surface S of codimension two. It turns out that the corresponding Noether charge is just proportional to the entropy

$$S_{Wald} = \frac{1}{8\kappa} \int_B J^{ab} dB_{ab}, \tag{16}$$

when χ^a is a timelike Killing vector (with vanishing norm at the Killing horizon), and the surface B is the cross-section of Killing horizon H. However one can formally define the quantity S_{Wald} on any closed spacelike surface Ω, and only in the end take the limit in which Ω approaches a section B of the Killing horizon H. In the following, we will define such a quantity on the section Ω of a stretched horizon Σ described by metric (12) and (13), and compare S_{Pad} with S_{Wald} in the limit $n \to 0$. On the stretched horizon Σ, the timelike Killing vector can be specified as

$$\chi^a = (1, 0, 0, 0, \ldots). \tag{17}$$

The proper velocity u^a of a fiducial observer moving along the orbit of χ^a is $u^a = \chi^a/\sqrt{-g_{ab}\chi^a\chi^b} = \left(\frac{d}{d\tau}\right)^a$ where τ is the proper time. The fiducial proper velocity u^a and unit normal n^a of stretched horizon Σ define $d\Omega_{ab} = n_{(a}u_{b)}dA$, where $dA = \sqrt{\sigma}d^{D-2}y$ is the area element on cross section Ω. We will evaluate both S_{Pad} and S_{Wald} for several typical diffeomorphism invariant theories of gravity. For simplicity, we will restrict on $D = 4$, but the results can be directly generalized to more higher dimensions.

A. $L \sim f(\phi, R)$

As the first example let us consider the following Lagrangian

$$L = \frac{1}{16\pi G} f(\phi, R).$$

Obviously, this example contains the popular $f(R)$ and scalar-tensor gravity as its special cases. From the Lagrangian the tensor P^{abcd} reads as

$$P^{abcd} = \frac{\partial L}{\partial R_{abcd}} = \frac{1}{32\pi G}\frac{\partial f}{\partial R}\left(g^{ac}g^{bd} - g^{ad}g^{bc}\right). \tag{18}$$

Substituting this tensor and the normal vector Eq. (14) into Eq. (15), and preserving the leading term of n in the end of the calculation, we obtain

$$\begin{aligned} S_{Pad} &= 4\int_\Sigma d^3x\sqrt{h}n_d\left(P^{abcd}\nabla_c n_a n_b\right) \\ &= \int_\Sigma d^3x\sqrt{g_H}\frac{\partial f}{\partial R}\left[\frac{\kappa}{8\pi G} + O\left(n^2\right)\right]. \end{aligned}$$

Restricting the t integral within the range $(0, 2\pi/\kappa)$ for the periodicity in Euclidean time [33, 34], we can obtain

$$S_{Pad} \simeq \int_\Omega \frac{1}{4G}\frac{\partial f}{\partial R}\sqrt{g_H}d^2y.$$

When $f(\phi, R) = R$, the entropy density $\frac{1}{4\pi G}\frac{\partial f}{\partial R}$ is $\frac{1}{4\pi G}$, and S_{Pad} is the well-known Gibbons-Hawking entropy. Substituting Eq. (18), Killing vector (17), and $d\Omega_{ab} = n_{(a}u_{b)}dA$ into Wald entropy (16), and preserving the leading term of n at last, we get

$$
\begin{aligned}
S_{Wald} &= \frac{1}{8\kappa}\int_{\Omega} J^{ab} d\Omega_{ab} \\
&= \int_{\Omega} \frac{\partial f}{\partial R}\left[\frac{1}{4G} + O\left(n^2\right)\right]\sqrt{g_H} d^2 y.
\end{aligned} \tag{19}
$$

We notice that although the higher-order terms $O\left(n^2\right)$ of S_{Pad} and S_{Wald} are different, their leading terms are exactly the same.

B. $L \sim \alpha R_{abcd}R^{abcd} + \beta R_{ab}R^{ab}$

Beside the term $f(R) \sim R^2$, the higher (quadratic) derivative interactions usually include

$$
L = \frac{1}{16\pi G}\left(\alpha R_{abcd}R^{abcd} + \beta R_{ab}R^{ab}\right),
$$

with arbitrary parameters α, β, which may depend on some scalar fields. Its derivative with respect to R_{abcd} is

$$
P^{abcd} = \frac{1}{8\pi G}\left[\alpha R^{abcd} + \frac{1}{4}\beta\left(g^{bd}R^{ac} - g^{ad}R^{bc} + g^{ac}R^{bd} - g^{bc}R^{ad}\right)\right].
$$

Similar to the above case, we can obtain

$$
\begin{aligned}
S_{Pad} = \int_{\Omega} [&\frac{-\beta}{2Gg_H}(\left[g_2\right]_{22}\left[g_H\right]_{33} + \left[g_H\right]_{22}\left[g_2\right]_{33} + \frac{2\kappa_2}{\kappa}\left[g_H\right]_{22}\left[g_H\right]_{33} - 2\left[g_2\right]_{23}\left[g_2\right]_{23} - \frac{2\kappa_2}{\kappa}\left[g_H\right]_{23}^2) \\
&-\frac{2\alpha\kappa_2}{G\kappa} + O\left(n^2\right)]\sqrt{g_H}d^2 y,
\end{aligned} \tag{20}
$$

which is identical with the evaluation of S_{Wald} up to $O\left(n^2\right)$.

In [18], it was proved that the Wald entropy of Gauss-Bonnet gravity $L \sim R_{abcd}R^{abcd} - 4R_{ab}R^{ab} + R^2$ is identical with the entropy derived from Eq. (15) (in Rindler frame). We have checked that this is a special case of combination of our discussions in A and B, since the formula S_{Wald} and S_{Pad} are both linear in the Lagrangian.

C. $L \sim R_{ab}\nabla^a\phi\nabla^b\phi$

It is known that there are some ambiguities in Wald entropy. Considering the following interaction involving the metric and a scalar field

$$
L = \frac{1}{16\pi G}R_{ab}\nabla^a\phi\nabla^b\phi,
$$

the corresponding P^{abcd} reads

$$P^{abcd}=-\frac{1}{16\pi G}\nabla^{[a}\phi g^{b][c}\nabla^{d]}\phi.$$

It is important to notice that this tensor is not altered if one adds some terms about scalar fields (but no more than their two order derivative) into L, such as,

$$L_i=\frac{1}{16\pi G}\left[\nabla^a\nabla^b\phi\nabla_a\nabla_b\phi-\left(\nabla^2\phi\right)^2+R_{ab}\nabla^a\phi\nabla^b\phi\right].$$

In [29], it was pointed out that, $L_i\epsilon$ (ϵ is the volume form) can be written as a total derivative $L_i\epsilon=d\alpha_i$, which yields a vanishing contribution to the Wald entropy. This contradicts with the direct evaluation using the tensor P^{abcd} in Wald entropy on the section of Killing horizon, until one realizes that $L_\chi\phi=0$ on it. For the entropy S_{Pad}, one can evaluate

$$\begin{aligned}S_{Pad}&=-\frac{1}{4\pi G}\int_\Sigma d^3x\sqrt{h}n_d\left(\nabla^{[a}\phi g^{b][c}\nabla^{d]}\phi\nabla_c n_a n_b\right)\\&=\int_\Omega\left[\frac{\partial_n^2\phi}{8G}+O\left(n\right)\right]\sqrt{g_H}d^2y,\end{aligned}\tag{21}$$

which is identical with the evaluation of S_{Wald} up to $O\left(n\right)$, and seems nonvanishing. In fact, one can make the similar understanding to that in [29], where $S_{Pad}\sim n_a\nabla^a\phi$ should vanish, since n_a will be a null Killing vector when $n\to 0$, and hence $n_a\nabla^a\phi\sim L_\chi\phi$ vanishes on the Killing horizon.

Hereto, we have shown that the surface term S_{Pad} can be reduced to Wald entropy near the static horizon in the leading order (the higher order terms are different). We will further show that this result can be generalized to any stationary but not static black holes. For such a black hole, it is expected to be axially symmetric [30]. Consider our spacetime is invariant under "time-reversal", it is convenient to write the spacetime metric near horizon $n=0$ as [31]

$$ds^2=-N\left(n,z\right)^2dt^2+g_{\phi\phi}\left(n,z\right)\left[d\varphi-\omega\left(n,z\right)dt\right]^2+dn^2+g_{zz}dz^2,$$

where N denotes the usual lapse function and ω is the angular-rotation parameter. The zeroth law of black hole mechanics and rigidity theorem [32] for axially symmetric (stationary, non-static) Killing horizons tell that surface gravity $\kappa\equiv\partial_n N|_{n\to 0}$ and $\omega_H\equiv\omega|_{n\to 0}$ must be a non-negative constant on the horizon, respectively. Moreover, a stationary Killing horizon is a geodesic submanifold which implies that the horizon is extrinsically flat, i.e. the extrinsic curvature of the horizon must be zero $K_{\mu\nu}|_{n\to 0}=0$. The above properties directly imply a set of necessary constraints

$$
\begin{aligned}
g_{\varphi\varphi}(n,z) &= [g_H]_{\varphi\varphi}(z) + \frac{[g_2]_{\varphi\varphi}(z)}{2!}n^2 + O(n^3), \\
g_{zz}(n,z) &= [g_H]_{zz}(z) + \frac{[g_2]_{zz}(z)}{2!}n^2 + O(n^3), \\
\omega(n,z) &= \omega_H + \frac{\omega_2(z)}{2!}n^2 + O(n^3), \\
N(n,z) &= \kappa n + \frac{\kappa_2(z)}{3!}n^3 + O(n^4),
\end{aligned}
\tag{22}
$$

where the first-order terms in ω and N are required to vanish to avoid a curvature singularity on the horizon. The Killing vector of this spacetime is

$$\chi^a = (1, \omega_H, 0, 0), \tag{23}$$

in terms of (t, φ, n, z) coordinate system.

Invoking the time integration carried out in Euclidean sector, and using the metric (22), the Killing vector (23) and the corresponding fiducial velocity u^a, we can evaluate the surface term S_{Pad} and Wald entropy near the horizon for different gravity theories. We find the entropy on horizon:

$$S_{Pad} = \int_\Omega \frac{\partial f}{\partial R}\left[\frac{1}{4G} + O\left(n^2\right)\right]\sqrt{g_H}dzd\varphi = S_{Wald} + O\left(n^2\right), \text{ for case A,}$$

$$
\begin{aligned}
S_{Pad} = &\int_\Omega [\frac{-\beta}{4Gg_H}([g_2]_{\varphi\varphi}[g_H]_{zz} - \frac{1}{\kappa^2}[g_H]_{\varphi\varphi}^2[g_H]_{zz}\omega_2(z)^2 + [g_H]_{\varphi\varphi}[g_2]_{zz} + \frac{\kappa_2}{\kappa}[g_H]_{\varphi\varphi}[g_H]_{zz}) \\
&+ \frac{\alpha}{4G\kappa^2}(3[g_H]_{\varphi\varphi}^2\omega_2 - 4\kappa\kappa_2) + O\left(n^2\right)]\sqrt{g_H}dzd\varphi \\
= &\, S_{Wald} + O\left(n^2\right), \text{ for case B,}
\end{aligned}
$$

$$S_{Pad} = \int_\Omega \left[\frac{\partial_n^2\phi}{8G} + O(n)\right]\sqrt{g_H}dzd\varphi = S_{Wald} + O(n), \text{ for case C.}$$

Now we can conclude that the boundary term S_{Pad} are the same as the standard Wald entropy for general static and stationary but non-static black holes. This is one of the key results of our paper, which justifies our function $S[\xi]$ as an authentic ‘entropy’.

V. CONCLUSION AND DISCUSSION

In this paper, we have generalized the analogy between the spacetime and solid developed in [18]. We have shown that the spacetime with generalized gravity theory can be analogized to the inhomogeneous elastic solid. Spacetimes described by the Einstein and Lovelock gravity are the limiting cases when we consider the solid being homogeneous. One key point in this work is that we have successfully constructed an entropy function of a displacement vector field to describe phenomenologically the macroscopic level of spacetime solid. Extremizing the total entropy function with respect to the displacement vector, we have found that the equilibrium equation can be identified with the field equation of general diffeomorphism-invariant gravity. Our generalization supports that there is an analogy between spacetime and solid and that the gravitational thermodynamics is not restricted on the concrete gravity theory [13]. We also expect that this approach is helpful to obtain the field equation in the scenarios of emergent gravity where the correct field equation is absent [17].

Beside providing an alternative approach to obtain the arbitrary gravitational field equation, we have shown that the entropy function on the boundary of any stationary spacetime is identical with the standard Wald horizon entropy in several typical higher derivative theories of gravity, though a general relationship between these two entropy expressions still needs further understanding. This provides a new method to calculate the horizon entropy for any stationary spacetime besides the Hamiltonian method, Noether charge method and the field redefinition method [29].

It should be stressed that the spacetime solid is very special. This solid acquires the equilibrium equation independent of the displacement vector field. We suspect that this acquirement may be related to certain underlying first principle to constrain the macroscopic description of spacetime solid. Moreover, the displacement vector field has the zero norm, which characterizes the key role taken by the null hypersurfaces in the spacetime solid [35, 36]. Furthermore, the spacetime solid has a special surface, the stretched horizon. Technically, we have only treated it as the near horizon to approach the true horizon. However, in the membrane paradigm, it has been treated as a dynamic fluid membrane, obeying such pre-relativistic equations as Ohm's law and the Navier-Stokes equation [26, 27]. Obviously, it is valuable to study the thermodynamical property of stretched horizon in the spacetime solid, comparing the differences and similarities with the fluid membrane. If one can understand these nontrivial properties of spacetime solid, the analogy might be of use in the context of the semiclassical limit and even the quantum gravity (spacetime) enigma. In fact, recently it have been proposed [37] that if the entropy function for Einstein and

Lovelock gravity is interpreted as an action in the semiclassical limit, its value will affect the phase of the semiclassical wave function, then the observer independent of the semiclassical gravity requires this phase (i.e. the Wald horizon entropy) to be quantized in units of 2π. Our work further suggests that the Wald entropy can be similarly quantized in arbitrary diffeomorphism-invariant theories of gravity.

ACKNOWLEDGMENTS

The work of BW was supported by the NSFC. The work of SFW and XHG were partially supported by NSFC under Grant Nos. 10905037 and 10947116. The work of GHY, XHG and SFW was partially supported by Shanghai Leading Academic Discipline Project (project number S30105) and Shanghai Research Foundation No 07dz22020.

[1] T. Jacobson, Phys. Rev. Lett. **75,** 1260 (1995) [arXiv:gr-qc/9504004].

[2] C. Eling, R. Guedens, and T. Jacobson, Phys. Rev. Lett. **96**, 121301 (2006) [arXiv:gr-qc/0602001]; E. Elizalde, and P. J. Silva, Phys. Rev. **D 78,** 061501(R) (2008) [arXiv:0804.3721].

[3] M. Akbar and R. G. Cai, Phys. Lett. **B 635,** 7 (2006) [arXiv:hep-th/0602156]; Phys. Rev. **D 75,** 084003 (2007) [arXiv:hep-th/0609128]; S. F. Wu, G. H Yang, P. M. Zhang, Prog. Theor. Phys. **120,** 615 (2008) [arXiv:0805.4044].

[4] T. Padmanabhan, Class. Quant. Grav. **19,** 5387 (2002) [arXiv:gr-qc/0204019]; Phys. Rept. **406**, 49 (2005) [arXiv:gr-qc/0311036]; Int. J. Mod. Phys. **D 15,** 1659 (2006) [arXiv:gr-qc/0606061].

[5] R. G. Cai, L. M. Cao, Y. P. Hu, S. P. Kim, Phys. Rev. **D 78,** 124012 (2008) [arXiv:0810.2610].

[6] A. V. Frolov, L. Kofman, JCAP **0305,** 009 (2003) [arXiv:hep-th/0212327]; U. H. Danielsson, Phys. Rev. **D 71,** 023516 (2005) [arXiv:hep-th/0411172]; R. Bousso, Phys. Rev. **D 71,** 064024 (2005) [arXiv:hep-th/0412197].

[7] R. G. Cai, S. P. Kim, JHEP **0502,** 050 (2005) [arXiv:hep-th/0501055]; R. G. Cai, L. M. Cao, Phys. Rev. **D 75,** 064008 (2007) [arXiv:gr-qc/0611071].

[8] Y. Gong, A. Wang, Phys. Rev. Lett. **99,** 211301 (2007).

[9] S. F. Wu, B. Wang, and G. H Yang, Nucl. Phys. **B 799,** 330 (2008) [arXiv:0711.1209]; S. F. Wu, B. Wang, and G. H Yang, P. M. Zhang, Class. Quant. Grav. **25,** 235018 (2008).

[10] R. G. Cai, L. M. Cao, Nucl. Phys. **B 785,** 135 (2007) [arXiv:hep-th/0612144].

[11] A. Sheykhi, B. Wang, and R. G. Cai, Nucl. Phys. **B 779,** 1 (2007) [arXiv:hep-th/0701198]; Phys. Rev. **D 76** (2007) 023515 [arXiv:hep-th/0701261].

[12] X. H. Ge, Phys. Lett. **B 651,** 49 (2007); S. F. Wu, G. H Yang, P. M. Zhang, arXiv:0710.5394.

[13] R. Brustein and M. Hadad, Phys. Rev. Lett. **103,** 101301 (2009) [arXiv:0903.0823].

[14] M. Parikh and S. Sarkar, arXiv:0903.1176.

[15] T. Padmanabhan, arXiv:0903.1254.

[16] E. Verlinde, arXiv:hep-th/0008140; B. Wang, E. Abdalla and R. K. Su, Phys. Lett. **B 503,** 394 (2001).

[17] O. Dreyer, arXiv:0710.4350; S. Liberati, F. Girelli, and L. Sindoni, arXiv:0909.3834.

[18] T. Padmanabhan, A. Paranjape, Phys. Rev. **D 75,** 064004 (2007) [arXiv:gr-qc/0701003]; T. Padmanabhan, arXiv:0807.2356.

[19] T. Padmanabhan, Gen. Rel. Grav. **40,** 529 (2008) [arXiv:0705.2533]; AIP. Conf. Proc. **939,** 114 (2007) [arXiv:0706.1654].

[20] A. D. Sakharov, Sov. Phys. Dokl. **12,** 1040 (1968); G. E. Volovik, Phys. Rept. **351,** 195 (2001); G E Volovik, *"The universe in a helium droplet"*, (Oxford University Press, 2003); B. L. Hu, Int. J. Theor. Phys. **44,** 1785 (2005) [arXiv:gr-qc/0503067].

[21] R M Wald, Phys. Rev. **D 48,** R3427 (1993) [arXiv:gr-qc/9307038].

[22] V. Iyer and R. M. Wald, Phys. Rev. **D 50,** 846 (1994) [arXiv:gr-qc/9403028].

[23] L. D. Landau and E. M. Lifshitz, *"Theory of Elasticity"*, 3rd ed., Butterworth-Heinemann, (Oxford, UK), (1986).

[24] *"Functionally Graded Materials 1996"*, edited by I. Shiota and M. Y. Miyamoto (Elsevier Science, Amsterdam, The Netherlands, 1997); R. Mirzaeifar, H. Bahai, and S. Shahab, Smart Mater. Struct. **17,** 045003 (2008).

[25] A. J. M. Medved, D. Martin, M. Visser, Class. Quantum Grav. **21,** 3111 (2004) [gr-qc/0402069].

[26] *"Black Holes: The Membrane Paradigm"*, edited by K. S. Thorne, R. H. Price, and D. A. Macdonald (Yale University Press, London, 1986).

[27] M. K. Parikh and Frank Wilczek, Phys. Rev. **D 58,** 064011 (1998); T. Damour and M. Lilley,

arXiv:0802.4169.

[28] G. L. Cardoso, B. de Wit, and T. Mohaupt, Fortsch. Phys. **48,** 49 (2000) [hep-th/9904005].

[29] T. Jacobson, G. Kang, R. C. Myers, Phys. Rev. **D 49,** 6587 (1994).

[30] R. M. Wald, "*General Relativity*", (University of Chicago Press, Chicago, 1984).

[31] A. J. M. Medved, D. Martin, M. Visser, Phys. Rev. **D 70,** 024009 (2004).

[32] B. Carter, in "*General Relativity*: *An Einstein centenary survey*", edited by S. W. Hawking and W. A. Israel (Cambridge, England, 1979).

[33] G. W. Gibbons, S. W. Hawking, Phys. Rev. **D 15,** 2752 (1977).

[34] D. Astefanesei, H. Yavartanoo, Nucl. Phys. **B 794**, 13 (2008).

[35] S, Frittelli, C, Kozameh, T, Newman, J. Math. Phys. **36,** 4975 (1995); S, Frittelli, C, Kozameh, T, Newman, J. Math. Phys. **36,** 4984 (1995); S. Frittelli, C. Kozameh, E. Newman, J. Math. Phys. **36,** 5005 (1995).

[36] D. Cremades, E. Lozano-Tellechea, Holography, JHEP **0701,** 045 (2007).

[37] D. Kothawala, T. Padmanabhan, S. Sarkar, Phys. Rev. **D 78,** 104018 (2008) [arXiv:0807.1481].