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Nonlinear Analysis 69 (2008) 1757–1767

Nonlinear Analysis

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Nonlinear parametric multi-valued variational inclusion systems involving (A, η)-accretive mappings in Banach spaces☆

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Received 22 August 2006; accepted 16 July 2007

Abstract

In this paper, by using the new parametric resolvent operator technique associated with (A, η)-accretive mappings, we analyze and establish an existence theorem for new nonlinear parametric multi-valued variational inclusion systems involving (A, η)-accretive mappings in Banach spaces. Our results generalize sensitivity analysis results of other recent works on strongly monotone quasi-variational inclusions, nonlinear implicit quasi-variational inclusions and nonlinear mixed quasi-variational inclusion systems.

MSC: 49J40; 47H05; 90C33

Keywords: Sensitive analysis; Relaxed cocoercive mapping; Nonlinear parametric multi-valued variational inclusion system; (A, η)-accretive mapping; Parametric resolvent operator technique

1. Introduction

It is well known that variational inequality type methods have been applied widely to problems arising from model equilibria problems in economics, optimization and control theory, operations research, transportation network modelling, and mathematical programming. Further, sensitivity analysis of solutions for variational inequalities with single-valued mappings have been studied by many authors via quite different techniques. For example, by using the project technique, Ding [3], Ding et al. [4], Moudafi [13] and Salahuddin [17] dealt with the sensitivity analysis of solutions for variational inequalities and nonlinear project equations in Hilbert spaces. By using the implicit function approach, Jittorntrum [8], Kyparisis [9], Robinson [16] studied the sensitivity analysis of solutions for variational inequalities under suitable second-order and regularity assumptions.

On the other hand, Dong et al. [5] analyzed solution sensitivity analysis for variational inequalities and variational inclusions by using the resolvent operator technique. Very recently, using the concept and technique of resolvent operators, Agarwal et al. [1] and Jeong [7] introduced and studied a new system of parametric generalized nonlinear mixed quasi-variational inclusions in a Hilbert space and in L_p $(p \geq 2)$ spaces, respectively. For some related work, we refer the reader to [2,6,15] and the references therein.

☆ This work was supported by the Scientific Research Fund of Sichuan Provincial Education Department (2006A106) and the Applied Foundation Research Project of Sichuan Province (07JY029-027).

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0362-546X/$ - see front matter
doi:10.1016/j.na.2007.07.021

Inspired and motivated by the above works, we intend in this paper to study the behavior and sensitivity analysis of the solution set for a new nonlinear parametric multi-valued variational inclusion system involving (A, η)-accretive mappings in Banach spaces. The results obtained generalize and improve the results on the sensitivity analysis for generalized nonlinear mixed quasi-variational inclusions [1,7] and others. For more details, we recommend [1–11,17–19].

2. Preliminaries

Let $\mathcal{B}$ be a real Banach space with dual space $\mathcal{B}^*$, $\langle \cdot, \cdot \rangle$ be the dual pair of $\mathcal{B}$ and $\mathcal{B}^*$, $CB(\mathcal{B})$ denote the family of all nonempty closed bounded subsets of $\mathcal{B}$ and $2^{\mathcal{B}}$ denote the family of all the nonempty subsets of $\mathcal{B}$. The generalized duality mapping $J_q : \mathcal{B} \to 2^{\mathcal{B}^*}$ is defined by

$$J_q(x) = \{f^* \in \mathcal{B}^* : \langle x, f^* \rangle = \|x\|^q, \|f^*\| = \|x\|^{q-1}\}, \quad \forall x \in \mathcal{B},$$

where $q > 1$ is a constant. In what follows we shall denote the single-valued generalized duality mapping by j_q.

Lemma 2.1 (*[20]*). *Let $\mathcal{B}$ be a real uniformly smooth Banach space. Then $\mathcal{B}$ is q-uniformly smooth if and only if there exists a constant $c_q > 0$ such that for all $x, y \in \mathcal{B}$,*

$$\|x + y\|^q \le \|x\|^q + q\langle y, j_q(x)\rangle + c_q\|y\|^q.$$

In the sequel, let Υ be a nonempty open subset of $\mathcal{B}$ in which the parameter ς takes values.

Definition 2.1. Let $\mathcal{B}$ be a q-uniformly smooth Banach space and $A : \mathcal{B} \to \mathcal{B}$ be a single-valued mapping. Then a mapping $T : \mathcal{B} \times \mathcal{B} \times \Upsilon \to \mathcal{B}$ is said to be

(i) m-relaxed accretive in the first argument if there exists a positive constant m such that

$$\langle T(x, u, \varsigma) - T(y, u, \varsigma), j_q(x - y)\rangle \ge -m\|x - y\|^q,$$

for all $(x, y, u, \varsigma) \in \mathcal{B} \times \mathcal{B} \times \mathcal{B} \times \Upsilon$;

(ii) s-cocoercive in the first argument if there exists a constant $s > 0$ such that

$$\langle T(x, u, \varsigma) - T(y, u, \varsigma), j_q(x - y)\rangle \ge s\|T(x, u, \varsigma) - T(y, u, \varsigma)\|^q,$$

for all $(x, y, u, \varsigma) \in \mathcal{B} \times \mathcal{B} \times \mathcal{B} \times \Upsilon$;

(iii) γ-relaxed cocoercive with respect to A in the first argument if there exists a positive constant γ such that

$$\langle T(x, u, \varsigma) - T(y, u, \varsigma), j_q(A(x) - A(y))\rangle \ge -\gamma\|T(x, u, \varsigma) - T(y, u, \varsigma)\|^q,$$

for all $(x, y, u, \varsigma) \in \mathcal{B} \times \mathcal{B} \times \mathcal{B} \times \Upsilon$;

(iv) (ϵ, α)-relaxed cocoercive with respect to A in the first argument if there exist positive constants ϵ and α such that

$$\langle T(x, u, \varsigma) - T(y, u, \varsigma), j_q(A(x) - A(y))\rangle \ge -\alpha\|T(x, u, \varsigma) - T(y, u, \varsigma)\|^q + \epsilon\|x - y\|^q,$$

for all $(x, y, u, \varsigma) \in \mathcal{B} \times \mathcal{B} \times \mathcal{B} \times \Upsilon$;

(v) μ-Lipschitz continuous in the first argument if there exists a constant $\mu > 0$ such that

$$\|T(x, u, \varsigma) - T(y, u, \varsigma)\| \le \mu\|x - y\|, \quad \forall (x, y, u, \varsigma) \in \mathcal{B} \times \mathcal{B} \times \mathcal{B} \times \Upsilon.$$

In a similar way, we can define (relaxed) cocoercivity and Lipschitz continuity of the mapping $T(\cdot, \cdot, \cdot)$ in the second and third arguments.

Definition 2.2. Let $\mathcal{B}$ be a q-uniformly smooth Banach space, $\eta : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ and $A, H : \mathcal{B} \to \mathcal{B}$ be single-valued mappings. Then multi-valued mapping $M : \mathcal{B} \to 2^{\mathcal{B}}$ is said to be

(i) accretive if

$$\langle u - v, j_q(x - y)\rangle \ge 0, \quad \forall x, y \in \mathcal{B}, u \in M(x), v \in M(y);$$

(ii) η-accretive if

$$\langle u - v, j_q(\eta(x, y))\rangle \ge 0, \quad \forall x, y \in \mathcal{B}, u \in M(x), v \in M(y);$$

(iii) strictly η-accretive if M is η-accretive and equality holds if and only if $x = y$;
(iv) r-strongly η-accretive if there exists a constant $r > 0$ such that

$$\langle u - v, j_q(\eta(x, y))\rangle \geq r\|x - y\|^q, \quad \forall x, y \in \mathcal{B}, u \in M(x), v \in M(y);$$

(v) α-relaxed η-accretive if there exists a constant $\alpha > 0$ such that

$$\langle u - v, j_q(\eta(x, y))\rangle \geq -\alpha\|x - y\|^q, \quad \forall x, y \in \mathcal{B}, u \in M(x), v \in M(y);$$

(vi) m-accretive if M is accretive and $(I + \rho M)(\mathcal{B}) = \mathcal{B}$ for all $\rho > 0$, where I denotes the identity operator on $\mathcal{B}$;
(vii) generalized m-accretive if M is η-accretive and $(I + \rho M)(\mathcal{B}) = \mathcal{B}$ for all $\rho > 0$;
(viii) H-accretive if M is accretive and $(H + \rho M)(\mathcal{B}) = \mathcal{B}$ for all $\rho > 0$;
(ix) (H, η)-accretive if M is η-accretive and $(H + \rho M)(\mathcal{B}) = \mathcal{B}$ for every $\rho > 0$.

In a similar way, we can define strict η-accretivity and strong η-accretivity of the single-valued mapping A.

Remark 2.1. When $X = \mathcal{H}$, (i)–(ix) of Definition 2.3 reduce to the definitions of monotone operators, η-monotone operators, strictly η-monotone operators, strongly η-monotone operators, relaxed η-monotone operators, maximal monotone operators, maximal η-monotone operators, H-monotone operators and (H, η)-monotone operators, respectively.

Definition 2.3. Let $F : \mathcal{B} \times \Upsilon \to 2^{\mathcal{B}}$ be a multi-valued mapping. Then F is called τ-$\hat{\mathbf{H}}$-Lipschitz continuous in the first argument if there exists a constant $\tau > 0$ such that

$$\hat{\mathbf{H}}(F(x, \varsigma), F(y, \varsigma)) \leq \tau\|x - y\|, \quad \forall x, y \in \mathcal{B},\ \varsigma \in \Upsilon,$$

where $\hat{\mathbf{H}} : 2^{\mathcal{B}} \times 2^{\mathcal{B}} \to (-\infty, +\infty) \cup \{+\infty\}$ is the Hausdorff metric, i.e.,

$$\hat{\mathbf{H}}(A, B) = \max\{\sup_{x\in A} \inf_{y\in B} \|x - y\|, \sup_{x\in B} \inf_{y\in A} \|x - y\|\}, \quad \forall A, B \in 2^{\mathcal{B}}.$$

In a similar way, we can define $\hat{\mathbf{H}}$-Lipschitz continuity of the mapping $F(\cdot, \cdot)$ in the second argument.

Lemma 2.2 ([12]). *Let $(\mathcal{X}, d)$ be a complete metric space and $T_1, T_2 : \mathcal{X} \to CB(\mathcal{X})$ be two set-valued contractive mappings with the same contractive constant $t \in (0, 1)$, i.e.,*

$$\hat{\mathbf{H}}(T_i(x), T_i(y)) \leq t d(x, y), \quad \forall x, y \in \mathcal{X}, i = 1, 2.$$

Then

$$\hat{\mathbf{H}}(F(T_1), F(T_2)) \leq \frac{1}{1 - t} \sup_{x\in\mathcal{X}} \hat{\mathbf{H}}(T_1(x), T_2(x)),$$

where $F(T_1)$ and $F(T_2)$ are fixed point sets of T_1 and T_2, respectively.

Definition 2.4. Let $A : \mathcal{B} \to \mathcal{B}$, $\eta : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ be two single-valued operators. Then a multi-valued mapping $M : \mathcal{B} \to 2^{\mathcal{B}}$ is called (A, η)-accretive if

(i) M is m-relaxed η-accretive, (ii) $(A + \rho M)(\mathcal{B}) = \mathcal{B}$ for every $\rho > 0$.

Remark 2.2. For appropriate and suitable choices of m, A, η and $\mathcal{B}$, it is easy to see that Definition 2.4 includes a number of definitions of monotone operators and accretive mappings (see [11]).

Proposition 2.1 ([11]). *Let $A : \mathcal{B} \to \mathcal{B}$ be an r-strongly η-accretive mapping, $M : \mathcal{B} \to 2^{\mathcal{B}}$ be an (A, η)-accretive mapping. Then the operator $(A + \rho M)^{-1}$ is single-valued for every $\rho > 0$.*

Definition 2.5. Let $A : \mathcal{B} \to \mathcal{B}$ be a strictly η-accretive mapping and $M : \mathcal{B} \to 2^{\mathcal{B}}$ be an (A, η)-accretive mapping. For any given constant $\rho > 0$, the resolvent operator $J^{\rho,A}_{\eta,M} : \mathcal{B} \to \mathcal{B}$ is defined by

$$J^{\rho,A}_{\eta,M}(u) = (A + \rho M)^{-1}(u), \quad \forall u \in \mathcal{B}.$$

Proposition 2.2 ([11]). *Let $\mathcal{B}$ be a q-uniformly smooth Banach space and $\eta : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ be τ-Lipschitz continuous, $A : \mathcal{B} \to \mathcal{B}$ be a r-strongly η-accretive mapping and $M : \mathcal{B} \to 2^{\mathcal{B}}$ be an (A, η)-accretive mapping. Then the resolvent operator $J_{\eta,M}^{\rho,A} : \mathcal{B} \to \mathcal{B}$ is $\frac{\tau^{q-1}}{r-\rho m}$-Lipschitz continuous, i.e.,*

$$\|J_{\eta,M}^{\rho,A}(x) - J_{\eta,M}^{\rho,A}(y)\| \le \frac{\tau^{q-1}}{r-\rho m}\|x-y\|, \quad \forall x, y \in \mathcal{B},$$

where $\rho \in (0, \frac{r}{m})$ is a constant.

Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two real Banach spaces, let Ω and Λ be two nonempty open subsets of $\mathcal{B}_1$ and $\mathcal{B}_2$ in which the parameter ω and λ take values, respectively, $E : \mathcal{B}_1 \times \mathcal{B}_2 \times \Omega \to \mathcal{B}_1$, $F : \mathcal{B}_1 \times \mathcal{B}_2 \times \Lambda \to \mathcal{B}_2$, $S : \mathcal{B}_1 \times \Omega \to 2^{\mathcal{B}_1}$ and $T : \mathcal{B}_2 \times \Lambda \to 2^{\mathcal{B}_2}$ are multi-valued mappings, $f : \mathcal{B}_1 \times \Omega \to \mathcal{B}_1$, $g : \mathcal{B}_2 \times \Lambda \to \mathcal{B}_2$, $\eta_1 : \mathcal{B}_1 \times \mathcal{B}_1 \times \Omega \to \mathcal{B}_1$ and $\eta_2 : \mathcal{B}_2 \times \mathcal{B}_2 \times \Lambda \to \mathcal{B}_2$ single-valued mappings. Suppose that $A_1 : \mathcal{B}_1 \to \mathcal{B}_1$, $A_2 : \mathcal{B}_2 \to \mathcal{B}_2$, $M : \mathcal{B}_1 \times \mathcal{B}_1 \times \Omega \to 2^{\mathcal{B}_1}$ and $N : \mathcal{B}_2 \times \mathcal{B}_2 \times \Lambda \to 2^{\mathcal{B}_2}$ are any nonlinear mappings such that for all $(z, \Omega) \in \mathcal{B}_1 \times \Omega$, $M(\cdot, z, \omega) : \mathcal{B}_1 \to 2^{\mathcal{B}_1}$ is an (A_1, η_1)-accretive mapping with $f(\mathcal{B}_1, \omega) \cap \mathrm{dom}(M(\cdot, z, \omega)) \neq \emptyset$ and for all $(t, \lambda) \in \mathcal{B}_2 \times \Lambda$, $N(\cdot, t, \lambda) : \mathcal{B}_2 \to 2^{\mathcal{B}_2}$ is an (A_2, η_2)-accretive mapping with $g(\mathcal{B}_2, \lambda) \cap \mathrm{dom}(N(\cdot, t, \lambda)) \neq \emptyset$, respectively. Throughout this paper, unless otherwise stated, we shall consider the following generalized parametric (A, η)-accretive variational inclusion systems.

For each fixed $(\omega, \lambda) \in \Omega \times \Lambda$, find $(x(\omega), y(\lambda)) \in \mathcal{B}_1 \times \mathcal{B}_2$ such that $u(\omega) \in S(x(\omega), \omega)$, $v(\lambda) \in T(y(\lambda), \lambda)$ and

$$\begin{cases} 0 \in E(x(\omega), v(\lambda), \omega) + M(f(x(\omega), \omega), x(\omega), \omega), \\ 0 \in F(u(\omega), y(\lambda), \lambda) + N(g(y(\lambda), \lambda), y(\lambda), \lambda). \end{cases} \tag{2.1}$$

Example 2.1. Let $S : \mathcal{B}_1 \times \Omega \to \mathcal{B}_1$ and $T : \mathcal{B}_2 \times \Lambda \to \mathcal{B}_2$ be single-valued mappings. Then for each fixed $(\omega, \lambda) \in \Omega \times \Lambda$, the problem (2.1) reduces to finding $(x(\omega), y(\lambda)) \in \mathcal{B}_1 \times \mathcal{B}_2$ such that

$$\begin{cases} 0 \in E(x(\omega), T(y(\lambda), \lambda), \omega) + M(f(x(\omega), \omega), x(\omega), \omega), \\ 0 \in F(S(x(\omega), \omega), y(\lambda), \lambda) + N(g(y(\lambda), \lambda), y(\lambda), \lambda). \end{cases} \tag{2.2}$$

Example 2.2. Suppose that $\mathcal{B}_1 = \mathcal{B}_2 = \mathcal{B}$, $f = g = I$, $M(x, y, \omega) = M(x, \omega)$ for all $(x, y, \omega) \in \mathcal{B} \times \mathcal{B} \times \Omega$ and $N(x, y, \lambda) = N(x, \lambda)$ for all $(x, y, \lambda) \in \mathcal{B} \times \mathcal{B} \times \Lambda$. Then there exist two constants $\rho, \mu > 0$ and nonlinear mappings G_i, V_i $(i = 1, 2)$ such that $E(x, T(y, \lambda), \omega) = \frac{1}{\rho}(x - y) + (G_1(y, \omega) + V_1(y, \omega))$ and $F(S(x, \omega), y, \lambda) = \frac{1}{\mu}(y - x) + (G_2(x, \lambda) + V_2(x, \lambda))$ for all $(x, y, \omega, \lambda) \in \mathcal{B} \times \mathcal{B} \times \Omega \times \Lambda$; then the problem (2.2) is equivalent to the following system of parametric general nonlinear mixed quasi-variational inclusions in Banach spaces: find $(x(\omega), y(\lambda)) \in \mathcal{B} \times \mathcal{B}$ such that

$$\begin{cases} 0 \in x(\omega) - y(\lambda) + \rho(G_1(y(\lambda), \omega) + V_1(y(\lambda), \omega)) + \rho M(x(\omega), \omega), \\ 0 \in y(\lambda) - x(\omega) + \mu(G_2(x(\omega), \lambda) + V_2(x(\omega), \lambda)) + \mu N(y(\lambda), \lambda), \end{cases} \tag{2.3}$$

which was studied by Jeong [7] for when M, N are m-accretive mappings in (2.3). Further, the problem (2.3) was introduced and studied by Agarwal et al. [1] for when $\mathcal{B} = \mathcal{H}$ is a Hilbert space, M, N are two maximal monotone mappings, $x(\omega) = x$ for all $\omega \in \Omega$ and $y(\lambda) = y$ for all $\lambda \in \Lambda$ in (2.3).

Remark 2.3. For appropriate and suitable choices of $E, F, M, N, S, T, f, g, A_i, \eta_i$ and $\mathcal{B}_i$ for $i = 1, 2$, it is easy to see that the problem (2.1) includes a number (systems) of (parametric) quasi-variational inclusions, (parametric) generalized quasi-variational inclusions, (parametric) quasi-variational inequalities, (parametric) implicit quasi-variational inequalities studied by many authors as special cases; see, for example, [1–13,15–19] and the references therein.

Now, for each fixed $(\omega, \lambda) \in \Omega \times \Lambda$, the solution set $Q(\omega, \lambda)$ of the problem (2.1) is denoted as

$$\begin{aligned} Q(\omega, \lambda) = \{&(x(\omega), y(\lambda)) \in \mathcal{B}_1 \times \mathcal{B}_2 : \exists u(\omega) \in S(x(\omega), \omega), \text{ and } v(\lambda) \in T(y(\lambda), \lambda), \\ &\text{such that } 0 \in E(x(\omega), v(\lambda), \omega) + M(f(x(\omega), \omega), x(\omega), \omega) \\ &\text{and } 0 \in F(u(\omega), y(\lambda), \lambda) + N(g(y(\lambda), \lambda), y(\lambda), \lambda)\}. \end{aligned}$$

In this paper, our main aim is to study the behavior of the solution set $Q(\omega, \lambda)$, and the conditions on these mappings $E, F, S, T, N, M, f, g, \eta_1, \eta_2, A_1, A_2$ under which the function $Q(\omega, \lambda)$ is continuous or Lipschitz continuous with respect to the parameter $(\omega, \lambda) \in \Omega \times \Lambda$.

3. Sensitivity analysis results

In the sequel, let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two real Banach spaces, Ω and Λ be two nonempty open subsets of $\mathcal{B}_1$ and $\mathcal{B}_2$ in which the parameters ω and λ take values, respectively. We shall first transform the problem (2.1) into a problem of finding the parametric fixed point of the associated (A, η)-resolvent operator.

Lemma 3.1. *For each fixed $(\omega, \lambda) \in \Omega \times \Lambda$, an element $(x(\omega), y(\lambda)) \in Q(\omega, \lambda)$ is a solution to (2.1) if and only if there are $(x(\omega), y(\lambda)) \in \mathcal{B}_1 \times \mathcal{B}_2$, $u(\omega) \in S(x(\omega), \omega)$ and $v(\lambda) \in T(y(\lambda), \lambda)$ such that*

$$\begin{cases} f(x(\omega), \omega) = J_{\rho, A_1}^{M(\cdot, x(\omega), \omega)}(A_1(f(x(\omega), \omega)) - \rho E(x(\omega), v(\lambda), \omega)), \\ g(y(\lambda), \lambda) = J_{\varrho, A_2}^{N(\cdot, y(\lambda), \lambda)}(A_2(g(y(\lambda), \lambda)) - \varrho F(u(\omega), y(\lambda), \lambda)), \end{cases} \tag{3.1}$$

where $J_{\rho, A_1}^{M(\cdot, x(\omega), \omega)} = (A_1 + \rho M(\cdot, x(\omega), \omega))^{-1}$ and $J_{\varrho, A_2}^{N(\cdot, y(\lambda), \lambda)} = (A_2 + \varrho N(\cdot, y(\lambda), \lambda))^{-1}$ are the corresponding resolvent operators in the first argument of an (A_1, η_1)-accretive mapping $M(\cdot, \cdot, \cdot)$ and an (A_2, η_2)-accretive mapping $N(\cdot, \cdot, \cdot)$, respectively, A_i is an r_i-strongly monotone mapping for $i = 1, 2$ and $\rho, \varrho > 0$ are two constants.

Proof. The fact directly follows from Definition 2.5 and some simple arguments. □

Theorem 3.1. *Let $\mathcal{B}_1$ be a q_1-uniformly smooth Banach space and $\mathcal{B}_2$ be a q_2-uniformly smooth Banach space, $A_i : \mathcal{B}_i \to \mathcal{B}_i$ be r_i-strongly monotone and s_i-Lipschitz continuous for all $i = 1, 2$, $S : \mathcal{B}_1 \times \Omega \to CB(\mathcal{B}_1)$ be κ_1-$\hat{\mathbf{H}}$-Lipschitz continuous in the first variable, $T : \mathcal{B}_2 \times \Lambda \to CB(\mathcal{B}_2)$ be κ_2-$\hat{\mathbf{H}}$-Lipschitz continuous in the first variable, $f : \mathcal{B}_1 \times \Omega \to \mathcal{B}_1$ be δ_1-strongly monotone and σ_1-Lipschitz continuous in the first variable, $g : \mathcal{B}_2 \times \Lambda \to \mathcal{B}_2$ be δ_2-strongly monotone and σ_2-Lipschitz continuous in the first variable, $M : \mathcal{B}_1 \times \mathcal{B}_1 \times \Omega \to 2^{\mathcal{B}_1}$ be (A_1, η_1)-accretive with constant m_1 in the first variable and $N : \mathcal{B}_2 \times \mathcal{B}_2 \times \Lambda \to 2^{\mathcal{B}_2}$ be (A_2, η_2)-accretive with constant m_2 in the first variable. Let $\eta_1 : \mathcal{B}_1 \times \mathcal{B}_1 \to \mathcal{B}_1$ be τ_1-Lipschitz continuous, $\eta_2 : \mathcal{B}_2 \times \mathcal{B}_2 \to \mathcal{B}_2$ be τ_2-Lipschitz continuous, $E : \mathcal{B}_1 \times \mathcal{B}_2 \times \Omega \to \mathcal{B}_1$ be (γ_1, α_1)-relaxed cocoercive with respect to f_1 and μ_1-Lipschitz continuous in the first variable, $F : \mathcal{B}_1 \times \mathcal{B}_2 \times \Lambda \to \mathcal{B}_2$ be (γ_2, α_2)-relaxed cocoercive with respect to g_2 and μ_2-Lipschitz continuous in the second variable, and let E be β_2-Lipschitz continuous in the second variable, and F be β_1-Lipschitz continuous in the first variable, where $f_1 : \mathcal{B}_1 \times \Omega \to \mathcal{B}_1$ is defined by $f_1(x) = A_1 \circ f(x, \omega) = A_1(f(x, \omega))$ for all $(x, \omega) \in \mathcal{B}_1 \times \Omega$, $g_2 : \mathcal{B}_2 \times \Lambda \to \mathcal{B}_2$ is defined by $g_2(x) = A_2 \circ g(x, \lambda) = A_2(g(x, \lambda))$ for all $(x, \lambda) \in \mathcal{B}_2 \times \Lambda$. If*

$$\|J_{\rho, A_1}^{M(\cdot, x, \omega)}(z) - J_{\rho, A_1}^{M(\cdot, y, \omega)}(z)\| \le \nu_1 \|x - y\|, \quad \forall (x, y, z, \omega) \in \mathcal{B}_1 \times \mathcal{B}_1 \times \mathcal{B}_1 \times \Omega, \tag{3.2}$$

$$\|J_{\varrho, A_2}^{N(\cdot, x, \lambda)}(z) - J_{\varrho, A_2}^{N(\cdot, y, \lambda)}(z)\| \le \nu_2 \|x - y\|, \quad \forall (x, y, z, \lambda) \in \mathcal{B}_2 \times \mathcal{B}_2 \times \mathcal{B}_2 \times \Lambda \tag{3.3}$$

and there exist constants $\rho \in (0, \frac{r_1}{m_1})$, $\varrho \in (0, \frac{r_2}{m_2})$ such that

$$\begin{cases} k_1 = \nu_1 + \sqrt[q_1]{1 - q_1\delta_1 + c_{q_1}\sigma_1^{q_1}} < 1, \qquad k_2 = \nu_2 + \sqrt[q_2]{1 - q_2\delta_2 + c_{q_2}\sigma_2^{q_2}} < 1, \\ \sqrt[q_1]{s_1^{q_1}\sigma_1^{q_1} - q_1\rho\gamma_1 + c_{q_1}\rho^{q_1}\mu_1^{q_1} + q_1\rho\alpha_1\mu_1^{q_1}} < \tau_1^{1-q_1}(r_1 - \rho m_1)\left(1 - k_1 - \dfrac{\varrho\beta_1\kappa_1\tau_2^{q_2-1}}{r_2 - \varrho m_2}\right), \\ \sqrt[q_2]{s_2^{q_2}\sigma_2^{q_2} - q_2\varrho\gamma_2 + c_{q_2}\varrho^{q_2}\mu_2^{q_2} + q_2\varrho\alpha_2\mu_2^{q_2}} < \tau_2^{1-q_2}(r_2 - \varrho m_2)\left(1 - k_2 - \dfrac{\rho\beta_2\kappa_2\tau_1^{q_1-1}}{r_1 - \rho m_1}\right), \end{cases} \tag{3.4}$$

where c_{q_1}, c_{q_2} are the constants as in Lemma 2.1, then for each $(\omega, \lambda) \in \Omega \times \Lambda$, the following results hold:

(1) *the solution set $Q(\omega, \lambda)$ of the problem (2.1) is nonempty;*
(2) *$Q(\omega, \lambda)$ is a closed subset of $\mathcal{B}_1 \times \mathcal{B}_2$.*

Proof. In the sequel, from (3.1), we first define mappings $\Phi_\rho : \mathcal{B}_1 \times \mathcal{B}_2 \times \Omega \to \mathcal{B}_1$ and $\Psi_\varrho : \mathcal{B}_1 \times \mathcal{B}_2 \times \Lambda \to \mathcal{B}_2$ as follows:

$$\begin{aligned}\Phi_\rho(x, v, \omega) &= x - f(x, \omega) + J_{\rho, A_1}^{M(\cdot, x, \omega)}(A_1(f(x, \omega)) - \rho E(x, v, \omega)),\\ \Psi_\varrho(u, y, \lambda) &= y - g(y, \lambda) + J_{\varrho, A_2}^{N(\cdot, y, \lambda)}(A_2(g(y, \lambda)) - \varrho F(u, y, \lambda))\end{aligned} \tag{3.5}$$

for all $(x, y, \omega, \lambda) \in \mathcal{B}_1 \times \mathcal{B}_2 \times \Omega \times \Lambda$.

Now define $\|\cdot\|_1$ on $\mathcal{B}_1 \times \mathcal{B}_2$ by

$$\|(x, y)\|_1 = \|x\| + \|y\|, \quad \forall (x, y) \in \mathcal{B}_1 \times \mathcal{B}_2.$$

It is easy to see that $(\mathcal{B}_1 \times \mathcal{B}_2, \|\cdot\|_1)$ is a Banach space (see [6]). By (3.5), for any given $\rho > 0$ and $\varrho > 0$, define $G : \mathcal{B}_1 \times \mathcal{B}_2 \times \Omega \times \Lambda \to 2^{\mathcal{B}_1} \times 2^{\mathcal{B}_2}$ by

$$\begin{aligned}G_{\rho,\varrho}(x, y, \omega, \lambda) = \{(&\Phi_\rho(x, v, \omega), \Psi_\varrho(u, y, \lambda)) : \forall u \in S(x, \omega), v \in T(y, \lambda), \text{ and}\\ &(x, y, \omega, \lambda) \in \mathcal{B}_1 \times \mathcal{B}_2 \times \Omega \times \Lambda\}.\end{aligned}$$

For any $(x, y, \omega, \lambda) \in \mathcal{B}_1 \times \mathcal{B}_2 \times \Omega \times \Lambda$, since $S(x, \omega) \in CB(\mathcal{B}_1)$, $T(y, \lambda) \in CB(\mathcal{B}_2)$, $f, g, A_1, A_2, \eta_1, \eta_2, E, F$, $J_{\rho,A_1}^{M(\cdot,x,\omega)}$, $J_{\rho,A}^{M(\cdot,x,\lambda)}$ are continuous, we have $G_{\rho,\varrho}(x, y, \omega, \lambda) \in CB(\mathcal{B}_1 \times \mathcal{B}_2)$. Now for each fixed $(\omega, \lambda) \in \Omega \times \Lambda$, we prove that $G_{\rho,\varrho}(x, y, \omega, \lambda)$ is a multi-valued contractive mapping.

In fact, for any $(x, y, \omega, \lambda), (\hat{x}, \hat{y}, \omega, \lambda) \in \mathcal{B}_1 \times \mathcal{B}_2 \times \Omega \times \Lambda$ and any $(a_1, a_2) \in G_{\rho,\varrho}(x, y, \omega, \lambda)$, there exist $u \in S(x, \omega)$, $v \in T(y, \lambda)$ such that

$$\begin{aligned}a_1 &= x - f(x, \omega) + J_{\rho, A_1}^{M(\cdot, x, \omega)}(A_1(f(x, \omega)) - \rho E(x, v, \omega)),\\ a_2 &= y - g(y, \lambda) + J_{\varrho, A_2}^{N(\cdot, y, \lambda)}(A_2(g(y, \lambda)) - \varrho F(u, y, \lambda)).\end{aligned}$$

Note that $S(\hat{x}, \omega) \in CB(\mathcal{B}_1)$, $T(\hat{y}, \lambda) \in CB(\mathcal{B}_2)$; it follows from Nadler's result [14] that there exist $\hat{u} \in S(\hat{x}, \omega)$ and $\hat{v} \in T(\hat{y}, \lambda)$ such that

$$\|u - \hat{u}\| \le \hat{\mathbf{H}}(S(x, \omega), S(\hat{x}, \omega)), \quad \|v - \hat{v}\| \le \hat{\mathbf{H}}(T(y, \lambda), T(\hat{y}, \lambda)). \tag{3.6}$$

Setting

$$\begin{aligned}b_1 &= \hat{x} - f(\hat{x}, \omega) + J_{\rho, A_1}^{M(\cdot, \hat{x}, \omega)}(A_1(f(\hat{x}, \omega)) - \rho E(\hat{x}, \hat{v}, \omega)),\\ b_2 &= \hat{y} - g(\hat{y}, \lambda) + J_{\varrho, A_2}^{N(\cdot, \hat{y}, \lambda)}(A_2(g(\hat{y}, \lambda)) - \varrho F(\hat{u}, \hat{y}, \lambda)),\end{aligned}$$

we have $(b_1, b_2) \in G_{\rho,\varrho}(\hat{x}, \hat{y}, \omega, \lambda)$. It follows from (3.2) and Proposition 2.2 that

$$\begin{aligned}\|a_1 - b_1\| \le\ & \|x - \hat{x} - [f(x, \omega) - f(\hat{x}, \omega)]\|\\ &+ \|J_{\rho, A_1}^{M(\cdot, x, \omega)}(A_1(f(x, \omega)) - \rho E(x, v, \omega)) - J_{\rho, A_1}^{M(\cdot, \hat{x}, \omega)}(A_1(f(x, \omega)) - \rho E(x, v, \omega))\|\\ &+ \|J_{\rho, A_1}^{M(\cdot, \hat{x}, \omega)}(A_1(f(x, \omega)) - \rho E(x, v, \omega)) - J_{\rho, A_1}^{M(\cdot, \hat{x}, \omega)}(A_1(f(\hat{x}, \omega)) - \rho E(\hat{x}, \hat{v}, \omega))\|\\ \le\ & \|x - \hat{x} - [f(x, \omega) - f(\hat{x}, \omega)]\| + \nu_1\|x - \hat{x}\| + \frac{\rho \tau_1^{q_1 - 1}}{r_1 - \rho m_1}\|E(\hat{x}, v, \omega) - E(\hat{x}, \hat{v}, \omega)\|\\ &+ \frac{\tau_1^{q_1 - 1}}{r_1 - \rho m_1}\|A_1(f(x, \omega)) - A_1(f(\hat{x}, \omega)) - \rho[E(x, v, \omega) - E(\hat{x}, v, \omega)]\|.\end{aligned} \tag{3.7}$$

By the assumptions on f, E, A_1, T and (3.6), we have

$$\|x - \hat{x} - [f(x, \omega) - f(\hat{x}, \omega)]\|^{q_1} \le (1 - q_1\delta_1 + c_{q_1}\sigma_1^{q_1})\|x - \hat{x}\|^{q_1}, \tag{3.8}$$

$$\|E(\hat{x}, v, \omega) - E(\hat{x}, \hat{v}, \omega)\| \le \beta_2\|v - \hat{v}\| \le \beta_2\hat{\mathbf{H}}(T(y, \lambda), T(\hat{y}, \lambda)) \le \beta_2\kappa_2\|y - \hat{y}\|, \tag{3.9}$$

$$\begin{aligned}&\|A_1(f(x, \omega)) - A_1(f(\hat{x}, \omega)) - \rho[E(x, v, \omega) - E(\hat{x}, v, \omega)]\|^{q_1}\\ &\quad \le \|A_1(f(x, \omega)) - A_1(f(\hat{x}, \omega))\|^{q_1} + c_{q_1}\rho^{q_1}\|E(x, v, \omega) - E(\hat{x}, v, \omega)\|^{q_1}\\ &\qquad - q_1\rho\langle E(x, v, \omega) - E(\hat{x}, v, \omega), A_1(f(x, \omega)) - A_1(f(\hat{x}, \omega))\rangle\\ &\quad \le (s_1^{q_1}\sigma_1^{q_1} - q_1\rho\gamma_1 + c_{q_1}\rho^{q_1}\mu_1^{q_1} + q_1\rho\alpha_1\mu_1^{q_1})\|x - \hat{x}\|^{q_1},\end{aligned} \tag{3.10}$$

where c_{q_1} is a constant as in Lemma 2.1. Combining (3.8)–(3.10) with (3.7), we infer

$$\|a_1 - b_1\| \le \theta_1 \|x - \hat{x}\| + \vartheta_1 \|y - \hat{y}\|, \tag{3.11}$$

where

$$\theta_1 = \nu_1 + \sqrt[q_1]{1 - q_1\delta_1 + c_{q_1}\sigma_1^{q_1}} + \frac{\tau_1^{q_1-1}}{r_1 - \rho m_1} \sqrt[q_1]{s_1^{q_1}\sigma_1^{q_1} - q_1\rho\gamma_1 + c_{q_1}\rho^{q_1}\mu_1^{q_1} + q_1\rho\alpha_1\mu_1^{q_1}},$$

$$\vartheta_1 = \frac{\rho\beta_2\kappa_2\tau_1^{q_1-1}}{r_1 - \rho m_1}.$$

On the other hand, by the assumptions of g, S, A_2, F and (3.6), we can obtain

$$\begin{aligned}
&\|y - \hat{y} - [g(y,\lambda) - g(\hat{y},\lambda)]\|^{q_2} \le (1 - q_2\delta_2 + c_{q_2}\sigma_2^{q_2})\|y - \hat{y}\|^{q_2},\\
&\|F(u,y,\lambda) - F(\hat{u},y,\lambda)\| \le \beta_1\kappa_1\|x - \hat{x}\|,\\
&\|A_2(g(y,\lambda)) - A_2(g(\hat{y},\lambda)) - \varrho(F(\hat{u},y,\lambda) - F(\hat{u},\hat{y},\lambda))\|^{q_2}\\
&\quad \le (s_2^{q_2}\sigma_2^{q_2} - q_2\varrho\gamma_2 + c_{q_2}\varrho^{q_2}\mu_2^{q_2} + q_2\varrho\alpha_2\mu_2^{q_2})\|y - \hat{y}\|^{q_2},
\end{aligned}$$

and

$$\begin{aligned}
\|a_2 - b_2\| &\le \|y - \hat{y} - [g(y,\lambda) - g(\hat{y},\lambda)]\|\\
&\quad + \|J_{\varrho,A_2}^{N(\cdot,y,\lambda)}(A_2(g(y,\lambda)) - \varrho F(u,y,\lambda)) - J_{\varrho,A_2}^{N(\cdot,\hat{y},\lambda)}(A_2(g(y,\lambda)) - \varrho F(u,y,\lambda))\|\\
&\quad + \|J_{\varrho,A_2}^{N(\cdot,\hat{y},\lambda)}(A_2(g(y,\lambda)) - \varrho F(u,y,\lambda)) - J_{\varrho,A_2}^{N(\cdot,\hat{y},\lambda)}(A_2(g(\hat{y},\lambda)) - \varrho F(\hat{u},\hat{y},\lambda))\|\\
&\le \theta_2\|x - \hat{x}\| + \vartheta_2\|y - \hat{y}\|,
\end{aligned} \tag{3.12}$$

where c_{q_2} is a constant as in Lemma 2.1 and

$$\theta_2 = \frac{\varrho\beta_1\kappa_1\tau_2^{q_2-1}}{r_2 - \varrho m_2},$$

$$\vartheta_2 = \nu_2 + \sqrt[q_2]{1 - q_2\delta_2 + c_{q_2}\sigma_2^{q_2}} + \frac{\tau_2^{q_2-1}}{r_2 - \varrho m_2} \sqrt[q_2]{s_2^{q_2}\sigma_2^{q_2} - q_2\varrho\gamma_2 + c_{q_2}\varrho^{q_2}\mu_2^{q_2} + q_2\varrho\alpha_2\mu_2^{q_2}}.$$

It follows from (3.11) and (3.12) that

$$\|a_1 - b_1\| + \|a_2 - b_2\| \le \upsilon(\|x - \hat{x}\| + \|y - \hat{y}\|), \tag{3.13}$$

where

$$\upsilon = \max\{\theta_1 + \theta_2, \vartheta_1 + \vartheta_2\}.$$

It follows from condition (3.4) that $\upsilon < 1$. Hence, from (3.13), we get

$$\begin{aligned}
d((a_1,a_2), G_{\rho,\varrho}(\hat{x},\hat{y},\omega,\lambda)) &= \inf_{(b_1,b_2)\in G_{\rho,\varrho}(\hat{x},\hat{y},\omega,\lambda)} (\|a_1 - b_1\| + \|a_2 - b_2\|)\\
&\le \upsilon\|(x,y) - (\hat{x} - \hat{y})\|_1.
\end{aligned}$$

Since $(a_1,a_2) \in G_{\rho,\varrho}(x,y,\omega,\lambda)$ is arbitrary, we obtain

$$\sup_{(a_1,a_2)\in G_{\rho,\varrho}(x,y,\omega,\lambda)} d((a_1,a_2), G_{\rho,\varrho}(\hat{x},\hat{y},\omega,\lambda)) \le \upsilon\|(x,y) - (\hat{x} - \hat{y})\|_1.$$

By using the same argument, we can prove

$$\sup_{(b_1,b_2)\in G_{\rho,\varrho}(\hat{x},\hat{y},\omega,\lambda)} d(G_{\rho,\varrho}(x,y,\omega,\lambda), (b_1,b_2)) \le \upsilon\|(x,y) - (\hat{x} - \hat{y})\|_1.$$

It follows from the definition of the Hausdorff metric $\hat{\mathbf{H}}$ on $CB(\mathcal{B}_1 \times \mathcal{B}_2)$ that

$$\hat{\mathbf{H}}(G_{\rho,\varrho}(x,y,\omega,\lambda), G_{\rho,\varrho}(\hat{x},\hat{y},\omega,\lambda)) \le \upsilon\|(x,y) - (\hat{x},\hat{y})\|_1$$

for all $(x, \hat{x}, \omega) \in \mathcal{B}_1 \times \mathcal{B}_1 \times \Omega$, $(y, \hat{y}, \lambda) \in \mathcal{B}_2 \times \mathcal{B}_2 \times \Lambda$, i.e., $G_{\rho,\varrho}(x, y, \omega, \lambda)$ is a multi-valued contractive mapping, which is uniform with respect to $(\omega, \lambda) \in \Omega \times \Lambda$. By a fixed point theorem of Nadler [14], for each $(\omega, \lambda) \in \Omega \times \Lambda$, $G_{\rho,\varrho}(x, y, \omega, \lambda)$ has a fixed point $(x(\omega), y(\lambda)) \in \mathcal{B}_1 \times \mathcal{B}_2$, i.e., $(x(\omega), y(\lambda)) \in G_{\rho,\varrho}(x(\omega), y(\lambda), \omega, \lambda)$. By the definition of G, we know that there exist $u(\omega) \in S(x(\omega), \omega)$ and $v(\lambda) \in T(y(\lambda), \lambda)$ such that (3.1) holds. Thus, it follows from Lemma 3.1 that $(x(\omega), y(\lambda)) \in Q(\omega, \lambda)$ is a solution of the problem (2.1) and so $Q(\omega, \lambda) \neq \emptyset$ for all $(\omega, \lambda) \in \Omega \times \Lambda$.

Next, we prove the conclusion (2). For each $(\omega, \lambda) \in \Omega \times \Lambda$, let $\{(x_n, y_n)\} \subset Q(\omega, \lambda)$ and $x_n \to x_0$, $y_n \to y_0$ as $n \to \infty$. Then we have $(x_n, y_n) \in G_{\rho,\varrho}(x_n, y_n, \omega, \lambda)$ for all $n = 1, 2, \ldots$. By the proof of conclusion (1), we have

$$\hat{\mathbf{H}}(G_{\rho,\varrho}(x_n, y_n, \omega, \lambda), G_{\rho,\varrho}(x_0, y_0, \omega, \lambda)) \le \upsilon \|(x_n, y_n) - (x_0, y_0)\|_1, \quad \forall (\omega, \lambda) \in \Omega \times \Lambda.$$

It follows that

$$\begin{aligned} d((x_0, y_0), G_{\rho,\varrho}(x_0, y_0, \omega, \lambda)) &\le \|(x_0, y_0) - (x_n, y_n)\|_1 + d((x_n, y_n), G_{\rho,\varrho}(x_n, y_n, \omega, \lambda)) \\ &\quad + \hat{\mathbf{H}}(G_{\rho,\varrho}(x_n, y_n, \omega, \lambda), G_{\rho,\varrho}(x_0, y_0, \omega, \lambda)) \\ &\le (1 + \upsilon)\|(x_n, y_n) - (x_0, y_0)\|_1. \end{aligned}$$

Hence, we have $(x_0, y_0) \in G_{\rho,\varrho}(x_0, y_0, \omega, \lambda)$ and $(x_0, y_0) \in Q(\omega, \lambda)$. Therefore, $Q(\omega, \lambda)$ is a nonempty closed subset of $\mathcal{B}_1 \times \mathcal{B}_2$. □

Theorem 3.2. *Under the hypotheses of Theorem 3.1, we suppose additionally*

(i) *for any $x \in \mathcal{B}_1$, $\omega \to S(x, \omega)$ is l_S-$\hat{\mathbf{H}}$-Lipschitz continuous (or continuous) and for any $y \in \mathcal{B}_2$, $\lambda \to T(y, \lambda)$ is l_T-$\hat{\mathbf{H}}$-Lipschitz continuous (or continuous);*

(ii) *for any $x, z \in \mathcal{B}_1$ and $y, t \in \mathcal{B}_2$, $\omega \to E(x, y, \omega)$, $\omega \to f(x, \omega)$, $\omega \to J_{\rho,A_1}^{M(\cdot,x,\omega)}(z)$, $\lambda \to F(x, y, \lambda)$, $\lambda \to g(y, \lambda)$ and $\lambda \to J_{\varrho,A_2}^{N(\cdot,y,\lambda)}(t)$ are Lipschitz continuous (or continuous) with Lipschitz constants l_E, l_f, l_{J_1}, l_F, l_g and l_{J_2}, respectively.*

Then, the solution map Q of the problem (2.1) is Lipschitz continuous (or continuous) from $\Omega \times \Lambda$ to $\mathcal{B}_1 \times \mathcal{B}_2$.

Proof. From the hypotheses of the theorem and Theorem 3.1, for any $(\omega, \lambda), (\bar{\omega}, \bar{\lambda}) \in \Omega \times \Lambda$, we know that $Q(\omega, \lambda)$ and $Q(\bar{\omega}, \bar{\lambda})$ are both nonempty closed subsets. By the proof of Theorem 3.1, $G_{\rho,\varrho}(x, y, \omega, \lambda)$ and $G_{\rho,\varrho}(x, y, \bar{\omega}, \bar{\lambda})$ are both multi-valued contractive mappings with the same contraction constant $\upsilon \in (0, 1)$ and have fixed points $(x(\omega, \lambda), y(\omega, \lambda))$ and $(x(\bar{\omega}, \bar{\lambda}), y(\bar{\omega}, \bar{\lambda}))$, respectively. From Lemmas 3.1 and 2.2, we get

$$\begin{aligned} x(\omega, \lambda) &= x(\omega, \lambda) - f(x(\omega, \lambda), \omega) + J_{\rho,A_1}^{M(\cdot,x(\omega,\lambda),\omega)}(A_1(f(x(\omega, \lambda), \omega)) - \rho E(x(\omega, \lambda), v(\omega, \lambda), \omega)), \\ y(\omega, \lambda) &= y(\omega, \lambda) - g(y(\omega, \lambda), \lambda) + J_{\varrho,A_2}^{N(\cdot,y(\omega,\lambda),\lambda)}(A_2(g(y(\omega, \lambda), \lambda)) - \varrho F(u(\omega, \lambda), y, \lambda)), \\ x(\bar{\omega}, \bar{\lambda}) &= x(\bar{\omega}, \bar{\lambda}) - f(x(\bar{\omega}, \bar{\lambda}), \bar{\omega}) + J_{\rho,A_1}^{M(\cdot,x(\bar{\omega},\bar{\lambda}),\bar{\omega})}(A_1(f(x(\bar{\omega}, \bar{\lambda}), \bar{\omega})) - \rho E(x(\bar{\omega}, \bar{\lambda}), v(\bar{\omega}, \bar{\lambda}), \bar{\omega})), \\ y(\bar{\omega}, \bar{\lambda}) &= y(\bar{\omega}, \bar{\lambda}) - g(y(\bar{\omega}, \bar{\lambda}), \bar{\lambda}) + J_{\varrho,A_2}^{N(\cdot,y(\bar{\omega},\bar{\lambda}),\bar{\lambda})}(A_2(g(y(\bar{\omega}, \bar{\lambda}), \bar{\lambda})) - \varrho F(u(\bar{\omega}, \bar{\lambda}), y(\bar{\omega}, \bar{\lambda}), \bar{\lambda})) \end{aligned} \tag{3.14}$$

and

$$\begin{aligned} &\hat{\mathbf{H}}(Q(\omega, \lambda), Q(\bar{\omega}, \bar{\lambda})) \\ &\quad \le \frac{1}{1 - \upsilon} \sup_{(x,y) \in \mathcal{B}_1 \times \mathcal{B}_2} \hat{\mathbf{H}}(G_{\rho,\varrho}(x(\omega, \lambda), y(\omega, \lambda), \omega, \lambda), G_{\rho,\varrho}(x(\bar{\omega}, \bar{\lambda}), y(\bar{\omega}, \bar{\lambda}), \bar{\omega}, \bar{\lambda})). \end{aligned} \tag{3.15}$$

Setting any $(a_1, a_2) \in G_{\rho,\varrho}(x(\omega, \lambda), y(\omega, \lambda), \omega, \lambda)$, there exist $u(\omega, \lambda) \in S(x(\omega, \lambda), \omega)$, $v(\omega, \lambda) \in T(y(\omega, \lambda), \lambda)$ such that

$$\begin{aligned} a_1 &= x(\omega, \lambda) - f(x(\omega, \lambda), \omega) + J_{\rho,A_1}^{M(\cdot,x(\omega,\lambda),\omega)}(A_1(f(x(\omega, \lambda), \omega)) - \rho E(x(\omega, \lambda), v(\omega, \lambda), \omega)), \\ a_2 &= y(\omega, \lambda) - g(y(\omega, \lambda), \lambda) + J_{\varrho,A_2}^{N(\cdot,y(\omega,\lambda),\lambda)}(A_2(g(y(\omega, \lambda), \lambda)) - \varrho F(u(\omega, \lambda), y, \lambda)). \end{aligned} \tag{3.16}$$

Since $S(x(\omega,\lambda),\omega), S(x(\bar{\omega},\bar{\lambda}),\bar{\omega}) \in CB(\mathcal{B}_1)$, $T(y(\omega,\lambda),\lambda), T(y(\bar{\omega},\bar{\lambda}),\bar{\lambda}) \in CB(\mathcal{B}_2)$, it follows from Nadler's result [14] that there exist $u(\bar{\omega},\bar{\lambda}) \in S(x(\bar{\omega},\bar{\lambda}),\bar{\omega})$ and $v(\bar{\omega},\bar{\lambda}) \in T(y(\bar{\omega},\bar{\lambda}),\bar{\lambda})$ such that

$$\|u(\omega,\lambda) - u(\bar{\omega},\bar{\lambda})\| \le \hat{\mathbf{H}}(S(x(\omega,\lambda),\omega), S(x(\bar{\omega},\bar{\lambda}),\bar{\omega})),$$
$$\|v(\omega,\lambda) - v(\bar{\omega},\bar{\lambda})\| \le \hat{\mathbf{H}}(T(y(\omega,\lambda),\lambda), T(y(\bar{\omega},\bar{\lambda}),\bar{\lambda})).$$

Let

$$\begin{aligned} b_1 &= x(\bar{\omega},\bar{\lambda}) - f(x(\bar{\omega},\bar{\lambda}),\bar{\omega}) + J_{\rho,A_1}^{M(\cdot,x(\bar{\omega},\bar{\lambda}),\bar{\omega})}(A_1(f(x(\bar{\omega},\bar{\lambda}),\bar{\omega})) - \rho E(x(\bar{\omega},\bar{\lambda}), v(\bar{\omega},\bar{\lambda}),\bar{\omega})), \\ b_2 &= y(\bar{\omega},\bar{\lambda}) - g(y(\bar{\omega},\bar{\lambda}),\bar{\lambda}) + J_{\varrho,A_2}^{N(\cdot,y(\bar{\omega},\bar{\lambda}),\bar{\lambda})}(A_2(g(y(\bar{\omega},\bar{\lambda}),\bar{\lambda})) - \varrho F(u(\bar{\omega},\bar{\lambda}), y(\bar{\omega},\bar{\lambda}),\bar{\lambda})). \end{aligned} \tag{3.17}$$

Then we have $(b_1, b_2) \in G_{\rho,\varrho}(x(\bar{\omega},\bar{\lambda}), y(\bar{\omega},\bar{\lambda}),\bar{\omega},\bar{\lambda})$. It follows from the assumptions on f, $J_{\rho,A_1}^{M(\cdot,\cdot,\cdot)}$, E, A_1, T, (3.16) and (3.17) that

$$\begin{aligned} \|a_1 - b_1\| \le\ & \|x(\omega,\lambda) - f(x(\omega,\lambda),\omega) + J_{\rho,A_1}^{M(\cdot,x(\omega,\lambda),\omega)}(A_1(f(x(\omega,\lambda),\omega)) - \rho E(x(\omega,\lambda), v(\omega,\lambda),\omega)) \\ & - \{x(\bar{\omega},\bar{\lambda}) - f(x(\bar{\omega},\bar{\lambda}),\omega) + J_{\rho,A_1}^{M(\cdot,x(\bar{\omega},\bar{\lambda}),\omega)}(A_1(f(x(\bar{\omega},\bar{\lambda}),\omega)) - \rho E(x(\bar{\omega},\bar{\lambda}), v(\bar{\omega},\bar{\lambda}),\omega))\}\| \\ & + \|f(x(\bar{\omega},\bar{\lambda}),\omega) - f(x(\bar{\omega},\bar{\lambda}),\bar{\omega})\| \\ & + \|J_{\rho,A_1}^{M(\cdot,x(\bar{\omega},\bar{\lambda}),\omega)}(A_1(f(x(\bar{\omega},\bar{\lambda}),\omega)) - \rho E(x(\bar{\omega},\bar{\lambda}), v(\bar{\omega},\bar{\lambda}),\omega)) \\ & - J_{\rho,A_1}^{M(\cdot,x(\bar{\omega},\bar{\lambda}),\bar{\omega})}(A_1(f(x(\bar{\omega},\bar{\lambda}),\omega)) - \rho E(x(\bar{\omega},\bar{\lambda}), v(\bar{\omega},\bar{\lambda}),\omega))\| \\ & + \|J_{\rho,A_1}^{M(\cdot,x(\bar{\omega},\bar{\lambda}),\bar{\omega})}(A_1(f(x(\bar{\omega},\bar{\lambda}),\omega)) - \rho E(x(\bar{\omega},\bar{\lambda}), v(\bar{\omega},\bar{\lambda}),\omega)) \\ & - J_{\rho,A_1}^{M(\cdot,x(\bar{\omega},\bar{\lambda}),\bar{\omega})}(A_1(f(x(\bar{\omega},\bar{\lambda}),\bar{\omega})) - \rho E(x(\bar{\omega},\bar{\lambda}), v(\bar{\omega},\bar{\lambda}),\bar{\omega}))\| \\ \le\ & \theta_1\|x(\omega,\lambda) - x(\bar{\omega},\bar{\lambda})\| + \vartheta_1\|y(\omega,\lambda) - y(\bar{\omega},\bar{\lambda})\| + k_1\|\omega - \bar{\omega}\|, \end{aligned} \tag{3.18}$$

where θ_1 and ϑ_1 are the same as in (3.11), and

$$k_1 = l_f + l_{J_1} + \frac{(\rho l_E + s_1 l_f)\tau_1^{q_1-1}}{r_1 - \rho m_1}.$$

Similarly, by the assumptions on g, $J_{\rho,A_2}^{N(\cdot,\cdot,\cdot)}$, F, A_2, S, (3.16) and (3.17), we have

$$\|a_2 - b_2\| \le \theta_2\|x(\omega,\lambda) - x(\bar{\omega},\bar{\lambda})\| + \vartheta_2\|y(\omega,\lambda) - y(\bar{\omega},\bar{\lambda})\| + k_2\|\lambda - \bar{\lambda}\|, \tag{3.19}$$

where θ_2 and ϑ_2 are the same as in (3.12), and

$$k_2 = l_g + l_{J_2} + \frac{(\varrho l_F + s_2 l_g)\tau_2^{q_2-1}}{r_2 - \varrho m_2}.$$

It follows from (3.16)–(3.19) and (3.14) that

$$\begin{aligned} \|a_1 - b_1\| + \|a_2 - b_2\| &\le (\theta_1 + \theta_2)\|x(\omega,\lambda) - x(\bar{\omega},\bar{\lambda})\| + (\vartheta_1 + \vartheta_2)\|y(\omega,\lambda) - y(\bar{\omega},\bar{\lambda})\| \\ &\quad + k_1\|\omega - \bar{\omega}\| + k_2\|\lambda - \bar{\lambda}\| \\ &\le \upsilon(\|a_1 - b_1\| + \|a_2 - b_2\|) + k_1\|\omega - \bar{\omega}\| + k_2\|\lambda - \bar{\lambda}\|, \end{aligned} \tag{3.20}$$

where υ is the same as in (3.13). (3.20) implies that

$$\|a_1 - b_1\| + \|a_2 - b_2\| \le \Theta(\|\omega - \bar{\omega}\| + \|\lambda - \bar{\lambda}\|), \tag{3.21}$$

where

$$\Theta = \frac{1}{1-\upsilon}\max\{k_1, k_2\}.$$

Hence, from (3.21), we obtain

$$\sup_{(a_1,a_2)\in G_{\rho,\varrho}(x,y,\omega,\lambda)} d((a_1,a_2), G_{\rho,\varrho}(x,y,\bar{\omega},\bar{\lambda})) \le \Theta\|(\omega,\lambda) - (\bar{\omega},\bar{\lambda})\|_1.$$

By using a similar argument to the above, we get

$$\sup_{(b_1,b_2)\in G_{\rho,\varrho}(x,y,\bar{\omega},\bar{\lambda})} d(G_{\rho,\varrho}(x,y,\omega,\lambda),(b_1,b_2)) \le \Theta\|(\omega,\lambda)-(\bar{\omega},\bar{\lambda})\|_1.$$

It follows that

$$\hat{\mathbf{H}}(G_{\rho,\varrho}(x,y,\omega,\lambda), G_{\rho,\varrho}(x,y,\bar{\omega},\bar{\lambda})) \le \Theta\|(\omega,\lambda)-(\bar{\omega},\bar{\lambda})\|_1,$$

for all $(x,y,\omega,\bar{\omega},\lambda,\bar{\lambda}) \in \mathcal{B}_1\times\mathcal{B}_2\times\Omega\times\Omega\times\Lambda\times\Lambda$. Thus, (3.15) implies

$$\hat{\mathbf{H}}(Q(\omega,\lambda), Q(\bar{\omega},\bar{\lambda})) \le \frac{\Theta}{1-\upsilon}\|(\omega,\lambda)-(\bar{\omega},\bar{\lambda})\|_1.$$

This proves that $Q(\omega,\lambda)$ is Lipschitz continuous with respect to $(\omega,\lambda)\in\Omega\times\Lambda$. If each mapping in conditions (i) and (ii) is assumed to be continuous with respect to $(\omega,\lambda)\in\Omega\times\Lambda$, then by a similar argument to the above, we can show that $Q(\omega,\lambda)$ is continuous with respect to $(\omega,\lambda)\in\Omega\times\Lambda$. □

Remark 3.1. If $\mathcal{B}_1$ and $\mathcal{B}_2$ are both 2-uniformly smooth Banach space, and $\varrho=\rho>0$ is a constant such that

$$\begin{cases} k_1=\nu_1+\sqrt{1-2\delta_1+c_2\sigma_1^2}<1, \qquad k_2=\nu_2+\sqrt{1-2\delta_2+c_2\sigma_2^2}<1,\\ s_1^2\sigma_1^2-2\rho(\gamma_1-\alpha_1\mu_1^2)+c_2\rho^2\mu_1^2<\dfrac{(r_1-\rho m_1)^2}{\tau_1^2}\left(1-k_1-\dfrac{\rho\beta_1\kappa_1\tau_2}{r_2-\rho m_2}\right)^2,\\ s_2^2\sigma_2^2-2\rho(\gamma_2-\alpha_2\mu_2^2)+c_2\rho^2\mu_2^2<\dfrac{(r_2-\rho m_2)^2}{\tau_2^2}\left(1-k_2-\dfrac{\rho\beta_2\kappa_2\tau_1}{r_1-\rho m_1}\right)^2, \end{cases}$$

then (3.4) holds. We note that Hilbert space and L_p (or l_p) $(2\le p<\infty)$ spaces are 2-uniformly smooth Banach spaces.

Remark 3.2. In Theorems 3.1 and 3.2, if E, F are strongly accretive in the first and second variables, i.e., when $\gamma_i=0$ $(i=1,2)$ in Theorems 3.1 and 3.2, respectively, then we can obtain the corresponding results. Our results improve and generalize the known results from [1,3–5,7,9,10,17–19].

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