Optim Lett
DOI 10.1007/s11590-011-0350-y

ORIGINAL PAPER

# Optimality of $E$-pseudoconvex multi-objective programming problems without constraint qualifications

GuoLin Yu · Yangyang Lu · Yan Jiang

Received: 15 July 2010 / Accepted: 23 May 2011


**Abstract** In this note, we consider the optimality criteria of multi-objective programming problems without constraint qualifications involving generalized convexity. Under the $E$-pseudoconvexity assumptions, the unified necessary and sufficient optimality conditions are established for weakly efficient and efficient solutions, respectively, in multi-objective programming problems.

**Keywords** Multi-objective programming · $E$-Convexity · $E$-Pseudoconvex function · Efficient solution

## 1 Introduction

Convexity and its various generalizations played a dominant role in multi-objective programming problems. In the past decades years, attempts have been made to weaken the convexity hypotheses and thus to explore the extent of optimality conditions and duality applicability. A significant generalization of convex functions is $E$ convex functions, introduced by Youness [9]. This kind of generalized convexity is based on the effect of an operator on the sets and domain of definition of functions. The initial results of Youness [9] inspired a great deal of subsequent work which has greatly expanded the role of $E$-convexity in optimization theory. Especially, much attention has been paid on extending the notion of $E$-convexity to the new classes of generalized $E$-convex functions and studied their properties. For example: semi-$E$-convex [1], semi strong $E$-convex [13,14], $E$-quasiconvex [15], $E$-pseduoconvex [5], $E$-$B$-vex [9] and $E$-preinvex [4], etc.

G. Yu (✉) · Y. Lu · Y. Jiang
Research Institute of Information and System Computation Science,
Beifang University of Nationalities, Yinchuan 750021, People's Republic of China
e-mail: guolin_yu@126.com

Published online: 10 June 2011 

Recently, in literatures [14] and [16], Youness investigated the characterization of efficient solutions for multi-objective programming problems involving $E$-convexity and semi strong $E$-convexity, respectively. In an earlier paper [17], we established the saddle points theorems and Lagrange dual theorems for multi-objective programming problems involving $E$-convexity. In this work, we shall propose the unified necessary and sufficient optimality conditions for weakly efficient and efficient solutions, respectively, in multi-objective programming problems under the assumptions of $E$-pseudoconvex functions. It is worth noticing that our results need not any constraint qualifications. This paper is divided into three sections. Section 2 includes preliminaries and related concepts which will be used in later sections. Section 3 is devoted to establish the unified necessary and sufficient optimality conditions.

## 2 Preliminaries

Let $R^n$ be the $n$-dimensional Euclidean space and $R^1$ be the set of all real numbers. Throughout this paper, the following convention for vectors in $R^n$ will be followed:

$$\begin{array}{lll} x < y & \text{if and only if} & x_i < y_i, i = 1, 2, \ldots, n, \\ x \leqq y & \text{if and only if} & x_i \leqq y_i, i = 1, 2, \ldots, n, \\ x \leqslant y & \text{if and only if} & x_i \leqq y_i, i = 1, 2, \ldots, n, \text{ but } x \neq y, \\ x \nless y & \text{is the negation of} & x < y. \end{array}$$

Now, let us recall the concepts of $E$-convex sets, $E$-convex functions and $E$-pseudoconvex functions, for more details, see [1–15].

**Definition 2.1** A set $X \subset R^n$ is said to be $E$-convex if there is a mapping $E: R^n \to R^n$ such that

$$\lambda E(x) + (1 - \lambda)E(y) \in X$$

for all $x, y \in X$ and $\lambda \in [0, 1]$.

It has been pointed out in Ref. [9] that if a set $X \subset R^n$ is $E$-convex, then $E(X) \subset X$; If $E(X)$ is a convex set and $E(X) \subset X$, then $X$ is $E$-convex.

**Definition 2.2** Let $X \subset R^n$ be a $E$-convex set. A real-valued function $f : X \to R^1$ is said to be $E$-convex if

$$f(\lambda E(x) + (1 - \lambda)E(y)) \leqq \lambda f(E(x)) + (1 - \lambda) f(E(y)),$$

for all $x, y \in X$ and $\lambda \in [0, 1]$.

**Definition 2.3** Let $f : R^n :\to R^1, E : R^n \to R^n$ are differentiable in an open $E$-convex set $X \subset R^n$. $f$ is said to be a $E$-quasiconvex function if

$$f(\lambda E(x) + (1 - \lambda)E(y)) \leqq \max\{f(E(x)), f(E(y)\},$$

for all $x, y \in X$ and $\lambda \in [0, 1]$; and strictly $E$-quasiconvex if strict inequality holds for all $x, y \in X$, $E(x) \neq E(y)$ and $\lambda \in (0, 1)$.

*Remark 2.1* It has been pointed out in Ref. [13] that any strictly $E$-quasiconvex function is $E$-quasiconvex.

**Definition 2.4** Let $f : R^n :\to R^1, E : R^n \to R^n$ are differentiable in an open $E$-convex set $X \subset R^n$. $f$ is said to be a $E$-pseudoconvex if

$$\nabla f(E(y))^T (E(x) - E(y)) \geqq 0 \Rightarrow f(E(x)) \geqq f(E(y)), \forall\, x, y \in X.$$

*Example 2.1* Let $E(x) = |x|$ and

$$f(x) = \begin{cases} 1 - \sqrt{1-(x-1)^2}, & 1 \leqq x < 2 \\ 0, & -1 < x < 1 \\ 1 - \sqrt{1-(x+1)^2}, & -2 < x \leqq -1 \end{cases}$$

Then, $f$ is a $E$-pseudoconvex function on the interval $(-2, 2)$. In fact, $f$ is also a $E$-quasiconvex function.

It is obviously that $E$-convex functions are $E$-pseudoconvex. But, the reverse is not correct.

*Example 2.2* Assume that $f(x) = \cos x, x \in [0, 2\pi]$, and

$$E(x) = \begin{cases} x, & 0 \leqq x \leqq \pi \\ x - \pi, & \pi < x \leqq 2\pi \end{cases}$$

Then $f$ is $E$-pseudoconvex on the interval $[0, 2\pi]$. However, $f$ is not $E$-convex on the interval $[0, \frac{\pi}{2}]$.

The following Lemma 2.1 shows that the $E$-pseudoconvex functions must be $E$-quasiconvex, and this fact will be used in the sequel.

**Lemma 2.1** *Let $f : R^n :\to R^1, E : R^n \to R^n$ are differentiable in an open $E$-convex set $X \subset R^n$. If $f$ is $E$-pseudoconvex then $f$ is strictly $E$-quasiconvex and $E$-quasiconvex.*

*Proof* Assuming that $f$ is not strictly $E$-quasiconvex on $X$, then there exist $x, y \in X$, $f(E(x)) < f(E(y))$ and $\lambda \in (0, 1)$ such that

$$f(\lambda E(x) + (1-\lambda)E(y)) \geqq f(E(y)) > f(E(x)). \tag{2.1}$$

Let $E(u) = \lambda E(x) + (1-\lambda)E(y)$. It follows from the $E$-pseudoconvexity of $f$ that

$$\nabla f(E(u))^T (E(x) - E(u)) < 0.$$

Since $E(x) - E(u) = -\frac{1-\lambda}{\lambda}(E(y) - E(u))$, we get

$$\nabla f(E(u))^T (E(y) - E(u)) > 0.$$

By the $E$-pseudoconvexity of $f$ again, we have

$$f(E(y)) \geqq f(E(u)).$$

Hence, it yields from (2.1) that $f(E(y)) = f(E(u))$. Noticing that

$$0 < \nabla f(E(u))^T (E(y) - E(u)) = \lim_{t \to 0^+} \frac{f(E(u) + t(E(y) - E(u))) - f(E(u))}{t}.$$

For small enough $1 > t > 0$, we get

$$f(E(u) + t(E(y) - E(u))) > f(E(u)) = f(E(y)).$$

Let $E(v) = tE(y) + (1-t)E(u)$, by the $E$-pseudoconvexity of $f$ again, it follows that

$$\nabla f(E(v))^T (E(y) - E(v)) < 0,$$
$$\nabla f(E(v))^T (E(u) - E(v)) < 0.$$

Since $E(u) - E(v) = \frac{t}{1-t}(E(v) - E(y))$, the above two inequalities can not hold simultaneously. Thus, we get a contradiction. So, $f$ is strictly $E$-quasiconvex. Finally, we get from Remark 2.1 that $f$ is also $E$-quasiconvex. This completes the proof.

Let $f = (f_1, f_2, \ldots, f_m)$ and $g = (g_1, g_2, \ldots, g_p)$ be vector valued functions on $R^n$, and $f$, $g$ be differentiable. Considering the following multi-objective programming problem (MP):

$$\begin{aligned} \min\ & f(x) = (f_1(x), f_2(x), \ldots, f_m(x)) \\ \text{s.t.}\ \ & x \in X = \{x \in R^n : \ g(x) \leqq 0\}. \end{aligned} \tag{MP}$$

We denote

$$M = \{1, 2, \ldots, m\}, \quad P = \{1, 2, \ldots, p\}.$$

**Definition 2.5** A point $\bar{x} \in X$ is said to be an efficient solution of problem (MP), if $f(x) \nleq f(\bar{x})$ for all $x \in X$.

**Definition 2.6** A point $\bar{x} \in X$ is said to be a weakly efficient solution of problem (MP), if $f(x) \nless f(\bar{x})$ for all $x \in X$.

## 3 Optimality criteria without constraint qualifications

In this section, we shall deal with the optimality conditions of problem (MP) (formulated in Sect. 2) under the assumptions of $E$-pseudoconvexity.

**Theorem 3.1** *In problem* (MP)*, let* $f_i : R^n \to R^1$ $(i \in M)$ *and* $g_j : R^n \to R^1$ $(j \in P)$ *be* $E$*-pseudoconvex real valued functions with respect to the same mapping* $E : R^n \to R^n$*, and* $E(X) \subset X$ *is convex and the mapping* $E$ *is one-to-one and onto. Suppose that* $\bar{x}$ *is a feasible solution of* (MP)*. Then,* $E(\bar{x}) \in X$ *is an efficient solution of* (MP) *if and only if for all nonempty subset* $M_1 \subset M$ *and subset* $P_1 \subset I(E(\bar{x})) = \{i \in P : g_i(E(\bar{x})) = 0, E(\bar{x}) \in X\}$*, the following inequalities*

$$[M_1, P_1] \begin{cases} z^T \nabla f_i(E(\bar{x})) < 0, & i \in M_1 \\ z^T g_j(E(\bar{x})) < 0, & j \in P_1 \\ f_k(E(\bar{x}) + \alpha z) = f_k(E(\bar{x})), & k \in M - M_1,\ \alpha \in [0, \bar{\alpha}],\ \bar{\alpha} > 0 \\ g_l(E(\bar{x}) + \alpha z) = g_l(E(\bar{x})), & l \in I(E(\bar{x})). \end{cases} \tag{3.1}$$

*have no solutions in* $R^n$.

*Proof* Necessity. Assuming that there exist a nonempty subset $M_1 \subset M$ and a subset $P_1 \subset I(E(\bar{x}))$ such that the inequalities $[M_1, P_1]$ have a solution $\bar{z}$. Since

$$f_i(E(\bar{x}) + \alpha\bar{z}) = f_i(E(\bar{x})) + \alpha\bar{z}^T \nabla f_i(E(\bar{x})) + O(\alpha), \quad i \in M_1$$

and $\bar{z}^T \nabla f_i(E(\bar{x})) < 0$, for small enough $\alpha > 0$ we get

$$f_i(E(\bar{x}) + \alpha\bar{z}) < f_i(E(\bar{x})), \quad i \in M_1.$$

Noticing that

$$f_k(E(\bar{x}) + \alpha\bar{z}) = f_k(E(\bar{x})), \quad k \in M - M_1.$$

Thus, we have

$$f(E(\bar{x}) + \alpha\bar{z}) \leqslant f(E(\bar{x})) \tag{3.2}$$

By the similar arguments, we can get

$$g(E(\bar{x}) + \alpha\bar{z}) \leqq g(E(\bar{x})) = 0. \tag{3.3}$$

It yields from (3.2) and (3.3) that $E(\bar{x})$ is not an efficient solution of problem (MP). This contradicts to the given conditions.

Sufficiency. We proceed by contradiction. Suppose that $E(\bar{x})$ is not an efficient solution of problem (MP), then there exists $\bar{\alpha} > 0$, $\bar{z} \in R^n$, such that

$$\begin{aligned} &f(E(\bar{x}) + \bar{\alpha}\bar{z}) \leqslant f(E(\bar{x})), \\ &g(E(\bar{x}) + \bar{\alpha}\bar{z}) \leqq 0. \end{aligned} \tag{3.4}$$

According to the $E$-pseudoconvex of $f$, we can get from (3.4) that

$$[(E(\bar{x})+\bar{\alpha}\bar{z})-E(\bar{x})]^T\nabla f(E(\bar{x}))=\bar{\alpha}\bar{z}^T\nabla f(E(\bar{x}))\leqq 0.$$

By $\bar{\alpha}>0$, that is

$$\bar{z}^T\nabla f(E(\bar{x}))\leqq 0.$$

In addition, there is at least an index $i$ in (3.4) such that $f_i(E(\bar{x})+\bar{\alpha}\bar{z})<f_i(E(\bar{x}))$. From the $E$-pseudoconvex of $f$ again, we can get $\bar{z}^T\nabla f_i(E(\bar{x}))<0$. Therefore, it follows that

$$\bar{z}^T\nabla f(E(\bar{x}))\leqslant 0. \tag{3.5}$$

Now, denote $M_1=\{i\in M:\ \bar{z}^T\nabla f_i(E(\bar{x}))<0\}$ and $M_0=\{i\in M:\ \bar{z}^T\nabla f_i(E(\bar{x}))=0\}$. It is clear that $M_1\neq\emptyset$ and $M_0\subset M-M_1$. Thus, the inequality (3.5) can be decomposed into the following two inequalities:

$$\bar{z}^T\nabla f_i(E(\bar{x}))<0,\quad i\in M_1, \tag{3.6}$$
$$\bar{z}^T\nabla f_k(E(\bar{x}))=0,\quad k\in M-M_1. \tag{3.7}$$

It yields from (3.6) that $\bar{z}$ is a solution of the first inequality in (3.1).

Next, we shall prove that $\bar{z}$ is a solution of the third inequality in (3.1). In fact, by (3.7), for any $\alpha>0$ we get

$$[(E(\bar{x})+\alpha\bar{z})-E(\bar{x})]^T\nabla f_k(E(\bar{x}))=0,\quad k\in M-M_1.$$

Since $f_k(x)$ is $E$-pseudoconvex, for all $\alpha\geqq 0$ we get from the above equation that

$$f_k(E(\bar{x})+\alpha\bar{z})\geqq f_k(E(\bar{x})),\quad k\in M-M_1. \tag{3.8}$$

Combining (3.8) with (3.4), we have

$$f_k(E(\bar{x})+\bar{\alpha}\bar{z})=f_k(E(\bar{x})),\quad k\in M-M_1. \tag{3.9}$$

On the other hand, any point in the line segment between $E(\bar{x})$ and $E(\bar{x})+\bar{\alpha}\bar{z}$ can be expressed as

$$(1-\lambda)(E(\bar{x})+\bar{\alpha}\bar{z})+\lambda E(\bar{x})=E(\bar{x})+(\bar{\alpha}-\bar{\alpha}\lambda)\bar{z},\quad \lambda\in[0,1]. \tag{3.10}$$

Let $\alpha=\bar{\alpha}-\lambda\bar{\alpha}$. If $\lambda\in[0,1]$, then $\alpha\in[0,\bar{\alpha}]$. Therefore, the Eq. (3.10) can be written as

$$(1-\lambda)(E(\bar{x})+\bar{\alpha}\bar{z})+\lambda E(\bar{x})=E(\bar{x})+\alpha\bar{z}.$$

Since $f_k$ is $E$-pseudoconvex, it follows from Lemma 2.1 that $f_k$ is $E$-quasiconvex. So we get from Definition 2.3 that

$$\begin{aligned} f_k(E(\bar{x})+\alpha\bar{z}) &= f_k((1-\lambda)(E(\bar{x})+\bar{\alpha}\bar{z})+\lambda E(\bar{x})) \\ &\leqq \max\{f_k(E(\bar{x}), f_k(E(\bar{x})+\bar{\alpha}\bar{z})), \\ &= f_k(E(\bar{x}), \quad k\in M-M_1,\ \alpha\in[0,\bar{\alpha}]. \end{aligned}$$

Combining the above inequality with (3.8), we have

$$f_k(E(\bar{x})+\alpha\bar{z}) = f_k(E(\bar{x}),\ k\in M-M_1, \quad \alpha\in[0,\bar{\alpha}],$$

which shows that $\bar{z}$ is a solution of the third inequality in (3.1).

Finally, it is obviously that

$$g_j(E(\bar{x})+\bar{\alpha}\bar{z}) \leqq g_j(E(\bar{x}))=0, \quad j\in I(E(\bar{x})).$$

This inequality is similar to (3.4), and $g_j(x)$ is also $E$-pseudoconvex. By the similarly argument, we can find a subset $P_1\subset I(E(\bar{x}))$, such that $\bar{z}$ is the solution of the second and the forth inequalities in (3.1). The proof is completed. □

**Theorem 3.2** *In problem (MP), let $f_i : R^n\to R^1$ ($i\in M$) and $g_j : R^n\to R^1$ ($j\in P$) be $E$-pseudoconvex real valued functions with respect to the same mapping $E : R^n\to R^n$, and $E(X)\subset X$ is convex and the mapping $E$ is one-to-one and onto. Suppose that $\bar{x}$ is a feasible solution of (MP). Then, $E(\bar{x})\in X$ is a weakly efficient solution of (MP) if and only if for all nonempty subset $\Omega\subset I(E(\bar{x}))=\{i\in P : g_i(E(\bar{x}))=0,\ E(\bar{x})\in M\}$, the following inequalities*

$$[\Omega]\begin{cases} z^T\nabla f(E(\bar{x}))<0 \\ z^T g_j(E(\bar{x}))<0, & j\in\Omega \\ g_l(E(\bar{x})+\alpha z)=0, & l\in I(E(\bar{x}))-\Omega,\ \alpha\in[0,\bar{\alpha}],\ \bar{\alpha}>0 \end{cases} \tag{3.11}$$

*has no solution in $R^n$.*

*Proof* Necessity. Assume that there exists some subset $\Omega\subset I(E(\bar{x}))$ such that (3.11) has a solution $\bar{z}$. Similar to the proof of Theorem 3.1, we can get

$$\begin{aligned} &f(E(\bar{x})+\alpha\bar{z}) < f(E(\bar{x})), \\ &g_j(E(\bar{x})+\alpha\bar{z}) < g_j(E(\bar{x})), \quad j\in\Omega, \\ &g_l(E(\bar{x})+\alpha\bar{z})=0, \quad l\in I(E(\bar{x}))-\Omega. \end{aligned}$$

Which shows that $E(\bar{x})$ is not weakly efficient solution for problem (MP), a contradiction.

Sufficiency. If $E(\bar{x}))$ is not a weakly efficient solution for problem (MP), then there exists $\bar{\alpha}>0,\ \bar{z}\in R^n$, such that

$$f(E(\bar{x})+\bar{\alpha}\bar{z}) < f(E(\bar{x})), \quad g(E(\bar{x})+\bar{\alpha}\bar{z}) \leqq 0.$$

It is still similar to the proof of Theorem 3.1, respectively, we can get

$$\begin{aligned} &\bar{z}^T \nabla f(E(\bar{x})) < 0, \\ &\bar{z}^T \nabla g_j(E(\bar{x})) \leqq 0, \quad j \in I(E(\bar{x})) \end{aligned} \tag{3.12}$$

Denote $\Omega = \{j \in (E(\bar{x}) : \bar{z}^T \nabla g_j(E(\bar{x})) < 0\}$, then the inequality (3.12) can be divided into the following two parts

$$\bar{z}^T \nabla g_j(E(\bar{x})) < 0, \quad j \in \Omega, \tag{3.13}$$

$$\bar{z}^T \nabla g_j(E(\bar{x})) = 0, \quad j \in I(E(\bar{x})) - \Omega. \tag{3.14}$$

From (3.14), we can also obtain that

$$g_l(E(\bar{x}) + \alpha \bar{z}) = 0, \quad l \in I(E(\bar{x})) - \Omega,\ \alpha \in [0, \bar{\alpha}],\ \bar{\alpha} > 0. \tag{3.15}$$

Combining with (3.13) and (3.15), it yields that there exists $\Omega \subset I(E(\bar{x}))$ such that $\bar{z}$ is a solution of inequalities (3.11). This is a contradiction to the given conditions. The proof is completed.

## 4 Conclusion

We have established the unified necessary and sufficient optimality conditions for weakly efficient and efficient solutions, respectively, in multi-objective programming problems under the $E$-pseudoconvexity assumptions. The results of this paper deepen and enrich the theory of multi-objective optimization.

On the other hand, there exist a great variety of literatures regarding the optimality conditions of multi-objective optimization. Readers who have a mind to comprehend recent developments in multi-objective optimization, can compare for the work of Chinchuluun and Pardalos [2], or consult for the monographs of Pardalos et al. [7] and Zopounidis and Pardalos [18].

**Acknowledgments** This research is supported by Zizhu Science Foundation of Beifang University of Nationalities; Natural Science Foundation for the Youth (No. 10901004).

## References

1. Chen, X.: Some properties of semi-$E$-convex functions. J. Math. Anal. Appl. **275**, 251–262 (2002)
2. Chinchuluun, A., Pardalos, P.M.: A survey of recent developments in multiobjective optimization. Ann. Oper. Res. **154**(1), 29–50 (2007)
3. Duca, D.I., Lupsa, L.L.: On the $E$-epigraph of an $E$-convex functions. J. Optim. Theory Appl. **129**(2), 341–348 (2006)
4. Fulga, C., Preda, V.: Nonlinear programming with $E$-preinvex and local $E$-preinvex functions. Eur. J. Oper. Res. **192**, 737–743 (2009)
5. Luo, J.: Some properties of $E$-pseudoconvex functions and the applications in mathematical programming. J. Congqing Technol. Business Univ. **25**(5), 449–453 (2008) (in Chinese)

6. Jian, J.B.: Incorrect result for $E$-convex functions and $E$-convex programming. Math. Res. Expo. **23**(3), 461–466 (2003)
7. Pardalos, P.M., Siskos, Y., Zopounidis, C.: Advances in Multicriteria Analysis. Kluwer, Dordrecht (1995)
8. Syaua, Y.R., Lee, S.: Some properties of $E$-convex functions. Appl. Math. Lett. **18**, 1074–1080 (2005)
9. Syaua, Y.R., Jia, L., Lee, S.: Generalizations of $E$-convex and B-vex functions. Comput. Math. Appl. **58**(4), 711–716 (2009)
10. Yang, X.M.: On $E$-convex set, $E$-convex functions, and $E$-convex programming. J. Optim. Theory Appl. **109**(3), 699–704 (2001)
11. Youness, E.A.: $E$-convex set, $E$-convex functions, and $E$-convex programming. J. Optim. Theory Appl. **102**(2), 439–450 (1999)
12. Youness, E.A.: Optimality criteria in $E$-convex programming[J]. Chaos Solitons Fractals **12**, 1737–1745 (2001)
13. Youness, E.A.: Quasi and strictly quasi $E$-convex functions. Stat. Manag. Syst. **4**, 201–210 (2001)
14. Youness, E.A.: Characterization of efficient solutions of multi-objective $E$-convex programming problems. Appl. Math. Comput. **151**, 755–761 (2004)
15. Youness, E.A., Emam, T.: Semi strongly $E$-convex functions. J. Math. Stat. **1**(1), 51–57 (2005)
16. Youness, E.A., Emam, T.: Characterization of efficient solutions for multi-objective optimization problems involving semi-strong and generalized semei-strong $E$-convexity. Acta Mathematica Scientia B **28**(1), 7–16 (2008)
17. Yu, G., Jiang, Y.: Saddle points optimality criteria and duality for multi-objective $E$-convex programming problems. Int. J. Optim Theory Methods Appl. (2010, in press)
18. Zopounidis, C., Pardalos, P.M.: Handbook of Multicriteria Analysis Series: Applied Optimization, vol. 103. Springer, Berlin (2010)