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Non-existence of first integrals in a Laurent polynomial ring for general semi-quasihomogeneous systems

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Abstract. In this paper, we give some simple criteria of non-integrability and partial integrability in a Laurent polynomial ring $C[u_1^{\pm}, \cdots, u_n^{\pm}]$ for general semi-quasihomogeneous systems.

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1. Introduction

An autonomous system of ordinary differential equations admitting a quasi-homogeneous group of symmetries is called a quasihomogeneous one. The interest of such systems lies in the existence of particular solution in the quasi-homogeneous ray form. Yoshida considered the algebraic integrability problem for quasi-homogeneous systems [12]. Using a singularity analysis type method, he was able to derive necessary conditions for algebraic integrability. Though some imperfections in his proof was found [4], Yoshida's ideas are quite fruitful and useful in this field. Inspired by Yoshida's ideas, Furta [3] made a further step in this direction. He suggested a simple and easily verifiable criterion of non-existence of nontrivial analytic integrals for general analytic autonomous systems. Based on his criterion, he also considered the non-integrability for general semi-quasihomogeneous systems(the definition will be given below). Some similar results related to non-existence of polynomial integrals, rational integrals and analytic integrals can be found in [2, 5, 6, 7, 8, 9, 10, 13].

In [11], we considered the non-existence and partial existence of Laurent polynomial first integrals for a general nonlinear system of ordinary differential equations

$$\dot{x} = Ax + \tilde{f}(x), \quad x = (x_1, \cdots, x_n) \in \mathbb{C}^n \tag{1}$$

in some neighborhood of the origin $x = 0$, where $\tilde{f}(x) = o(x)$. Here, A *Lau-*

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rent polynomial $P(u)$ in the n variables $u = (u_1, \cdots, u_n)$ is given by $P(u) = \sum_{(k_1,\cdots,k_n)\in\mathcal{A}} P_{k_1\cdots k_n} u_1^{k_1} \cdots u_n^{k_n}$, where $P_{k_1\cdots k_n} \in \mathbb{C}$ and $\mathcal{A}$, the support of $P(u)$, is a finite subset of the integer group $\mathbb{Z}^n$.

Let $G = \{\mathbf{k} = (k_1, \cdots, k_n) \in \mathbb{Z}^n : \sum_{j=1}^{n} k_j\lambda_j = 0\}$. We proved the following results [11]:

Theorem A. *If the eigenvalues $\lambda_1, \cdots, \lambda_n$ of A are $\mathbb{Z}$-independent, i.e., they do not satisfy any resonant equality of the following type*

$$\sum_{j=1}^{n} k_j\lambda_j = 0, \quad k_j \in \mathbb{Z}, \quad \sum_{j=1}^{n} |k_j| > 0,$$

then system (1) does not have any nontrivial Laurent polynomial integral.

Theorem B. *Assume system (1) has $s(s < n)$ nontrivial Laurent polynomial integrals $P^1(x), \cdots, P^s(x)$ and matrix A is diagonalizable. If $P^1_{l_1}(x), \cdots, P^s_{l_s}(x)$ are functionally independent and* rank $G = s$*, then any other nontrivial Laurent polynomial integral $Q(x)$ of system (1) must be a function of $P^1(x), \cdots, P^s(x)$.*

In the present paper, we consider the Laurent polynomial first integrals for general semi-quasihomogeneous systems. By using the so-called Kowalevsky exponents, we will give some criteria of non-existence and partial existence of nontrivial Laurent polynomial integrals for general semi-quasihomogeneous systems.

The paper is organized as follows. For completeness, we first describe some elementary definitions and results for semi-quasihomogeneous systems in section 2, which can also be found in [3]. In section 3, we discuss the partial integrability for semi-quasihomogeneous systems. Some examples will be given in the last section to illustrate our results.

2. Quasi-homogeneous and semi-quasihomogeneous systems

Consider a system of differential equations

$$\dot{u} = g(u), \quad u = (u_1, \cdots, u_n) \in \mathbb{C}^n, \tag{2}$$

where $g(u) = (g^1(u), \cdots, g^n(u))$ is a vector-valued function of dimension n.

Definition 1. *System (2) is called a quasi-homogeneous one of degree m with exponents $s_1, \cdots, s_n \in \mathbb{Z}, m > 1$, if for any $\rho \in \mathbb{Z}^+$ and $u = (u_1, \cdots, u_n)$,*

$$g^j(\rho^{s_1}u_1, \cdots, \rho^{s_n}u_n) = \rho^{s_j+m-1} g^j(u_1, \cdots, u_n), \tag{3}$$

i.e., $\rho^{E-S}g(u) = (\rho^{1-s_1}g_1(u), \cdots, \rho^{1-s_n}g_n(u))$ is quasi-homogeneous of degree m, here E is the unit matrix, $S = \mathrm{diag}(s_1, \cdots, s_n)$ and $\rho^{E-S} = \mathrm{diag}(\rho^{1-s_1}, \cdots, \rho^{1-s_n})$.

Definition 2. *We will say that system (2) is semi-quasihomogeneous if*

$$g(u) = g_m(u) + \tilde{g}(u),$$

where $g_m(u)$ is a quasi-homogeneous vector field of degree m with exponents s_1, $\cdots$, s_n and $\rho^{E-S}\tilde{g}(u)$ is the sum of quasi-homogeneous polynomials of degree all larger than m or all less than m. In the former case (respectively, latter), we say that (2) is positively (respectively, negatively) semi-quasihomogeneous.

Let system (2) be semi-quasihomogeneous. Then under the transformation

$$u \to \rho^S u, \quad t \to \rho^{-\alpha} t, \quad \alpha = \frac{1}{m-1}, \tag{4}$$

it becomes

$$\dot{u} = g_m(u) + \tilde{g}(u, \rho), \tag{5}$$

where $\tilde{g}(u,\rho)$ is a formal power series either with respect to ρ (positive semi-quasihomogeneity) or respect to ρ^{-1} (negative semi-quasihomogeneity) without any constant term.

First of all we consider the quasi-homogeneous cut of system (2)

$$\dot{u} = g_m(u). \tag{6}$$

System (6) has particular solutions of the quasi-homogeneous ray form

$$u_0(t) = t^{-H}\xi,$$

where $H = \alpha S$ and the coefficients $\xi \in \mathbb{C}^n$ are given by the algebraic equation $H\xi + g_m(\xi) = 0$. For a given $g(u)$, there may exist different sets of values ξ which will be referred to as different *balances.*

Make the change of variables

$$u = t^{-H}(\xi + x), \quad t = \ln \tau, \tag{7}$$

then system (6) reads

$$x' = Kx + \tilde{f}(x), \tag{8}$$

where prime means the derivative with respect to τ, $K = H + \frac{\partial g_m}{\partial u}(\xi)$ is the so-called Kowalevsky matrix associated to the balance ξ and $\tilde{f}(x) = H\xi + g_m(\xi + x) - \frac{\partial g_m}{\partial u}(\xi)x = o(x)$.

The following statement was shown in [12].

Lemma 1. *$\lambda = -1$ is an eigenvalue of the Kowalevsky matrix K and $\eta = H\xi$ is a corresponding eigenvector.*

Without loss of generality, we assume $\lambda_n = -1$. Our first result is the following

Theorem 1. *Let system (2) be semi-quasihomogeneous system with balance ξ, and $\lambda_1, \cdots, \lambda_n$ be eigenvalues of Kowalevsky matrix K associated to the balance ξ.*

If $\lambda_1, \cdots, \lambda_n$ *are* $\mathbb{Z}$*-independent, i.e., they do not satisfy any resonant condition*

$$\sum_{j=1}^{n} k_j \lambda_j = 0, \quad k_j \in \mathbb{Z}, \quad \sum_{j=1}^{n} |k_j| \geq 1, \tag{9}$$

then system (2) does not have any nontrivial Laurent polynomial integral.

Proof. Assume that system (2) has a Laurent polynomial first integral

$$\Phi(u) = \sum_{(k_1, \cdots, k_n) \in \mathcal{A}} \Phi_{k_1 \cdots k_n} u_1^{k_1} \cdots u_n^{k_n},$$

where $\mathcal{A}$ is a finite subset of $\mathbb{Z}^n$.

In the case of positive semi-quasihomogeneity, $\Phi(u)$ can be rewritten as

$$\Phi(u) = \Phi_L(u) + \Phi_{L+1}(u) + \cdots + \Phi_M(u), \quad L \leq M, \ L, M \in \mathbb{Z},$$

where

$$\Phi_{L+i}(u) = \sum_{\substack{k_1 s_1 + \cdots + k_n s_n = L+i \\ (k_1, \cdots, k_n) \in \mathcal{A}}} \Phi_{k_1 \cdots k_n} u_1^{k_1} \cdots u_n^{k_n}$$

are quasi-homogeneous functions of the degree $L+i$, i.e., $\Phi_{L+i}(\rho^S u) = \rho^{L+i} \Phi_{L+i}(u)$. Under the transformation (4), the system (2) changes to (5), $\Phi(\rho^S u)$ changes to

$$\tilde{\Phi}(u, \rho) = \rho^L (\Phi_L(u) + \rho \Phi_{L+1}(u) + \cdots + \rho^{M-L} \Phi_M(u)),$$

so $\tilde{\Phi}(u, \rho)$ is an integral of the system (5). Note that this integral exists for any value of ρ, thus the shortened system (6) has to have a quasi-homogeneous integral $\tilde{\Phi}(u, 0) = \Phi_L(u)$.

In the case of negative semi-quasihomogeneity, we rewrite $\Phi(u)$ as

$$\Phi(u) = \Phi_M(u) + \Phi_{M+1}(u) + \cdots + \Phi_L(u), \quad M \leq L, \quad M, L \in \mathbb{Z}.$$

Then system (5) has an integral

$$\tilde{\Phi}(u, \rho) = \rho^L (\Phi_L(u) + \rho^{-1} \Phi_{L-1}(u) + \cdots + \rho^{M-L} \Phi_M(u)),$$

which is also defined for any ρ. Thus (6) has to have a quasi-homogeneous integral $\tilde{\Phi}(u, \infty) = \Phi_L(u)$.

Now we make the change of variables (7). After this transformation, (6) becomes (8), the integral $\Phi_L(u)$ becomes

$$\Phi_L(u) = t^{-\alpha L} \Phi_L(\xi + x) = e^{-\alpha L \tau} \Phi_L(\xi + x),$$

So $e^{-\alpha L \tau} \Phi_L(\xi + x)$ is a nonautonomous integral of system(8).

Let $x_0 = e^{-\alpha \tau}$ be a new auxiliary variable. Then the augmented system

$$x_0{}' = -\alpha x_0, \quad x' = Kx + \tilde{f}(x) \tag{10}$$

has a Laurent polynomial integral $P(x_0, x) = x_0^L \Phi_L(\xi + x)$. Write $P(x_0, x)$ as

$$P(x_0, x) = x_0^L (P_l(x) + P_{l+1}(x) + \cdots + P_p(x)), \quad l \leq p, \quad l, p \in \mathbb{Z}, \tag{11}$$

where $P_k(x)$ are homogeneous Laurent polynomials of degree k in x and $P_l(x) \not\equiv 0, P_p(x) \not\equiv 0$. Now we can easily see that $Q(x_0, x) = x_0^L P_l(x)$ is a homogeneous Laurent polynomial integral of linear system

$$x_0{}' = -\alpha x_0, \quad x' = Kx. \tag{12}$$

By Theorem A, if there is no any resonance condition of the following type

$$-k_0\alpha + \sum_{j=1}^{n} k_j\lambda_j = 0, \quad k_0, k_j \in \mathbb{Z}, \quad \sum_{j=0}^{n} |k_j| \geq 1 \tag{13}$$

is fulfilled, then system (10) does not have any Laurent polynomial integral. Note that $\alpha = \frac{1}{m-1}$ and $\lambda_n = -1$, therefore (13) can be rewritten as

$$-[k_0 + (m-1)k_n] + (m-1)\sum_{j=1}^{n-1} k_j\lambda_j = 0,$$

a contradiction.

Remark 1. In general, the balance of (2) is not unique. According to Theorem 1, if we can find a balance such that the non-resonance condition (9) holds, then it is enough to conclude that the system under consideration has no Laurent polynomial first integrals.

Remark 2. If system (2) has a nontrivial Laurent polynomial integral $\Phi(u)$, then system (10) has a Laurent polynomial integral $P(x_0, x)$, and thus the linear system (12) has also a homogeneous Laurent polynomial integral $Q(x_0, x)$.

3. Partial integrability for semi-quasihomogeneous systems

Now we assume that system (2) has s Laurent polynomial integrals $\Phi^1(u), \cdots, \Phi^s(u)$. By remark 1, we know that system (10) has s Laurent polynomial integrals $P^1(x_0, x), \cdots, P^s(x_0, x)$, and the linear systems (12) has s homogeneous Laurent polynomial integrals $Q^1(x_0, x), \cdots, Q^s(x_0, x)$. By Theorem 1, there is at least one resonant relationship of type (9) must be satisfied. Therefore the set

$$G = \left\{ (k_1, \cdots, k_n) \in \mathbb{Z}^n : \sum_{j=1}^{n} k_j\lambda_j = 0 \right\}$$

is a nonempty subgroup of $\mathbb{Z}^n$.

Lemma 2. *Let system (2) have s nontrivial Laurent polynomial integrals $\Phi^1(u), \cdots, \Phi^s(u)$ and let any nontrivial quasi-homogeneous integral $\Phi_L(u)$ of the cut system (6) be a smooth function of the $\Phi^1_{L_1}(u), \cdots, \Phi^s_{L_s}(u)$, i.e., $\Phi_L = \mathcal{H}(\Phi^1_{L_1}, \cdots, \Phi^s_{L_s})$. Then any nontrivial Laurent polynomial integral $\Phi(u)$ of system (2) is a smooth function of $\Phi^1(u), \cdots, \Phi^s(u)$.*

Proof. We prove the lemma only for positive semi-quasihomogeneous system. The negative semi-quasihomogeneous case can be proved similarly.

Under the transformation (7), system (2) can be rewritten in the following form

$$\dot{u} = g_m(u) + \sum_{j=1}^{\infty} \rho^j g_{m+j}(u), \tag{14}$$

where g_{m+j} are quasi-homogeneous vector fields.

Similarly, the integral $\Phi^i(u)$ and $\Phi(u)$ of system (2) can be rewritten as

$$\tilde{\Phi}^i(u,\rho) = \Phi^i(\mu^H u) = \rho^{L_i}(\Phi^i_{L_i}(u) + \rho\Phi^i_{L_i+1}(u) + \cdots + \rho^{M_i-L_i}\Phi^i_{M_i}(u)),$$
$$\tilde{\Phi}(u,\rho) = \Phi(\mu^H u) = \rho^{L}(\Phi_{L}(u) + \rho\Phi_{L+1}(u) + \cdots + \rho^{M-L}\Phi_{L}(u)),$$

where $\Phi^i_{L_i+j}(u)$ and $\Phi_{L+j}(u)$ are the corresponding quasi-homogeneous functions.

Let $\mathcal{H}^{(0)} = \mathcal{H}$. Then the function

$$\tilde{\Phi}^{(1)}(u,\rho) = \tilde{\Phi}(u,\rho) - \mathcal{H}^{(0)}(\tilde{\Phi}^1(u,\rho), \cdots, \tilde{\Phi}^s(u,\rho))$$

is an integral of system (14), since $\tilde{\Phi}(u,\rho)$ and $\tilde{\Phi}^1(u,\rho), \cdots, \tilde{\Phi}^s(u,\rho)$ are all integrals of system (14).

It is not difficult to see that $\tilde{\Phi}^{(1)}(u,\rho)$ is at least of $L+1$ order with respect to ρ, and $\tilde{\Phi}^{(1)}(u,\rho)$ can be rewritten as

$$\tilde{\Phi}^{(1)}(u,\rho) = \rho^{\tilde{L}_1}(\Phi^{(1)}_{\tilde{L}_1}(u) + \sum_{j=1}^{\infty} \rho^j \Phi^{(1)}_{\tilde{L}_1+j}(u)),$$

where $\tilde{L}_1 \geq L+1$ is an integer, $\Phi^{(1)}_{\tilde{L}_1+j}(u)$ is a homogeneous form of degree $\tilde{L}_1 + j$.

Obviously, $\Phi^{(1)}_{\tilde{L}_1}(u)$ is an integral of system (14) as $\rho = 0$, i.e., an integral of the system (6). According to the assumptions of the lemma, $\Phi^{(1)}_{\tilde{L}_1}(u) = \mathcal{H}^{(1)}(\Phi^1_{L_1}(u), \cdots, \Phi^s_{L_s}(u))$. So the function

$$\tilde{\Phi}^{(2)}(u,\rho) = \tilde{\Phi}^{(1)}(u,\rho) - \mathcal{H}^{(1)}(\tilde{\Phi}^1(u,\rho), \cdots, \tilde{\Phi}^s(u,\rho))$$

is also an integral of system (14) which is at least of order $\tilde{L}_1 + 1$ with respect to ρ.

By repeating infinitely many times this process, we obtain that

$$\tilde{\Phi}(u,\rho) = \sum_{j=0}^{\infty} \mathcal{H}^{(j)}(\tilde{\Phi}^1(u,\rho), \cdots, \tilde{\Phi}^s(u,\rho)),$$

which is equivalent to the fact that

$$\Phi(u) = \mathcal{F}(\Phi^1(u), \cdots, \Phi^s(u)),$$

where $\mathcal{F}$ is some smooth function.

Lemma 3. *Assume the Kowalevsky matrix K is diagonalizable. Let $Q(x_0, x)$ be an integral of the augmented linear system (12) which is generated by the quasi-homogeneous integral $\Phi_L(u)$ of system (6). Then $Q(x_0, x)$ does not depend on the last variable x_n.*

Proof. Without loss of generality, we assume that K has already a diagonal form $\mathrm{diag}(\lambda_1, \cdots, \lambda_n)$.

Since $\Phi_L(u)$ is a integral of system (6), we have

$$\left\langle \frac{\mathrm{d}\Phi_L}{\mathrm{d}u}(u), g_m(u) \right\rangle \equiv 0. \tag{15}$$

Under the transformation $u = \xi + x$, (15) is changed to

$$\left\langle \frac{\mathrm{d}\Phi_L}{\mathrm{d}x}(\xi + x), g_m(\xi + x) \right\rangle \equiv 0, \quad \text{for all } x.$$

By (11) we get

$$\left\langle \frac{\mathrm{d}P_l}{\mathrm{d}x}(x), g_m(\xi) \right\rangle = \left\langle \frac{\mathrm{d}P_l}{\mathrm{d}x}(x), -H\xi \right\rangle \equiv 0.$$

By Lemma 1, $H\xi = \eta = (0, \cdots, 0, \eta_n)$ is the eigenvector corresponding to eigenvalue $\lambda_n = -1$, therefore

$$\frac{\partial P_l}{\partial x_n}(x) \equiv 0.$$

So the integral $Q(x_0, x) = x_0^L P_l(x)$ of system (12) does not depend on x_n.

Theorem 2. *Let system (2) be a semi-quasihomogeneous system with balance ξ, and let $\Phi^1(u)$, $\cdots$, $\Phi^s(u)$ be nontrivial Laurent polynomial integrals of (2). Assume that the Kowalevsky matrix K associated to the balance ξ is diagonalizable and* rank $G = s$. *If the integrals $Q^1(x_0, x)$, $\cdots$, $Q^s(x_0, x)$ of system (12) are functionally independent, then any other nontrivial Laurent polynomial integral $\Phi(u)$ of system (2) is a function of $\Phi^1(u)$, $\cdots$, $\Phi^s(u)$.*

Proof. By Lemma 2, we need only show that any quasi-homogeneous integral $\Phi_L(u)$ of (6) is a smooth function of the $\Phi^1_{L_1}(u), \cdots, \Phi^s_{L_s}(u)$. This is equivalent to prove that any integral $P(x_0, x)$ of (10) is a smooth function of the $P^1(x_0, x), \cdots, P^s(x_0, x)$.

For simplicity, we assume that K has already a diagonal form $\mathrm{diag}(\lambda_1, \cdots, \lambda_n)$. By Lemma 3, the integrals $Q(x_0, x)$ and $Q^i(x_0, x)$ of linear system (12) do not depend on the last variable x_n, so they are also homogeneous integrals of the linear system

$$x_0' = -\alpha x_0, \quad x_1' = \lambda_1 x_1, \cdots, x_{n-1}' = \lambda_{n-1} x_{n-1}.$$

Let

$$G_0 = \left\{ (k_0, \cdots, k_{n-1}) \in \mathbb{Z}^n : -\alpha k_0 + \sum_{j=1}^{n-1} k_j \lambda_j = 0 \right\}.$$

Then G_0 is a nonempty subgroup of $\mathbb{Z}^n$ and rank $G_0 = s$, since $\alpha = \frac{1}{m-1}$, $\lambda_n = -1$ and rank $G = s$. Since $Q^1(x_0, x)$, $\cdots$, $Q^s(x_0, x)$ are functionally independent, by

Lemma 2 in [11], there exists a function $\mathcal{H}$ such that

$$Q(x_0, x) = \mathcal{H}(Q^1(x_0, x), \cdots, Q^s(x_0, x)).$$

Now by the same method as used in the proof of Lemma 1 in [11] and using Lemma 3 in each step, we can prove that any integral $P(x_0, x)$ of (10) is a smooth function of the $P^1(x_0, x), \cdots, P^s(x_0, x)$, and this completes the proof.

4. Examples

Example 1. Consider the following quadratic system

$$\dot{u}_i = u_i(a_{i1}u_1 + a_{i2}u_2 + \cdots + a_{in}u_n), \quad i = 1, 2, \cdots, n, \tag{16}$$

where a_{ij} are real constants.

System (16) can be treated as semi-quasihomogeneous system with exponents $s_1 = s_2 = \cdots = s_n = 1$. So $S = E, m = 2, \alpha = 1, H = E$.

Note that if $a_{jj} \neq 0$, then system (16) has a balance $\xi = (0, \cdots, -\frac{1}{a_{jj}}, \cdots, 0)$, and thus (16) has a particular solution $u(t) = (0, \cdots, -\frac{1}{a_{jj}t}, \cdots, 0)$. For simplicity, we consider the case that $j = n$. In this case the Kowalevsky matrix is

$$K = \begin{pmatrix} 1 - \frac{a_{1n}}{a_{nn}} & 0 & \cdots & 0 & 0 \\ 0 & 1 - \frac{a_{2n}}{a_{nn}} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 - \frac{a_{(n-1)n}}{a_{nn}} & 0 \\ -\frac{a_{n1}}{a_{nn}} & -\frac{a_{n2}}{a_{nn}} & \cdots & -\frac{a_{n(n-1)}}{a_{nn}} & -1 \end{pmatrix}.$$

Obviously,

$$\lambda_1 = 1 - \frac{a_{1n}}{a_{nn}}, \quad \lambda_2 = 1 - \frac{a_{2n}}{a_{nn}}, \cdots, \lambda_{n-1} = 1 - \frac{a_{(n-1)n}}{a_{nn}}, \quad \lambda_n = -1$$

are n eigenvalues of K.

According to Theorem 1, system (16) does not have any Laurent polynomial integral if there is no resonance condition

$$k_1\left(1 - \frac{a_{1n}}{a_{nn}}\right) + \cdots + k_{n-1}\left(1 - \frac{a_{(n-1)n}}{a_{nn}}\right) - k_n = 0, \quad k_j \in \mathbb{Z}, \quad \sum_{j=1}^{n} |k_j| \geq 1$$

is fulfilled. This is equivalent to that for any $\tilde{k}_j \in \mathbb{Z}$, $\sum\limits_{j=1}^{n} |\tilde{k}_j| \geq 1$,

$$\tilde{k}_1 a_{1n} + \tilde{k}_2 a_{2n} + \cdots + \tilde{k}_n a_{nn} \neq 0.$$

Corollary 1. *If for some j $(1 \leq j \leq n)$, $a_{1j}, a_{2j}, \cdots, a_{nj}$ are $\mathbb{Z}-$independent, i.e., they do not satisfy any resonant condition*

$$k_1 a_{1j} + k_2 a_{2j} + \cdots + k_n a_{nj} = 0, \quad k_j \in \mathbb{Z}, \quad \sum_{j=1}^{n} |k_j| \geq 1,$$

then system (16) does not have any Laurent polynomial integrals.

Example 2. To illustrate the Theorem 2, we consider the following Euler-Poincaré equations on Lie algebras [1]

$$\begin{aligned}
\dot{x}_1 &= -s(x)(\alpha_1 x_1 + \beta_1 x_2 + \gamma_1 x_3),\\
\dot{x}_2 &= -s(x)(\beta_2 x_2 + \gamma_2 x_3),\\
\dot{x}_3 &= -s(x)(\beta_3 x_2 + \gamma_3 x_3),\\
\dot{x}_4 &= p(x)(\alpha_1 x_1 + \beta_1 x_2 + \gamma_1 x_3) + q(x)(\beta_2 x_2 + \gamma_2 x_3) + r(x)(\beta_3 x_2 + \gamma_3 x_3),
\end{aligned} \tag{17}$$

where $p(x) = ax_1 + ex_2 + fx_3 + gx_4, q(x) = ex_1 + bx_2 + hx_3 + ix_4, r(x) = fx_1 + hx_2 + cx_3 + jx_4, s(x) = gx_1 + ix_2 + jx_3 + dx_4$.

System (17) is a quasi-homogeneous one with exponents $s_1 = s_2 = s_3 = s_4 = 1$. So $S = E, m = 2, \alpha = 1, H = E$. Obviously, it has an integral

$$\begin{aligned}
T &= \frac{1}{2}(x_1 p(x) + x_2 q(x) + x_3 r(x) + x_4 s(x))\\
&= \frac{1}{2}(ax_1^2 + bx_2^2 + cx_3^2 + dx_4^2 + 2ex_1x_2 + 2fx_1x_3 + 2gx_1x_4\\
&\quad +2hx_2x_3 + 2ix_2x_4 + 2jx_3x_4).
\end{aligned} \tag{18}$$

Notice that system (17) has a particular solution $x(t) = (\frac{1}{\alpha_1 g t}, 0, 0, 0)$ if $a = 0, \alpha_1 g \neq 0$. The Kowalevsky matrix of system (17) corresponding to this particular solution is

$$K = \begin{pmatrix}
-1 & -\frac{i}{g} - \frac{\beta_1}{\alpha_1} & -\frac{j}{g} - \frac{\gamma_1}{\alpha_1} & -\frac{d}{g}\\
0 & 1 - \frac{\beta_2}{\alpha_1} & -\frac{\gamma_2}{\alpha_1} & 0\\
0 & -\frac{\beta_3}{\alpha_1} & 1 - \frac{\gamma_3}{\alpha_1} & 0\\
0 & \frac{\alpha_1 e + \beta_2 e + \beta_3 f}{\alpha_1 g} & \frac{\alpha_1 f + \gamma_2 e + \gamma_3 f}{\alpha_1 g} & 2
\end{pmatrix}$$

with the following four eigenvalues

$$\lambda_1 = 2, \quad \lambda_{2,3} = 1 - \frac{\beta_2 + \gamma_3 \pm \sqrt{(\beta_2 - \gamma_3)^2 + 4\beta_3\gamma_2}}{2\alpha_1}, \quad \lambda_4 = -1.$$

According to Theorem 2, any Laurent polynomial integral of system (17) is functionally dependent on T if rank $G = 1$, where

$$G = \{(k_1, k_2, k_3, k_4) \in \mathbb{Z} : k_1\lambda_1 + k_2\lambda_2 + k_3\lambda_3 + k_4\lambda_4 = 0\}.$$

This is equivalent to

$$\tilde{k}_2 \cdot \frac{\beta_2 + \gamma_3}{2\alpha_1} + \tilde{k}_3 \cdot \frac{\sqrt{(\beta_2 - \gamma_3)^2 + 4\beta_3\gamma_2}}{2\alpha_1} + \tilde{k}_4 \neq 0.$$

for any $\tilde{k}_2, \tilde{k}_3, \tilde{k}_4 \in \mathbb{Z}, |\tilde{k}_2| + |\tilde{k}_3| + |\tilde{k}_4| \geq 1$.

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