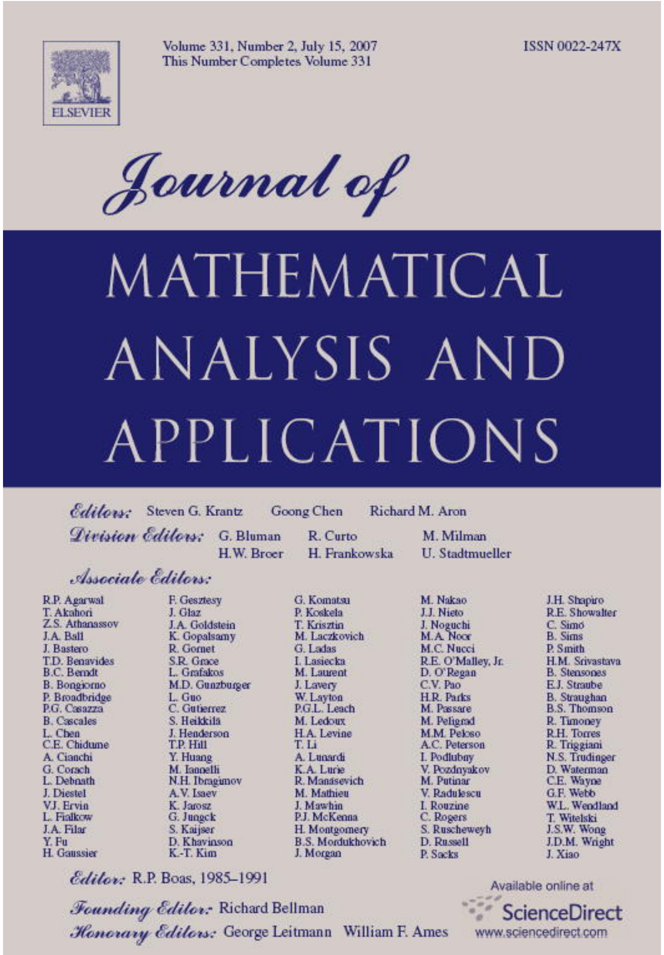
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A SIRS epidemic model with infection-age dependence ☆

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Abstract

Based on J. Mena-Lorca and H.W. Hethcote's epidemic model, a SIRS epidemic model with infection-age-dependent infectivity and general nonlinear contact rate is formulated. Under general conditions, the unique existence of its global positive solutions is obtained. Moreover, under more general assumptions than the existing, the existence and asymptotical stability of its equilibria are discussed. In the end, the condition on the stability of endemic equilibrium is verified by a special model.

Keywords: Epidemiology; SIRS epidemic model; Infection-age; Equilibria; Asymptotic stability

1. Introduction

In most epidemiological models, it has been assumed that all infected individuals are equally infectious during their infectivity period. The assumption has been proved to be reasonable in the study of the dynamics of communicable diseases such as influenza [8] and the sexually transmitted diseases such as gonorrhea [9]. However, the early infectivity experiments [10] reported in Francis together with the measurements of HIV antigen and antibody titers have supported the possibility of an early infectivity peak (a few weeks after exposure) and a late infectivity plateau (one year or so before the onset of "full-blown" AIDS) [11]. Therefore, there are enough

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reasons to study the possible effects of variable infectivity on the epidemic dynamics. The initial work was owed to Kermack and Mckendrick [12–15], where the infectivity was allowed to depend on infection-age (namely, the time that has passed since the moment of infection). However, the infection-age-dependent epidemic models were largely neglected until 1970s [16,17]. Recently, the infection-age-dependent epidemic models have been extensively considered (see, for examples, [2–7]).

In [4], H.R. Thieme and C. Castillo-Chavez explored the role of variable infectivity together with a variable incubation period in the dynamics of HIV transmission in a homogeneously mixing population, and discussed both the asymptotical behavior of the equilibria and the possibility whether the undamped oscillations occur or not. In [2], M.Y. Kim and F.A. Milner formulated a SIR epidemic model with screening together with variable infectivity and proved the global existence and uniqueness of the positive solution. In a sequent paper [3], M.Y. Kim discussed the asymptotical properties of equilibria of the model. In [5], C.M. Kribs-Zaleta and M. Martcheva modelled a disease with acute, chronic infective stages, variable infectivity, and recovery rates and exhibited backward bifurcations under some conditions. In [6], H. Inaba and H. Sekine studied the stability of the equilibria of an infection-age-dependent model for Chagas disease. In [7], Jia Li et al. considered epidemiological models for the transmission of a pathogen that can mutate in the host to create a second infectious mutand strain and showed that there exists a Hopf bifurcation where the endemic equilibrium loses its stability under certain circumstances.

In [9], considering temporary immunity, Mena-Lorca and Hethcote founded a kind of SIRS model. Indeed, it is a quite general model for epidemic transmission and many traditional epidemic models can be viewed as its special examples. However, in that model the infection-age did not be considered. In this paper, by incorporating the infection-age and general contact rate into the Mena-Lorca and Hethcote's model, we formulate a SIRS epidemic model, and then study the existence and uniqueness of the positive solutions together with the existence and stability of equilibria.

The remainder of this paper is organized as follows. Section 2 is devoted to the model derivation. Section 3 proves the global existence and uniqueness of positive solutions. Section 4 studies the existence together with the asymptotical stability of the equilibria and the condition on the stability of endemic equilibrium is verified by a special model.

2. Model formulation

Mena-Lorca and Hethcote's SIRS epidemic model [1] is described by

$$\begin{cases} \frac{dS(t)}{dt} = \Lambda - \mu S(t) - \beta S(t)I(t) + \delta R(t), \\ \frac{dI(t)}{dt} = \beta S(t)I(t) - \mu I(t) - \alpha I(t) - \epsilon I(t), \\ \frac{dR(t)}{dt} = \epsilon I(t) - \delta R(t) - \mu R(t), \end{cases} \tag{2.1}$$

where $S(t)$, $I(t)$ and $R(t)$ respectively denote the numbers of susceptibles, infectives and recovered at time t, Λ is the input flow, μ the natural mortality rate, β the transmission rate, α the death rate due to disease, ϵ the recovery rate, δ the rate that the removed return to the susceptible and β the infection rate. Obviously, no infection-age was considered in this model. As stated in the introduction, for infections, specially those may last for a long time (relative to the life span of the infected individuals), and it is necessary for variable infectivity to predict accurately the spread of the infection in population [18]. Now, we concentrate our attention on the SIRS model (2.1) in which $I(t)$, α, ϵ and β are all structured by the infection-age. Denote by τ the infection-age variable, by $i(t,\tau)$ the distribution function of $I(t)$ over infection-age τ at time t, and by

$\beta(\tau)$, $\alpha(\tau)$ and $\epsilon(\tau)$ the distribution functions of β, α and ϵ over τ, respectively. Then, the infected individuals $I(t)$ at time t equals to $\int_0^\infty i(t,\tau)\,d\tau$, the nonlinear incidence is characterized by

$$\frac{C(S(t),I(t),R(t))}{S(t)+I(t)+R(t)}S(t)\int_0^\infty \beta(\tau)i(t,\tau)\,d\tau$$

where $C(S(t),I(t),R(t))$ is the contact rate (i.e., the mean number contacts per individual per unit time), the recovery individuals from disease is characterized by $\int_0^\infty \epsilon(\tau)i(t,\tau)\,d\tau$, and the continuous-time dynamics of the infected individuals is governed by a partial differential equation other than an ordinary equation (that is, the second equation in the model (2.1) will be replaced by a partial differential equation of the distribution $i(t,\tau)$). Correspondingly, we derive a new SIRS epidemic model with general nonlinear contact rate

$$\begin{cases} \frac{dS(t)}{dt} = \Lambda - \mu S(t) - \frac{C(S(t),I(t),R(t))}{N(t)}S(t)\int_0^\infty \beta(\tau)i(t,\tau)\,d\tau + \delta R(t),\\ \frac{\partial i(t,\tau)}{\partial t} + \frac{\partial i(t,\tau)}{\partial \tau} = -(\mu+\alpha(\tau)+\epsilon(\tau))i(t,\tau),\\ \frac{dR(t)}{dt} = \int_0^\infty \epsilon(\tau)i(t,\tau)\,d\tau - (\mu+\delta)R(t),\\ N(t) = S(t)+I(t)+R(t),\\ i(t,0) = \frac{C(S(t),I(t),R(t))}{N(t)}S(t)\int_0^\infty \beta(\tau)i(t,\tau)\,d\tau,\\ i(0,\tau) = \eta(\tau), \qquad S(0)=S_0, \qquad R(0)=R_0, \end{cases} \tag{2.2}$$

where Λ is the input flow into susceptible, μ the natural mortality rate, δ the rate for loss of immunity (return to susceptibles), $\eta(\tau)$ the initial distribution of infected individuals with infection-age, S_0 and R_0 are the initial susceptibles and recovered, respectively.

In model (2.2), if all of the individuals are assumed to have the same mortality rate (i.e. $\alpha(\tau)=0$ for all τ) and the individuals recovered from the disease are assumed to immediately enter the class of removed individuals and not to participate in the dynamics of transmission (i.e. $\delta=0$), then the model is reduced to Kim–Milner model [2] with screening individuals (the infected individuals who are screened and enter into the compartment of the removed)

$$\begin{cases} \frac{dS(t)}{dt} = \Lambda - \mu S(t) - \frac{C(S(t),I(t))}{S(t)+I(t)}S(t)\int_0^\infty \beta(\tau)i(t,\tau)\,d\tau,\\ \frac{\partial i(t,\tau)}{\partial t} + \frac{\partial i(t,\tau)}{\partial \tau} = -(\mu+\epsilon(\tau))i(t,\tau) - \frac{\sigma(S(t),I(t))}{S(t)+I(t)}i(t,\tau),\\ \frac{dR(t)}{dt} = \int_0^\infty \epsilon(\tau)i(t,\tau)\,d\tau - \mu R(t) + \frac{\sigma(S(t),I(t))}{S(t)+I(t)}I(t),\\ i(t,0) = \frac{C(S(t),I(t))}{S(t)+I(t)}S(t)\int_0^\infty \beta(\tau)i(t,\tau)\,d\tau,\\ i(0,\tau) = \eta(\tau), \qquad S(0)=S_0, \qquad R(0)=R_0, \end{cases} \tag{2.3}$$

where $\frac{\sigma(S(t),I(t))}{S(t)+I(t)}$ is the number of individuals screened per unit time. Furthermore, if the infected individuals are additionally assumed not to be cured and the contact rate is assumed to be a function of only the total of $S(t)$ and $I(t)$, then the model is reduced to Thieme–Chavez model [18].

Throughout the remainder of the paper, we adopt the following assumptions similar to Kim and Milner's [2]. The contact rate C is a nonnegative and partially differentiable function of S, I and R such that $\frac{\partial C}{\partial S}, \frac{\partial C}{\partial I}, \frac{\partial C}{\partial R} \in L^\infty([0,\infty)\times[0,\infty)\times[0,\infty))$; the nonnegative functions α, ϵ and β satisfy $\alpha(\cdot), \epsilon(\cdot) \in C^1[0,\infty)\cap L^\infty[0,\infty)$ with $\alpha'(\cdot), \epsilon'(\cdot) \in L^\infty[0,\infty)$, and $\beta(\cdot) \in C^2[0,\infty)\cap L^\infty[0,\infty)$ with $\beta'(\cdot), \beta''(\cdot) \in L^\infty[0,\infty)$; Λ, μ and δ are positive constants. Denote by $\|\cdot\|_1$ the norm of Banach space $L^1[0,\infty)$, by $\|\cdot\|_\infty$ the norm of Banach space $L^\infty[0,\infty)$, and by $L^1_+[0,\infty)$ the positive cone of nonnegative functions in $L^1[0,\infty)$.

3. Global existence and uniqueness of positive solution

For convenience, we denote $B(t) = i(t,0) = \frac{C(S(t),I(t),R(t))}{N(t)} S(t) \int_0^\infty \beta(\tau) i(t,\tau)\,d\tau$ and $m = \mu + \delta$. Integrating the second equation of (2.2) along the characteristic line $t = \tau$ yields

$$i(t,\tau) = \begin{cases} B(t-\tau)\pi(\tau), & t-\tau \geqslant 0; \\ \eta(\tau - t)\pi(\tau - t, \tau), & \tau - t > 0, \end{cases} \tag{3.1}$$

where

$$\pi(\tau) = e^{-\int_0^\tau (\mu+\alpha(\rho)+\epsilon(\rho))\,d\rho}, \qquad \pi(\tau_1,\tau_2) = e^{-\int_{\tau_1}^{\tau_2} (\mu+\alpha(\rho)+\epsilon(\rho))\,d\rho}. \tag{3.2}$$

Thus, we get an equivalent system to (2.2)

$$\begin{cases} S(t) = e^{-\mu t - \int_0^t B_1(s)\,ds}[S_0 + \int_0^t (\Lambda + \delta R(s)) e^{\mu s + \int_0^s B_1(\rho)\,d\rho}\,ds], \\ I(t) = \int_0^t B_1(t-\tau)S(t-\tau)\pi(\tau)\,d\tau + \int_t^\infty \eta(\tau - t)\pi(\tau - t, \tau)\,d\tau, \\ R(t) = e^{-mt}[R_0 + \int_0^t e^{ms} f(s)\,ds], \\ B_1(t) = \frac{C(S(t),I(t),R(t))}{N(t)} \int_0^t \beta(\tau) B_1(t-\tau) S(t-\tau)\pi(\tau)\,d\tau + g(t), \end{cases} \tag{3.3}$$

where

$$B(t) = B_1(t)S(t), \tag{3.4}$$

$$f(t) = \int_0^t \epsilon(\tau) B_1(t-\tau) S(t-\tau)\pi(\tau)\,d\tau + \int_t^\infty \epsilon(\tau)\eta(\tau - t)\pi(\tau - t, \tau)\,d\tau,$$

$$g(t) = \frac{C(S(t),I(t),R(t))}{N(t)} \int_t^\infty \beta(\tau)\eta(\tau - t)\pi(\tau - t, \tau)\,d\tau.$$

Lemma 1. *Let $S_0 \geqslant 0$, $R_0 \geqslant 0$ and $\eta \in L^1_+[0,\infty)$. Then, there exists a constant $T > 0$ such that the model* (2.2) *admits a unique nonnegative solution $(S(t), I(t), R(t))$ on $[0,T)$.*

Proof. It is clear that for each $T > 0$ and any triple $(S,I,R) \in (C_+[0,T])^3$ (where $C_+[0,T]$ stands for the positive cone of nonnegative functions in $C[0,T]$), both the convolution kernel and $g(t)$ in the forth equation of system (3.3) are nonnegative and continuous. By the general theory of Volterra equations [20, p. 13] we conclude that for any given triple $(S,I,R) \in (C_+[0,T])^3$, the forth equation of system (3.3) has a nonnegative solution $B_1(t)$, which is denoted by $H(S,I,R)(t)$ below. Substituting $H(S,I,R)(t)$ into (3.3) leads to the following integral equations with respect to $(S(t), I(t), R(t))$

$$\begin{cases} S(t) = e^{-\mu t - \int_0^t H(S,I,R)(s)\,ds}[S_0 + \int_0^t (\Lambda + \delta R(s)) e^{\mu s + \int_0^s H(S,I,R)(\rho)\,d\rho}\,ds], \\ I(t) = \int_0^t H(S,I,R)(t-\tau)S(t-\tau)\pi(\tau)\,d\tau + \int_t^\infty \eta(\tau - t)\pi(\tau - t, \tau)\,d\tau, \\ R(t) = e^{-mt}R_0 + e^{-mt}\int_0^t [e^{ms}\int_0^s \epsilon(\tau)H(S,I,R)(s-\tau)S(s-\tau)\pi(\tau)\,d\tau \\ \qquad + \int_s^\infty \epsilon(\tau)\eta(\tau - s)\pi(\tau - s, \tau)\,d\tau]\,ds. \end{cases} \tag{3.5}$$

Define three operators $\mathbb{S}, \mathbb{I}, \mathbb{R}$ from $(C^+[0,T])^3$ into $C^+[0,T]$ as follows

$$\mathbb{S}(S,I,R)(t)=e^{-\mu t-\int_0^t H(S,I,R)(s)\,ds}\left[S_0+\int_0^t \big(\Lambda+\delta R(s)\big)e^{\mu s+\int_0^s H(S,I,R)(\rho)\,d\rho}\,ds\right], \tag{3.6}$$

$$\mathbb{I}(S,I,R)(t)=\int_0^t H(S,I,R)(t-\tau)S(t-\tau)\pi(\tau)\,d\tau+\int_0^\infty \eta(\tau)\pi(\tau,\tau+t)\,d\tau \tag{3.7}$$

and

$$\mathbb{R}(S,I,R)(t)=e^{-mt}\left[R_0+\int_0^t e^{ms}\left(\int_0^s \epsilon(\tau)H(S,I,R)(s-\tau)S(s-\tau)\,d\tau + \int_s^\infty \epsilon(\tau)\eta(\tau-s)\pi(\tau-s,\tau)\,d\tau\right)ds\right]. \tag{3.8}$$

Then, $S(t), I(t), R(t)\geqslant 0$ solve the equations of (2.2) over $[0,T]$ iff the triple $(S,I,R)\in(C_+[0,T])^3$ is a fixed point of the map $\mathbb{F}:(C_+[0,T])^3\to(C_+[0,T])^3$, $\mathbb{F}(S,I,R)=(\mathbb{S}(S,I,R),\mathbb{I}(S,I,R),\mathbb{R}(S,I,R))$.

Without loss of generality, we assume that $S_0+I_0+R_0\neq 0$, where $I_0=\int_0^\infty \eta(\tau)\,d\tau$. Endow the space $(C[0,T])^3$ with the norm $\|(x_1,x_2,x_3)\|_{T,3}=\sum_{j=1}^3\|x_j\|_T$, where $\|x_j\|_T=\sup_{t\in[0,T]}|x_j(t)|$, and let $\mathbb{O}_{T,r}\subset(C_+[0,T])^3$ be a closed ball of radius $r\leqslant\frac{S_0+I_0+R_0}{2}$, centered at (S_0,I_0,R_0). It can be shown for each $(S,I,R)\in\mathbb{O}_{T,r}$,

$$\|S+I+R\|_T\geqslant(S_0+I_0+R_0)-r:=K>0$$

and

$$\|S\|_T\leqslant\frac{R_0+I_0}{2}+\frac{3}{2}S_0,\qquad \|I\|_T\leqslant\frac{S_0+R_0}{2}+\frac{3}{2}I_0,\qquad \|R\|_T\leqslant\frac{S_0+I_0}{2}+\frac{3}{2}R_0.$$

Therefore, it follows from the forth equation of system (3.3) that

$$H(S,I,R)(t)\leqslant\frac{C_T\beta_\infty}{K}\left[\frac{2S_0+N_0}{2}\int_0^t H(S,I,R)(s)\,ds+I_0\right], \tag{3.9}$$

for all $(S,I,R)\in\mathbb{O}_{T,r}$, where $C_T=\sup_{t\in[0,T]}C(S(t),I(t),R(t))$, $\beta_\infty=\|\beta\|_\infty$ and $N_0=S_0+I_0+R_0$. Applying Gronwall's inequality lemma [19, p. 23] to (3.9) yields

$$H(S,I,R)(t)\leqslant K_1e^{K_2t},\quad K_1=\frac{C_T\beta_\infty I_0}{K},\ K_2=\frac{C_T\beta_\infty(2S_0+N_0)}{2K}. \tag{3.10}$$

Now, we show that the operator $\mathbb{F}$ maps $\mathbb{O}_{T,r}$ into itself for some $T>0$. Let $(S,I,R)\in\mathbb{O}_{T,r}$. Then, by the formulae (3.6)–(3.8), for all $t\in[0,T]$ we can show,

$$\begin{aligned}&\big|\mathbb{S}(S,I,R)(t)-S_0\big|\\&\quad\leqslant S_0\big|1-e^{-\mu t-\int_0^t H(S,I,R)(s)\,ds}\big|+\left|\int_0^t\big(\Lambda+\delta R(s)\big)e^{-\mu(t-s)-\int_s^t H(S,I,R)(\rho)\,d\rho}\,ds\right|\\&\quad\leqslant\big(1-e^{-\mu t-\int_0^t K_1e^{K_2s}\,ds}\big)S_0+\left(\Lambda+\frac{N_0+2R_0}{2}\delta\right)T\end{aligned}$$

$$\leqslant \left[\left(\mu + K_1 e^{K_2 T} \right) S_0 + \left(\Lambda + \frac{N_0 + 2R_0}{2} \delta \right) \right] T,$$

$$\begin{aligned}
&\left| \mathbb{I}(S, I, R)(t) - I_0 \right| \\
&\quad \leqslant \left| \int_0^t H(S, I, R)(t-\tau) S(t-\tau) \pi(\tau)\, d\tau \right| + \left| \int_0^\infty \eta(\tau) \big(\pi(\tau, \tau + t) - 1 \big)\, d\tau \right| \\
&\quad \leqslant \frac{2S_0 + N_0}{2} \int_0^t K_1 e^{K_2 \tau}\, d\tau + \left| \int_0^\infty \eta(\tau) \big[1 - e^{-\int_\tau^{\tau+t} (\mu + \alpha(\rho) + \epsilon(\rho))\, d\rho} \big]\, d\tau \right| \\
&\quad \leqslant \left[\frac{2S_0 + N_0}{2} K_1 e^{K_2 T} + I_0 (\mu + \alpha_\infty + \epsilon_\infty) \right] T
\end{aligned}$$

and

$$\begin{aligned}
&\left| \mathbb{R}(S, I, R)(t) - R_0 \right| \\
&\quad \leqslant \left| \left(e^{-mt} - 1 \right) R_0 \right| + \left| \int_0^t e^{-m(t-s)} \left[\int_0^s \epsilon(\tau) \pi(\tau) H(S, I, R)(s-\tau) S(s-\tau)\, d\tau \right. \right. \\
&\qquad \left. \left. + \int_s^\infty \epsilon(\tau) \eta(\tau - s) \pi(\tau - s, \tau)\, d\tau \right] ds \right| \\
&\quad \leqslant m R_0 T + \epsilon_\infty T \left[\frac{2S_0 + N_0}{2} K_1 e^{K_2 T} T + I_0 \right],
\end{aligned}$$

where $\epsilon_\infty = \|\epsilon\|_\infty$ and $\alpha_\infty = \|\alpha\|_\infty$. Therefore, for sufficiently small $T > 0$, $\mathbb{F}$ really maps $\mathbb{O}_{T,r}$ into itself.

Let $(S_i, I_i, R_i) \in \mathbb{O}_{T,r}$ $(i = 1, 2)$ and

$$\begin{aligned}
\Delta_1(t) &= \left| \mathbb{S}(S_1, I_1, R_1)(t) - \mathbb{S}(S_2, I_2, R_2)(t) \right|, \\
\Delta_2(t) &= \left| \mathbb{I}(S_1, I_1, R_1)(t) - \mathbb{I}(S_2, I_2, R_2)(t) \right|, \\
\Delta_3(t) &= \left| \mathbb{R}(S_1, I_1, R_1)(t) - \mathbb{R}(S_2, I_2, R_2)(t) \right|.
\end{aligned}$$

Obviously, $\mathbb{F}$ is a contraction mapping on $\mathbb{O}_{T,r}$. In fact, for $t \in [0, T]$ we have

$$\begin{aligned}
&\Delta_1(t) \\
&\quad \leqslant S_0 e^{-\mu t} \left| e^{-\int_0^t H(S_1, I_1, R_1)(s)\, ds} - e^{-\int_0^t H(S_2, I_2, R_2)(s)\, ds} \right| \\
&\qquad + \Lambda \left| \int_0^t e^{-\mu(t-s)} \left(e^{-\int_s^t H(S_1, I_1, R_1)(s)\, ds} - e^{-\int_s^t H(S_2, I_2, R_2)(s)\, ds} \right) ds \right| \\
&\qquad + \delta \left| \int_0^t \left(R_1(s) e^{-\mu(t-s) - \int_s^t H(S_1, I_1, R_1)(\rho)\, d\rho} - R_2(s) e^{-\mu(t-s) - \int_s^t H(S_2, I_2, R_2)(\rho)\, d\rho} \right) ds \right| \\
&\quad := e_1(t) + e_2(t) + e_3(t),
\end{aligned}$$

$$\Delta_2(t) \leqslant \left| \int_0^t \big(S_1(s)H(S_1, I_1, R_1)(s) - S_2(s)H(S_2, I_2, R_2)(s)\pi(t-s)\big)\, ds \right|$$

and

$$\Delta_3(t) \leqslant \left| \int_0^t e^{-m(t-s)} \int_0^s \epsilon(\tau)\pi(\tau)\big[S_1(s-\tau)H(S_1, I_1, R_1)(s-\tau) - S_2(s-\tau)H(S_2, I_2, R_2)(s-\tau)\big]\, d\tau\, ds \right|.$$

Let $h(t) = H(S_1, I_1, R_1)(t) - H(S_2, I_2, R_2)(t)$, $t \in [0, T]$. Since $H(S, I, R)(t)$ solves the forth equation of system (3.3), it follows that for all $t \in [0, T]$,

$$\begin{aligned} \big|h(t)\big| \leqslant{} & \big|\Delta_4(t)\big| \left| \int_0^t \beta(\tau)H(S_2, I_2, R_2)(t-\tau)S_2(t-\tau)\pi(\tau)\, d\tau \right| \\ & + \big|\Delta_4(t)\big| \left| \int_t^\infty \beta(\tau)\eta(\tau - t)\pi(\tau - t, \tau)\, d\tau \right| \\ & + \left| \frac{C(S_1, I_1, R_1)}{N_1} \int_0^t \beta(t-\tau)\pi(t-\tau)\Delta_5(\tau)\, d\tau \right|, \end{aligned} \tag{3.11}$$

where $N_j(t) = S_j(t) + I_j(t) + R_j(t)$, $j = 1, 2$,

$$\Delta_4(t) = \frac{C(S_1(t), I_1(t), R_1(t))}{N_1(t)} - \frac{C(S_2(t), I_2(t), R_2(t))}{N_2(t)}, \quad t \in [0, T]$$

and

$$\Delta_5(t) = \big[H(S_2, I_2, R_2)(t)S_2(t) - H(S_1, I_1, R_1)(t)S_1(t)\big].$$

By (3.10), make an upper estimation of $\Delta_5(t)$

$$\big|\Delta_5(t)\big| \leqslant K_1 e^{K_2 T} \big\|(S_2 - S_1, I_2 - I_1, R_2 - R_1)\big\|_T + \frac{2S_0 + N_0}{2}\big|h(t)\big|, \quad t \in [0, T].$$

Applying the mean value theorem of differentials to Δ_4, for all $t \in [0, T]$ we obtain,

$$\begin{aligned} \|\Delta_4\|_T &\leqslant \big\|\nabla F(\theta) \cdot (S_2 - S_1, I_2 - I_1, R_2 - R_1)\big\|_T \\ &\leqslant \frac{\frac{5}{2}N_0 \sum_{j=1}^3 C_j^T + 3C_T}{K^2} \big\|(S_2 - S_1, I_2 - I_1, R_2 - R_1)\big\|_T, \end{aligned}$$

where $\theta \in (0, 1)$, $C_1 = \frac{\partial C}{\partial S}$, $C_2 = \frac{\partial C}{\partial I}$, $C_3 = \frac{\partial C}{\partial R}$, $C_j^T = \sup_{t \in [0,T]} |C_j(t)|$, $j = 1, 2, 3$,

$$\nabla F(\theta) = \nabla \left(\frac{C(S_1 + \theta(S_2 - S_1), I_1 + \theta(I_2 - I_1), R_1 + \theta(R_2 - R_1))}{N_1 + \theta(N_2 - N_1)} \right)$$

and ∇ is the gradient operator. Combining the inequalities about $\|\Delta_4\|_T$ and $\|\Delta_5\|_T$ with the inequality (3.11), we have

$$\left|h(t)\right| \leqslant P_2 \int_0^t \left|h(s)\right| ds + P_1 \left\|(S_2 - S_1, I_2 - I_1, R_2 - R_1)\right\|_T, \quad t \in [0, T]$$

where

$$P_1 = \beta_\infty \left(I_0 + \frac{2S_0 + N_0}{2} K_1 T e^{K_2 T}\right) \frac{\frac{5}{2} N_0 \sum_{j=1}^3 C_j^T + 3C_T}{K^2} + \frac{C_T \beta_\infty}{K} K_1 T e^{K_2 T}$$

and

$$P_2 = \frac{(2S_0 + N_0) C_T \beta_\infty}{2K}.$$

Solving the above inequality yields

$$\left|h(t)\right| \leqslant P_1 e^{P_2 t} \left\|(S_2 - S_1, I_2 - I_1, R_2 - R_1)\right\|_T, \quad t \in [0, T].$$

Therefore, for all $t \in [0, T]$ we have,

$$e_1(t) \leqslant S_0 P_1 T e^{P_2 T} \left\|(S_2 - S_1, I_2 - I_1, R_2 - R_1)\right\|_T,$$
$$e_2(t) \leqslant \Lambda P_1 T^2 e^{P_2 T} \left\|(S_2 - S_1, I_2 - I_1, R_2 - R_1)\right\|_T,$$
$$e_3(t) \leqslant \delta T \left(1 + \frac{2R_0 + N_0}{2} P_1 T e^{P_2 T}\right) \left\|(S_2 - S_1, I_2 - I_1, R_2 - R_1)\right\|_T.$$

From the above inequalities, it follows

$$\Delta_1(t) \leqslant T \left[P_1 e^{P_2 T} \left(S_0 + \Lambda T + \frac{2R_0 + N_0}{2} T\right) + \delta\right] \left\|(S_2 - S_1, I_2 - I_1, R_2 - R_1)\right\|_T,$$
$$\Delta_2(t) \leqslant T \left[K_1 e^{K_2 T} + \frac{2S_0 + N_0}{2} P_1 e^{P_2 T}\right] \left\|(S_2 - S_1, I_2 - I_1, R_2 - R_1)\right\|_T,$$
$$\Delta_3(t) \leqslant \epsilon_\infty T^2 \left[\frac{2S_0 + N_0}{2} P_1 e^{P_2 T} + K_1 e^{K_2 T}\right] \left\|(S_2 - S_1, I_2 - I_1, R_2 - R_1)\right\|_T.$$

These three inequalities imply that the component operators $\mathbb{S}$, $\mathbb{I}$ and $\mathbb{R}$ of $\mathbb{F}$ are contraction mappings in $\mathbb{O}_{T,r}$ for sufficiently small $T > 0$. Accordingly, Brouwer's fixed point theorem gives the unique existence of the fixed point of $\mathbb{F}$, which is just the unique positive solution of (2.2) on $[0, T)$. □

Theorem 3.1. *For all initial values $S_0 \geqslant 0$, $R_0 \geqslant 0$ and $\eta \in L_+^1[0, \infty)$, the model* (2.2) *admits a unique nonnegative solution $(S(t), I(t), R(t))$ on* $[0, \infty)$.

Proof. Let $[0, T)$ be the maximal existence interval of the positive solution $(S(t), I(t), R(t))$ initiated from (S_0, I_0, R_0), where $I_0 = \int_0^\infty \eta(\tau)\, d\tau$. By the proof of Lemma 1 it is plain that T is a continuous function of the initial values (S_0, I_0, R_0). For convenience, we denote T by $T(S_0, I_0, R_0)$.

Let $i(t, \tau)$ be defined by (3.1), with $B(t)$ being the corresponding solution of the equation of (3.4). Then, it is easy to check that $i(t, \tau)$ is just the distribution of $I(t)$ over the infection-age τ. Obviously, $i(t, \tau)$ is nonnegative for all $\tau \geqslant 0$ and for all $t \in [0, T)$. Hence, integrating the second equation of system (2.2) over τ from 0 to ∞ yields

$$\frac{dI(t)}{dt} \leqslant B(t) - \mu I(t) - \int_0^\infty \big(\alpha(\tau) + \epsilon(\tau)\big) i(t, \tau)\, d\tau, \quad t \in [0, T). \tag{3.12}$$

Combining the first and the third equations of (2.2) with Eq. (3.12) leads to

$$\frac{dN(t)}{dt} \leqslant \Lambda - \mu N(t) - \int_0^\infty \alpha(\tau) i(t,\tau)\, d\tau, \quad t \in [0, T). \tag{3.13}$$

For all $t \in [0, T)$, from (3.12) and (3.13) we derive the following priori estimations of $N(t)$ and $I(t)$

$$N(t) \leqslant N_0 + \Lambda t, \qquad I(t) \leqslant I_0 + C_T \beta_T \left(N_0 t + \frac{\Lambda t^2}{2}\right). \tag{3.14}$$

Define

$$B_S(t) = N_0 + (\Lambda + \delta N_0)t + \frac{1}{2}\delta \Lambda t^2,$$
$$B_I(t) = N_0 + C_T \beta_T \left(N_0 t + \frac{1}{2}\Lambda t^2\right),$$
$$B_R(t) = N_0 + N_0 t + \frac{1}{2}\Lambda t^2.$$

Obviously, these functions are strictly monotone increasing with respect to t in $[0, \infty)$, and $S_0 \leqslant B_S(t)$, $I_0 \leqslant B_I(t)$ and $R_0 \leqslant B_R(t)$ for all $t \geqslant 0$. Let $\widetilde{T} > T(S_0, I_0, R_0)$ and

$$T^* = \min_{(\phi_1,\phi_2,\phi_3)\in[0,B_S(\widetilde{T})]\times[0,B_I(\widetilde{T})]\times[0,B_R(\widetilde{T})]} T(\phi_1, \phi_2, \phi_3).$$

By Lemma 1 and the continuity of $T(\cdot,\cdot,\cdot)$, we get $0 < T^* \leqslant T(S_0, I_0, R_0)$. Moreover, from (3.14) it follows that

$$S(T^*) < B_S(T(S_0, I_0, R_0)) \leqslant B_S(\widetilde{T}),$$
$$I(T^*) < B_I(T(S_0, I_0, R_0)) \leqslant B_I(\widetilde{T}),$$
$$R(T^*) < B_R(T(S_0, I_0, R_0)) \leqslant B_R(\widetilde{T}).$$

Hence, $T^* \leqslant T(S(T^*), I(T^*), R(T^*))$. This indicates that the model (2.2) admits a unique positive solution on the interval $[T^*, 2T^*]$, initiated from $(S(T^*), I(T^*), R(T^*))$. Similarly, we can show that the model (2.2) admits a unique positive solution on any interval $[kT^*, (k+1)T^*]$ as long as $(k+1)T^* \leqslant \widetilde{T}$. So, by the arbitrariness of $\widetilde{T}$ we conclude the theorem. $\square$

Remark 1. It should be noted that in our theorem there is not any assumption imposed on the distribution function $i(t,\tau)$ of the infected individuals. However, in [5] $i(t,\tau)$ is assumed to satisfy $\lim_{\tau\to\infty} i(t,\tau) = 0$ for all $t \geqslant 0$.

4. Asymptotic stability analysis

In epidemic dynamic, the existence and stability of equilibrium are important research topics because of the equilibrium standing for the possible ultima states of the special disease, and the asymptotic stability of an equilibrium revealing the capability of disease that tends to the ultima state corresponding to the equilibrium. This section is mainly devoted to the existence and asymptotic stability of equilibria of the model (2.2). Moreover, the reproductive number $\mathbb{R}_0$ (namely, the number of secondary cases produced in a completely susceptible population by a typical infected individual during its whole period of infection) is found, which is usually intimately connected with the existence of equilibria.

Obviously, $(S, i(\cdot), R)$ is an equilibrium of system (2.2) iff it solves the following equations

$$\begin{cases} \Lambda - \mu S - B + \delta R = 0, \\ \frac{di(\tau)}{d\tau} = -(\mu + \alpha(\tau) + \epsilon(\tau))i(\tau), \\ \int_0^\infty \epsilon(\tau)i(\tau)\,d\tau - mR = 0, \\ B = \frac{C(S,I,R)}{N} S \int_0^\infty \beta(\tau)i(\tau)\,d\tau = i(0), \end{cases} \tag{4.1}$$

where $I = \int_0^\infty i(\tau)\,d\tau$. Hence, system (2.2) always possesses the disease-free equilibrium $(\frac{\Lambda}{\mu}, 0, 0)$. In the following we discuss the existence of endemic equilibria of (2.2).

Solving the second equation of (4.1), we have

$$i(\tau) = i(0)\pi(\tau), \tag{4.2}$$

where $\pi(\tau)$ is defined by (3.2). Substituting (4.2) into Eqs. (4.1), we further have

$$\begin{cases} S = \frac{\Lambda - (1 - \frac{\delta}{m} \int_0^\infty \epsilon(\tau)\pi(\tau)\,d\tau) B}{\mu}, \\ I = B \int_0^\infty \pi(\tau)\,d\tau, \\ R = \frac{B \int_0^\infty \pi(\tau)\epsilon(\tau)\,d\tau}{m}, \\ B = \frac{C(S,I,R)}{S+I+R} S B \int_0^\infty \beta(\tau)\pi(\tau)\,d\tau. \end{cases} \tag{4.3}$$

Define a real function G on $[0, \infty)$

$$G(B) = \frac{C(S(B), I(B), R(B))}{S(B) + I(B) + R(B)} S(B) \int_0^\infty \beta(\tau)\pi(\tau)\,d\tau, \quad B \geqslant 0 \tag{4.4}$$

where $S = S(B)$, $I = I(B)$ and $R = R(B)$ are defined by (4.3). It is clear that G is a continuous real function.

Theorem 4.1. *Let*

$$\mathbb{R}_0 = C\left(\frac{\Lambda}{\mu}, 0, 0\right) \int_0^\infty \beta(\tau)\pi(\tau)\,d\tau. \tag{4.5}$$

Then, in the case that $\mathbb{R}_0 > 1$*, there exists at least an endemic equilibrium, while in the case that* $\mathbb{R}_0 \leqslant 1$*, no endemic equilibrium exists if the function G define by* (4.4) *is strictly decrease.*

Proof. By (4.3), we know that the existence of endemic equilibria is equivalent to the existence of positive parameter B that solves the forth equation of (4.3). Obviously, a positive B solves the forth equation of (4.3) iff it satisfies $G(B) = 1$. It is clear that $G(0) = \mathbb{R}_0$ and $G(\frac{\Lambda}{1 - \frac{\delta}{m}\int_0^\infty \epsilon(\tau)\pi(\tau)\,d\tau}) = 0$. Hence, if $\mathbb{R}_0 (= G(0)) > 1$, there exists at least a B such that $G(B) = 1$. Corresponding to this determined B, an endemic equilibrium is specified by (4.3). If $\mathbb{R}_0 \leqslant 1$ (that is, $G(0) = \mathbb{R}_0 \leqslant 1$), then only $B = 0$ satisfies $G(B) = 1$ provided G is strictly decrease. □

Remark 2. The strict decrease of function G is not a restrictive assumption for a practical epidemic model. In fact, it is not hard to check that if $C(S, I, R) = C(N)$ and $C'(N) \geqslant 0$ then G is strictly decreasing, where $C(S, I, R) = C(N)$ and $C'(N) \geqslant 0$, that is, the contact rate C is a function of the total population and increased with the total population.

In the remainder of this section, we study the asymptotic behavior of (2.2). First, similar to the proof of Theorem 2 of [4], we obtain the global asymptotic stability of disease-free equilibrium provided $\mathbb{R}_0 < 1$. Specifically,

Theorem 4.2. *Assume that $C = C(N)$, and $C'(N) \geqslant 0$. If $\mathbb{R}_0 < 1$, the disease-free equilibrium of the model* (2.2) *is globally asymptotical stability.*

Similar to the proof of Theorem 3.1 of [3], we can prove the following more general stability theorem.

Theorem 4.3. *Assume that $C(S, I, R) \leqslant C(\frac{\Lambda}{\mu}, 0, 0)$ for all nonnegative S, I and R. If $\mathbb{R}_0 < 1$, then the disease-free equilibrium of the model* (2.2) *is globally asymptotically stable.*

Next, we analyze the stability of endemic equilibrium if it exists. Let $(S^*, i^*(\tau), R^*)$ be an endemic equilibrium of (2.2). Set $\overline{S}(t) = S(t) - S^*$, $\bar{i}(t,\tau) = i(t,\tau) - i^*(\tau)$ and $\overline{R}(t) = R(t) - R^*$. Clearly, system (2.2) is equivalently transferred into

$$\begin{cases} \frac{d\overline{S}(t)}{dt} = -\mu\overline{S}(t) + \delta\overline{I}(t) - \bar{i}(t,0), \\ \frac{\partial\bar{i}(t,\tau)}{\partial t} + \frac{\partial\bar{i}(t,\tau)}{\partial\tau} = -(\mu + \alpha(\tau) + \epsilon(\tau))\bar{i}(t,\tau), \\ \frac{\overline{R}(t)}{dt} = \int_0^\infty \epsilon(\tau)\bar{i}(t,\tau)\,d\tau - m\overline{R}(t), \\ \bar{i}(t,0) = i(t,0) - i^*(0), \\ \overline{S}(0) = S_0 - S^*, \qquad \bar{\eta}(\tau) = \eta(\tau) - i^*(\tau), \\ \overline{I}(0) = I(0) - I^*, \qquad \overline{R}(0) = R_0 - R^*, \end{cases} \tag{4.6}$$

where

$$\bar{i}(t,0) = M(t)\int_0^\infty \beta(\tau)\big(\bar{i}(t,\tau) + i^*(\tau)\big)\,d\tau - M^*\int_0^\infty \beta(\tau)i^*(\tau)\,d\tau,$$

$$M(t) = \frac{C(S,I,R)S}{N}, \quad M^* = \frac{C^*(S^*,I^*,R^*)S^*}{N^*}. \tag{4.7}$$

Integrating the second equation of system (4.6) along the characteristic $t = \tau$, we get

$$\bar{i}(t,\tau) = \begin{cases} \overline{B}(t-\tau)\pi(\tau), & t \geqslant \tau, \\ \bar{\eta}(\tau - t)\pi(\tau - t, \tau), & t < \tau, \end{cases} \tag{4.8}$$

where $\overline{B}(t) = \bar{i}(t,0)$. Additionally, by (4.7) we get

$$\begin{aligned} \bar{i}(t,0) = {} & M^*\int_0^\infty \beta(\tau)\bar{i}(t,\tau)\,d\tau + \nabla M^* \cdot (\overline{S}, \overline{I}, \overline{R})\int_0^\infty \beta(\tau)i^*(\tau)\,d\tau \\ & + \big(M(t) - M^* - \nabla M^* \cdot (\overline{S}, \overline{I}, \overline{R})\big)\int_0^\infty \beta(\tau)\big(\bar{i}(t,\tau) + i^*(\tau)\big)\,d\tau \\ & + \nabla M^* \cdot (\overline{S}, \overline{I}, \overline{R})\int_0^\infty \beta(\tau)\bar{i}(t,\tau)\,d\tau. \end{aligned}$$

Substituting (4.8) into (4.6) leads to the equivalent system

$$\begin{cases} \frac{d\bar{S}(t)}{dt} = -\mu\bar{S}(t) + \delta\bar{R}(t) - \bar{B}(t), \\ \bar{I}(t) = \int_0^t \bar{B}(t-\tau)\pi(\tau)\,d\tau + \int_t^\infty \bar{\eta}(\tau-t)\pi(\tau-t,\tau)\,d\tau, \\ \frac{d\bar{R}(t)}{dt} = \int_0^t \epsilon(\tau)\bar{B}(t-\tau)\pi(\tau)\,d\tau + \int_0^\infty \epsilon(\tau)\bar{\eta}(\tau-t)\pi(\tau-t,\tau)\,d\tau - m\bar{R}(t), \\ \bar{B}(t) = M^*\int_0^\infty \beta(\tau)\bar{i}(t,\tau)\,d\tau + \nabla M^*\cdot(\bar{S},\bar{I},\bar{R})\int_0^\infty \beta(\tau)i^*(\tau)\,d\tau + \Psi(t), \end{cases} \tag{4.9}$$

where

$$\Psi(t) = \big[M(t) - M^* - \nabla M^*\cdot(\bar{S},\bar{I},\bar{R})\big]\int_0^\infty \beta(\tau)\big(\bar{i}(t,\tau) + i^*(\tau)\big)\,d\tau$$
$$+ \nabla M^*\cdot(\bar{S},\bar{I},\bar{R})\int_0^\infty \beta(\tau)\bar{i}(t,\tau)\,d\tau.$$

Solving Eqs. (4.9) yields

$$\begin{cases} \bar{S}(t) = e^{-\mu t}\bar{S}(0) + \int_0^t e^{-\mu(t-\tau)}[\delta\bar{R}(\tau) - \bar{B}(\tau)]\,d\tau, \\ \bar{I}(t) = \int_0^t \bar{B}(t-\tau)\pi(\tau)\,d\tau + \int_t^\infty \bar{\eta}(\tau-t)\pi(\tau-t,\tau)\,d\tau, \\ \bar{R}(t) = e^{-mt}\bar{R}(0) + \int_0^t e^{-m(t-\tau)}\int_0^\tau \epsilon(s)\bar{B}(\tau-s)\pi(s)\,ds\,d\tau \\ \qquad + \int_0^t e^{-m(t-\tau)}\int_\tau^\infty \epsilon(s)\bar{\eta}(s-\tau)\pi(s-\tau,s)\,ds\,d\tau. \end{cases} \tag{4.10}$$

Let $\bar{X} = (\bar{S},\bar{I},\bar{R},\bar{B})^T$. It is clear that the above system is equivalent to the compact form

$$A\bar{X} + \int_0^t K(t-\tau)\bar{X}(\tau)\,d\tau = f(t), \tag{4.11}$$

where

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -M_1^*\int_0^\infty \beta(\tau)i^*(\tau)\,d\tau & -M_2^*\int_0^\infty \beta(\tau)i^*(\tau)\,d\tau & -M_3^*\int_0^\infty \beta(\tau)i^*(\tau)\,d\tau & 1 \end{pmatrix},$$

$$K(t) = \begin{pmatrix} 0 & 0 & \delta e^{-\mu t} & e^{-\mu t} \\ 0 & 0 & 0 & -\pi(t) \\ 0 & 0 & 0 & e^{-mt}\int_0^t e^{m\tau}\epsilon(\tau)\pi(\tau)\,d\tau \\ 0 & 0 & 0 & -M^*\beta(t)\pi(t) \end{pmatrix},$$

$$f(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \\ f_4(t) \end{pmatrix},$$

$$f_1(t) = e^{-\mu t}\bar{S}(0),$$

$$f_2(t) = \int_t^\infty \bar{\eta}(\tau-t)\pi(\tau-t,\tau)\,d\tau,$$

$$f_3(t) = e^{-mt}\overline{R}(0) + \int_0^t e^{-m(t-\tau)} \int_\tau^\infty \epsilon(s)\bar{\eta}(s-\tau)\pi(s-\tau, s)\,ds\,d\tau,$$

$$f_4(t) = M^* \int_t^\infty \beta(\tau)\bar{\eta}(\tau - t)\pi(\tau - t, \tau)\,d\tau + \Psi(t),$$

$$M_1^* = \frac{\partial M(S^*, I^*, R^*)}{\partial S},$$

$$M_2^* = \frac{\partial M(S^*, I^*, R^*)}{\partial I},$$

$$M_3^* = \frac{\partial M(S^*, I^*, R^*)}{\partial R}.$$

It is a routine matter to show that there exists a positive constant D such that

$$\|K(t)\|, \|K'(t)\|, \|K''(t)\| \leqslant De^{-\mu t}. \tag{4.12}$$

Hence, the Laplace transform $\widehat{K}(s)$ of $K(t)$ is analytic in the right half plane $\Re(s) > -\mu$. Moreover, it is easy to verify $\lim_{|s|\to+\infty} \widehat{K}(s) = 0$ and then $\lim_{|s|\to+\infty} \det(A + \widehat{K}(s)) = 1$. Therefore, all of the roots of $\det(A + \widehat{K}(s))$ are isolated and lie in a certain ball centered at $(0, 0, 0)$.

Assume that all roots of $\det(A + \widehat{K}(s))$ have negative real parts, that is, there exists a μ^* and $0 < \mu^* < \mu$ such that no root of $\det(A + \widehat{K}(s))$ lies outside $\Re(s) < -\mu^*$. Let $L(s)$ denote the analytic reverse matrix of $A + \widehat{K}(s)$ in $\Re(s) \geqslant -\mu^*$. Since A is invertible and $\lim_{|s|\to+\infty} \widehat{K}(s) = 0$, for sufficiently large $|s|$ and $\Re(s) > -\mu$ we have

$$L(s) = A^{-1}\bigl(I + A^{-1}\widehat{K}(s)\bigr)^{-1} = A^{-1}\sum_{j=0}^{\infty}\bigl(A^{-1}\widehat{K}(s)\bigr)^j$$

and

$$\lim_{|s|\to\infty} L(s) = \lim_{|s|\to\infty} A^{-1}\bigl(I + A^{-1}\widehat{K}(s)\bigr)^{-1} = A^{-1}.$$

Moreover, by Taylor theorem it can be shown that

$$\widehat{K}(s) = \frac{K(0)}{s} + \frac{K'(0)}{s^2} + o\bigl(s^{-2}\bigr), \quad \text{for } |s| \to \infty \text{ in } \Re(s) > -\mu^*.$$

Therefore, we get a constant matrix J_0 such that

$$L(s) = A^{-1} + \frac{1}{s}J_0 + O\bigl(s^{-2}\bigr), \quad \text{for } |s| \to \infty \text{ in } \Re(s) > \mu^*.$$

Which implies that $\widehat{J}(s) := L(s) - A^{-1}$ is the Laplace transform of $J(t)$, where

$$J(t) = \frac{1}{2\pi}e^{-\mu^* t}\int_{-\infty}^{\infty} e^{i\xi t}\widehat{J}\bigl(-\mu^* + i\xi\bigr)\,d\xi, \quad t \geqslant 0. \tag{4.13}$$

It is easy to obtain from (4.12) that there exists a constant D_1 such that

$$\|J(t)\| \leqslant D_1 e^{-\mu^* t}, \quad t \geqslant 0.$$

To discuss the asymptotical stability of endemic equilibrium, we use the following assumption, which is a reduction of the assumption $H5$ of [3].

A. $|M - M^* - \nabla M^* \cdot (\bar{S}, \bar{I}, \bar{R})| = o(|\bar{S}| + |\bar{I}| + |\bar{R}|)$*, as* $|\bar{S}| + |\bar{I}| + |\bar{R}| \to 0$*, that is, for* $\epsilon_0 > 0$*, there exists* $\bar{\delta}(\epsilon_0) > 0$ *such that* $|\bar{S} + \bar{I} + \bar{R}| < \bar{\delta}(\epsilon_0)$ *implies that* $|M - M^* - \nabla M^* \cdot (\bar{S}, \bar{I}, \bar{R})| < \epsilon_0(|\bar{S}| + |\bar{I}| + |\bar{R}|)$.

Remark 3. It should be pointed out that this assumption is obviously satisfied if $C(S, I, R) = \beta_1 N$, where β_1 is a nonnegative number and N is the population size.

Theorem 4.4. *Under the assumption* A. *If all roots of* $\det(A + \widehat{K}(s))$ *have negative real parts, then there exist positive numbers* a*,* b *and* δ *such that for initial value* S_0*,* $\eta(\tau)$*, and* R_0 *with* $|S_0| + |R_0| + \|\eta(\tau)\|_1 < \delta$*, the solutions for* (2.2) *satisfy* $|S(t) - S^*| + \|i(t, \cdot) - i^*(\cdot)\|_1 + |R(t) - R^*| + |i(t, 0) + i^*(0)| < ae^{-bt}$*,* $t \geqslant 0$.

Proof. Denote by $\widehat{\overline{X}}(s)$ the Laplace transform of $\overline{X}(t)$ and by $\widehat{f}(s)$ the Laplace transform of $f(t)$. Then, it follows from (4.11) that

$$A\widehat{\overline{X}}(s) + \widehat{K}(s)\widehat{\overline{X}}(s) = \widehat{f}(s).$$

Hence, we have

$$\begin{aligned} \overline{X}(t) &= \mathfrak{L}^{-1}\big(\big(A + \widehat{K}(s)\big)^{-1}\widehat{f}(s)\big) \\ &= A^{-1}f(t) + \int_0^\infty J(t-\tau)f(\tau)\,d\tau, \quad t \geqslant 0, \end{aligned}$$

where $\mathfrak{L}^{-1}$ represents the inverse Laplace transform. To obtain a bound of $\overline{X}(t)$, we consider the following system

$$\begin{cases} \frac{d\underline{S}(t)}{dt} = -\mu\underline{S}(t) + \delta\underline{R}(t) - \underline{B}(t), \\ \frac{\partial\underline{i}(t,\tau)}{\partial t} + \frac{\partial\underline{i}(t,\tau)}{\partial\tau} = -(\mu + \epsilon(\tau) + \alpha(\tau))\underline{i}(t,\tau), \\ \frac{d\underline{R}(t)}{dt} = \int_0^\infty \epsilon(\tau)\underline{i}(t,\tau)\,d\tau - m\underline{R}(t), \\ \underline{B}(t) = \underline{i}(t,0) = M^*\int_0^\infty \beta(\tau)\underline{i}(t,\tau)\,d\tau + \nabla M^* \cdot (\underline{S}, \underline{I}, \underline{R})\int_0^\infty \beta(\tau)i^*(\tau)\,d\tau, \\ \underline{S}(0) = \bar{S}(0), \qquad \underline{i}(0,\tau) = \bar{i}(0,\tau) - \Psi(\tau), \qquad \underline{R}(0) = \overline{R}(0), \qquad \underline{\eta}(\tau) = \bar{\eta}(\tau). \end{cases} \tag{4.14}$$

Clearly,

$$\underline{i}(t,\tau) = \begin{cases} \underline{B}(t-\tau)\pi(\tau), & t - \tau \geqslant 0; \\ \bar{\eta}(\tau - t)\pi(\tau - t, \tau), & \tau - t > 0, \end{cases}$$

where $\underline{B}(t) = \underline{i}(t, 0)$. Let $\underline{Y}(t) = (\underline{S}(t), \underline{I}(t), \underline{R}(t), \underline{B}(t))^T$. Then, the system (4.14) can be written as

$$A\underline{Y}(t) + \int_0^t K(t-\tau)\underline{Y}(\tau)\,d\tau = \underline{l}(t), \tag{4.15}$$

where

$$\underline{l} = \begin{pmatrix} e^{-\mu t}\overline{S}(0) \\ \int_t^\infty \overline{\eta}(\tau - t)\pi(\tau - t, \tau)\,d\tau \\ e^{-mt}\overline{R}(0) + \int_0^t e^{-m(t-\tau)} \int_\tau^\infty \epsilon(s)\overline{\eta}(s-\tau)\pi(s-\tau, s)\,ds\,d\tau \\ M^* \int_t^\infty \beta(\tau)\overline{\eta}(\tau - t)\pi(\tau - t, \tau)\,d\tau \end{pmatrix}, \tag{4.16}$$

and A and K are defined by (4.11). Similar to the previous discussion, we have

$$\underline{Y}(t) = A^{-1}\underline{l}(t) + \int_0^t J(t-\tau)\underline{l}(\tau)\,d\tau. \tag{4.17}$$

Let us endow 4-dimensional real space $\mathbb{R}^4$ with the norm $\|x\| = \sum_{j=1}^4 |x_j|$. Then, it follows from (4.16) that

$$\begin{aligned} \|\underline{l}\| &\leqslant e^{-\mu t}\big[\big|\overline{S}(0)\big| + \big|\overline{R}(0)\big| + \big(1 + M^*\beta_\infty\big)\|\overline{\eta}\|_1\big] \\ &\quad + \epsilon_\infty \left| \int_0^t e^{-m(t-\tau)-\mu\tau} \int_\tau^\infty \overline{\eta}(s-\tau)\,ds\,d\tau \right| \\ &\leqslant \big(1 + M^*\beta_\infty\big)\overline{N}(0)e^{-\mu t} + \frac{\epsilon_\infty}{m-\mu}\|\overline{\eta}\|_1 e^{-\mu t} \\ &\leqslant \left(1 + M^*\beta_\infty + \frac{\epsilon_\infty}{m-\mu}\right)\overline{N}(0)e^{-\mu t} := D_2\overline{N}(0)e^{-\mu t}, \end{aligned} \tag{4.18}$$

where $\overline{N}(0) = |\overline{S}(0)| + |\overline{R}(0)| + \|\overline{\eta}\|_1$. Considering (4.17), (4.18) and the second equation of (4.14), we get the following estimations of $\|\underline{Y}\|$ and $\|\underline{i}(t,\cdot)\|$

$$\begin{aligned} \|\underline{Y}\| &\leqslant \big\|A^{-1}\big\|\,\big\|\underline{l}(t)\big\| + D_1 \int_0^t e^{-\mu^*(t-\tau)}\|\underline{l}\|\,d\tau \\ &\leqslant D_2\overline{N}(0)\left[\big\|A^{-1}\big\|e^{-\mu t} + D_1 e^{-\mu^* t}\int_0^t e^{-(\mu-\mu^*)s}\,ds\right] \\ &< D_2\left[\big\|A^{-1}\big\| + \frac{D_1}{\mu-\mu^*}\right]\overline{N}(0)e^{-\mu^* t} := D_3\overline{N}(0)e^{-\mu^* t} \end{aligned} \tag{4.19}$$

and

$$\begin{aligned} \big\|\underline{i}(t,\cdot)\big\|_1 &= \left|\int_0^t \underline{B}(t-\tau)\pi(\tau)\,d\tau\right| + \left|\int_t^\infty \underline{\eta}(\tau - t)\pi(\tau - t, \tau)\,d\tau\right| \\ &\leqslant \int_0^t \big\|\underline{Y}(t-\tau)\big\|\pi(\tau)\,d\tau + e^{-\mu t}\big\|\underline{\eta}(t)\big\|_1 \\ &\leqslant \frac{D_3}{\mu}\overline{N}(0)e^{-\mu^* t} + \overline{N}(0)e^{-\mu^* t} := D_4\overline{N}(0)e^{-\mu^* t}. \end{aligned} \tag{4.20}$$

Let $W(t)=|\bar{S}(t)|+|\bar{I}(t)|+|\bar{R}(t)|+\|\bar{i}(t,\cdot)\|_1$. Without loss of generality, for some ϵ_0, choosing $\bar{\delta}(\epsilon_0)<\epsilon_0$, under the assumption A, we get

$$\begin{aligned}
|\Psi(t)| &\leqslant \max\{|M_1^*|,|M_2^*|,|M_3^*|\}\beta_\infty\big(|\bar{S}(t)|+|\bar{I}(t)|+|\bar{R}(t)|\big)\|\bar{i}(t,\cdot)\|_1 \\
&\quad + \epsilon_0\big(|\bar{S}(t)|+|\bar{I}(t)|+|\bar{R}(t)|\big)\left(\beta_\infty\|\bar{i}(t,\cdot)\|_1+\int_0^\infty \beta(\tau)i^*(\tau)\,d\tau\right) \\
&\leqslant \big(|\bar{S}(t)|+|\bar{I}(t)|+|\bar{R}(t)|\big)\left(D_5\|\bar{i}(t,\cdot)\|_1+\epsilon_0\int_0^\infty \beta(\tau)i^*(\tau)\,d\tau\right) \\
&\leqslant \epsilon_0(D_5+D_6)W(t) := \epsilon_0 D_7 W(t),
\end{aligned} \tag{4.21}$$

where $D_5=\beta_\infty(\epsilon_0+\max\{|M_j^*|, j=1,2,3\})$ and $D_6=\int_0^\infty \beta(\tau)i^*(\tau)\,d\tau$. By (4.21) and the expressions of $f(t)$ and $\underline{l}(t)$, we obtain

$$\|f(t)-\underline{l}(t)\| \leqslant \epsilon_0 D_7 W(t). \tag{4.22}$$

Note that

$$\bar{X}(t)=\underline{Y}(t)+A^{-1}\big(f(t)-\underline{l}(t)\big)+\int_0^t J(t-\tau)\big(f(\tau)-\underline{l}(\tau)\big)\,d\tau,$$

we get

$$\begin{aligned}
\|\bar{X}\| &\leqslant \|\underline{Y}\|+\epsilon_0 D_7\|A^{-1}\|W(t)+\epsilon_0 D_1 D_7\int_0^t e^{-\mu^*(t-\tau)}W(\tau)\,d\tau \\
&:= \|\underline{Y}\|+\epsilon_0 D_8 W(t)+\epsilon_0 D_9\int_0^t e^{-\mu^*(t-\tau)}W(\tau)\,d\tau,
\end{aligned} \tag{4.23}$$

and

$$\begin{aligned}
\|\bar{i}(t,\cdot)-\underline{i}(t,\cdot)\|_1 &= \int_0^t |\bar{B}(t-\tau)-\underline{B}(t-\tau)|\pi(\tau)\,d\tau \\
&\leqslant \epsilon_0 D_7\int_0^t W(t-\tau)\pi(\tau)\,d\tau.
\end{aligned} \tag{4.24}$$

Thus, by (4.20), (4.23) and (4.24), we get

$$\begin{aligned}
W(t) &\leqslant \|\bar{X}\|+\|\bar{i}(t,\cdot)\|_1 \\
&\leqslant \|\underline{Y}\|+\epsilon_0 D_8 W(t)+\epsilon_0 D_9\int_0^t e^{-\mu^*(t-\tau)}W(\tau)\,d\tau+\|\bar{i}(t,\cdot)-\underline{i}(t,\cdot)\|_1+\|\underline{i}(t,\cdot)\|_1 \\
&\leqslant (D_3+D_4)\bar{N}(0)e^{-\mu^* t}+\epsilon_0 D_8 W(t)+\epsilon_0(D_7+D_9)\int_0^t e^{-\mu^*(t-\tau)}W(\tau)\,d\tau.
\end{aligned} \tag{4.25}$$

Let $\epsilon_0 D_8 < 1$. It follows from Gronwall's lemma that

$$\begin{aligned} W(t) &\leqslant \frac{D_3+D_4}{1-\epsilon_0 D_8}\overline{N}(0)\left[e^{-\mu^* t} + \frac{\epsilon_0(D_7+D_9)}{1-\epsilon_0 D_8}\int_0^t e^{-\mu^* t + \frac{\epsilon_0(D_7+D_9)}{1-\epsilon_0 D_8}(t-s)}\,ds\right] \\ &\leqslant \frac{D_3+D_4}{1-\epsilon_0 D_8}\overline{N}(0)\left[e^{-\mu^* t} + e^{-(\mu^* - \frac{\epsilon_0(D_7+D_9)}{1-\epsilon_0 D_8})t}\right] \\ &\leqslant 2\frac{D_3+D_4}{1-\epsilon_0 D_8}\overline{N}(0)e^{-(\mu^* - \frac{\epsilon_0(D_7+D_9)}{1-\epsilon_0 D_8})t}. \end{aligned} \tag{4.26}$$

Substituting (4.19) and (4.26) into (4.23) yields

$$\begin{aligned} \|\overline{X}\| \leqslant{}& D_3\overline{N}(0)e^{-\mu^* t} + \frac{2(D_3+D_4)\epsilon_0 D_8}{1-\epsilon_0 D_8}\overline{N}(0)e^{-(\mu^* - \frac{\epsilon_0(D_7+D_9)}{1-\epsilon_0 D_8})t} \\ &+ \frac{2\epsilon_0 D_9(D_3+D_4)}{1-\epsilon_0 D_8}\overline{N}(0)\int_0^t e^{-\mu^* t + \frac{\epsilon_0(D_7+D_9)}{1-\epsilon_0 D_8}s}\,ds. \end{aligned}$$

Clearly, there exists positive constant ξ such that

$$\left\|\overline{X}(t)\right\| \leqslant \xi\overline{N}(0)e^{-(\mu^* - \frac{\epsilon_0(D_7+D_9)}{1-\epsilon_0 D_8})t}. \tag{4.27}$$

Next, we prove that there exist positive numbers a, b and δ such that for $|S_0| + \|\eta(\cdot)\|_1 + |R_0| < \delta$, the solution to (2.2) satisfy $|\overline{X}(t)| < ae^{-bt}$, $t \geqslant 0$.

Let $\delta < \min\{\bar{\delta}(\epsilon_0), \frac{\bar{\delta}(\epsilon_0)}{\xi}\}$. By continuity of $(S(t), I(t), R(t))$, if $|S_0| + \|\eta(\cdot)\|_1 + |R_0| < \delta < \bar{\delta}(\epsilon_0)$ there exists a constant $T > 0$ such that $|\overline{S}(t)| + |\overline{I}(t)| + |\overline{R}(t)| < \bar{\delta}(\epsilon_0)$ on $[0, T]$. According to the above discussion, we have

$$\left\|\overline{X}(t)\right\| \leqslant \xi\overline{N}(0)e^{-(\mu^* - \frac{\epsilon_0(D_7+D_9)}{1-\epsilon_0 D_8})t}, \quad t \in [0, T]. \tag{4.28}$$

It is clear that the inequality is satisfied for $t \geqslant 0$. Overwise, there exists a constant $t_1 > 0$ such that

$$\left|\overline{S}(t_1)\right| + \left|\overline{I}(t_1)\right| + \left|\overline{R}(t_1)\right| \geqslant \bar{\delta}(\epsilon_0) \tag{4.29}$$

and

$$\left|\overline{X}(t_1)\right| \leqslant \xi\overline{N}(0)e^{-(\mu^* - \frac{\epsilon_0(D_7+D_9)}{1-\epsilon_0 D_8})t_1} < \bar{\delta}(\epsilon_0)e^{-(\mu^* - \frac{\epsilon_0(D_7+D_9)}{1-\epsilon_0 D_8})t_1}. \tag{4.30}$$

Inequalities (4.29) and (4.30) lead to a contradiction iff $\epsilon_0 < \frac{\mu^*}{\mu^* D_8 + D_7 + D_9}$.

Let $a = \xi\overline{N}(0)$ and $b = \mu^* - \frac{\epsilon_0(D_7+D_9)}{1-\epsilon_0 D_8}$ with $\epsilon_0 < \frac{\mu^*}{\mu^* D_8 + D_7 + D_9}$, if $|S_0| + \|\eta(\cdot)\|_1 + |R_0| < \delta$, we have

$$\left|\overline{X}(t)\right| \leqslant ae^{-bt}, \quad t \in [0, \infty).$$

This completes the proof. □

Indeed, for special $\alpha(\tau)$, $\epsilon(\tau)$, $\beta(\tau)$ and $C(S, I, R)$, the condition of Theorem 4.4 is satisfied.

Example 1. Let $\epsilon(\tau) \equiv \epsilon$, $\alpha(\tau) \equiv \alpha$, $\beta(\tau) = e^{-\beta\tau}$ and $C(S, I, R) = \beta' N$, where α, ϵ, β and β' are positive numbers. Correspondingly, we have $M = \beta' S$, $R_0 = \frac{\beta' \Lambda}{\mu(\mu+\alpha+\epsilon+\beta)}$, $S^* = \frac{\mu+\alpha+\epsilon+\beta}{\beta'}$, $B^* = \frac{m\mu(\mu+\alpha+\epsilon)(\mu+\alpha+\epsilon+\beta)(R_0-1)}{\beta'(m(\mu+\alpha+\epsilon)-\delta\epsilon)}$, $I^* = \frac{B^*}{\mu+\alpha+\epsilon}$, $R^* = \frac{B^*\epsilon}{m(\mu+\epsilon+\alpha)}$. Then, we further have

$$A + \widehat{K}(s) = \begin{pmatrix} 1 & 0 & \frac{\delta}{s+\mu} & \frac{1}{s+\mu} \\ 0 & 1 & 0 & -\frac{1}{s+\mu_1} \\ 0 & 0 & 1 & \frac{\epsilon}{(s+m)(s+\mu_1)} \\ -k & 0 & 0 & 1 - \frac{\mu_2}{s+\mu_2} \end{pmatrix},$$

where $\mu_1 = \mu + \alpha + \epsilon$, $\mu_2 = \mu + \alpha + \epsilon + \beta$ and $k = \frac{m\mu\mu_1(R_0-1)}{m\mu_1 - \epsilon\delta}$.

Clearly,

$$\det\bigl(A + \widehat{K}(s)\bigr) = \frac{s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4}{(s+m)(s+\mu)(s+\mu_1)(s+\mu_2)},$$

where $a_1 = k + \mu + \mu_1 + m$, $a_2 = \mu\mu_1 + m\mu + \mu_1 m + k(\mu_1 + \mu_2 + m)$, $a_3 = \mu\mu_1 m + k\mu_2(\mu_1 + m) + k(\mu_1 m - \delta\epsilon)$ and $a_4 = k\mu_1(\mu_1 m - \delta\epsilon)$.

For convenience, we introduce the following denotations

$$\Omega_1 = a_1, \qquad \Omega_2 = \begin{pmatrix} a_1 & a_3 \\ 1 & a_2 \end{pmatrix},$$

$$\Omega_3 = \begin{pmatrix} a_1 & a_3 & 0 \\ 1 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{pmatrix} \quad \text{and} \quad \Omega_4 = \begin{pmatrix} a_1 & a_3 & 0 & 0 \\ 1 & a_2 & a_4 & 0 \\ 0 & a_1 & a_3 & 0 \\ 0 & 1 & a_2 & a_4 \end{pmatrix}.$$

It is easy to obtain that

$$\begin{aligned}
\det(\Omega_1) &= k + \mu + \mu_1 + m, \\
\det(\Omega_2) &= (\mu_1 + \mu_2 + m)k^2 + \bigl(m^2 + \mu\mu_2 + 2\mu\mu_1 + 2\mu_1 m + \delta\epsilon + 2m\mu + \mu_1^2\bigr)k \\
&\quad + \mu\mu_1^2 + \mu_1^2 m + 2\mu\mu_1 m + m^2\mu + \mu_1 m^2 + \mu^2\mu_1 + m\mu^2, \\
\det(\Omega_3) &= \bigl(\mu_2^2\mu_1 + \mu_1^2\mu_2 + \mu_2^2 m + m^2\mu_2 + \mu_1 m^2 + 2\mu_1 m\mu_2 - \mu_1\delta\epsilon - m\delta\epsilon + \mu_1^2 m\bigr)k^3 \\
&\quad + \bigl(2\mu_1^2 m^2 + \mu_1^3\mu_2 + \mu_1^3 m + 4\mu\mu_1 m\mu_2 - \delta^2\epsilon^2 + \mu\mu_2^2 m - m^2\delta\epsilon + 2\mu\mu_1^2\mu_2 \\
&\quad + \mu\mu_2^2\mu_1 + \mu\mu_2\delta\epsilon - 2\mu\mu_1\delta\epsilon - 2m\mu\delta\epsilon + 3mu\mu_1^2 m - \mu_1^2\delta\epsilon + \mu_1 m^2\mu_2 \\
&\quad + 2m^2\mu\mu_2 - \mu_1 m\delta\epsilon + m^3\mu_2 + m^3\mu_1 + 3m\mu_2\delta\epsilon + 3m^2\mu\mu_1 + 3\mu_1\mu_2\delta\epsilon \\
&\quad + \mu_1^2 m\mu_2\bigr)k^2 + \bigl(\mu_1^2 m^3 + \mu_1^3 m^2 + 3\mu^2\mu_1^2 m + 3m^2\mu^2\mu_1 + 4\mu_1^2 m^2\mu + m^3\mu\mu_2 \\
&\quad + 2m^3\mu\mu_1 + \mu^2\mu_1^2\mu_2 + m^2\mu^2\mu_2 + \mu\mu_1^2 m\mu_2 + \mu\mu_1 m^2\mu_2 - m^2\mu\delta\epsilon \\
&\quad - \mu_1 m^2\delta\epsilon + 2\mu^2\mu_1 m\mu_2 - \mu^2\mu_1\delta\epsilon - m\mu^2\delta\epsilon - \mu\mu_1^2\delta\epsilon - \mu_1^2 m\delta\epsilon + \mu_2 m^2\delta\epsilon \\
&\quad + \mu_2\mu^2\delta\epsilon + \mu_2\mu_1^2\delta\epsilon + \mu\mu_1^3\mu_2 + 2\mu\mu_1^3 m - \mu\mu_1 m\delta\epsilon + 2\mu_2 m\mu\delta\epsilon \\
&\quad + 2\mu_2\mu_1 m\delta\epsilon + 2\mu_2\mu\mu_1\delta\epsilon\bigr)k + \mu_1^2 m^3\mu + \mu^3\mu_1^2 m + 2\mu^2\mu_1^2 m^2 + m^3\mu^2\mu_1 \\
&\quad + \mu_1^3 m^2\mu + m^2\mu^3\mu_1 + \mu^2\mu_1^3 m, \\
\det(\Omega_4) &= a_4 \det(\Omega_3).
\end{aligned}$$

Clearly, if $R_0 > 1$, we have $a_j > 0$ and $\det(\Omega_j) > 0$, $j = 1, 2, 3, 4$. By Routh–Hurwitz Criterion, we get that all of roots of $s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4$ have negative real parts. Then, we arrive at that all of roots of $\det(A + \widehat{K}(s))$ have negative real parts.

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