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Blow-up criteria of solutions to a modified two-component Camassa–Holm system

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ABSTRACT

In this paper, we consider a modified two-component Camassa–Holm (MCH2) system which arises in shallow water theory. We analyze the wave breaking mechanism by establishing some new blow-up criteria for this system formulated either on the line or with space-periodic initial condition.

1. Introduction

The two-component Camassa–Holm (CH2) system reads:

$$\begin{cases} u_t - u_{xxt} + 3uu_x - 2u_x u_{xx} - uu_{xxx} + g\rho\rho_x = 0, & x \in \mathbb{R}, t > 0; \\ \rho_t + (\rho u)_x = 0, & x \in \mathbb{R}, t > 0. \end{cases}$$

This system appears initially in [1], and recently Constantin and Ivanov in [2] gave a demonstration about its derivation in view of the fluid shallow water theory from the hydrodynamic point of view. This generalization, similarly to the Camassa–Holm equation, possessed the peakon, multi-kink solutions and the bi-Hamiltonian structure [3,4] and is integrable. Well-posedness and wave breaking mechanism were discussed in [5–7] and the existence of global solutions was analyzed in [2,6,8]. Particularly, Guo in the recent paper [8] established a new blow-up criterion via the associated potential where the global existence result leads to a better understanding for this problem.

Obviously, under the constraint of $\rho(x, t) = 0$, this system reduces to the Camassa–Holm equation, which was derived physically by Camassa and Holm in [9] by approximating directly the Hamiltonian for Euler's equation in the shallow water region with $u(x, t)$ representing the free surface above a flat bottom. Some satisfactory results have been obtained recently, for instance, see Refs. [10–16]. Moreover, wave breaking criteria for a large class of initial data have been established in [11,14–16]. In [17], Xin and Zhang showed global existence of weak solutions but uniqueness was obtained only under a priori assumption that is known to hold only for initial data $u_0(x) \in H^1$ such that $u_0(x) - u_{0xx}(x)$ is a sign-definite Random measure. The solitary waves of the Camassa–Holm equation are peaked solitons and are orbitally stable [18]; see also [19] for a very related Rod equation. If $\rho(x, t) \neq 0$, this CH2 system which includes both velocity and density variables in the dynamics is actually an extension of the CH equation. Although possessing peaked solutions in the velocity, the CH2 system does not admit singular solutions in the density profile. Its mathematical properties have been studied further in many works [2–6,20–22]. We note that some other related two-component models appeared in [23–25] were also investigated.

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However, in this paper, we consider the Cauchy problem of the following modified two-component Camassa–Holm (MCH2) system:

$$\begin{cases} u_t - u_{xxt} + 3uu_x - 2u_x u_{xx} - uu_{xxx} + g\rho\bar{\rho}_x = 0, & x \in \mathbb{R}, t > 0; \\ \rho_t + (\rho u)_x = 0, & x \in \mathbb{R}, t > 0, \end{cases} \tag{1.1}$$

where u denotes the velocity field, g is the downward constant acceleration of gravity in applications to shallow water waves, and $\rho = (1 - \partial_x^2)(\bar{\rho} - \bar{\rho}_0)$, where $\bar{\rho}_0$ is taken to be a constant. This MCH2 system does admit peaked solutions in the velocity and average density; we refer this to Ref. [26] for details. There the authors analytically identified the steepening mechanism that allows the singular solutions to emerge from smooth spatially confined initial data. They found that wave breaking in the fluid velocity does not imply singularity in the pointwise density ρ at the point of vertical slope. Some other recent work can be found in [27,28]. We find that the MCH2 system is expressed in terms of an averaged or filtered density $\bar{\rho}$ in analogy to the relation between momentum and velocity by setting $\rho = (1 - \partial_x^2)(\bar{\rho} - \bar{\rho}_0)$, but it may not be integrable unlike the CH2 system. The characteristic is that it will amount to strengthening the norm for $\bar{\rho}$ from L^2 to H^1 in the potential energy term. Note that this MCH2 system is a modified version of the CH2 system to allow a dependence on the average density $\bar{\rho}$ (or depth, in the shallow water interpretation) as well as the pointwise density ρ. It is written in terms of velocity u and locally averaged density $\bar{\rho}$. From the geometric points of view, it is defined as geodesic motion on the corresponding semidirect-product Lie group [29] with respect to a certain metric and is given as a set of Euler–Poincaré equations on the dual of the corresponding Lie algebra. In the general case, for a Lagrangian $L(u, \bar{\rho})$, the corresponding semidirect-product Euler–Poincaré equation are written as [30]

$$\frac{\partial}{\partial t}\frac{\delta L}{\delta u} = -\Lambda_u \frac{\delta L}{\delta u} - \frac{\delta L}{\delta \bar{\rho}} \nabla \bar{\rho}, \qquad \frac{\partial}{\partial t}\frac{\delta L}{\delta u} = -\Lambda_u \frac{\delta L}{\delta \bar{\rho}},$$

where $\Lambda_u(\delta L/\delta u)$ is the Lie derivative of the one-form density $u - u_{xx} = \delta L/\delta u$ with respect to the vector field u and $\Lambda_u(\delta L/\delta u)$ is the corresponding Lie derivative of the scalar density $\delta L/\delta \bar{\rho}$. By setting $y = u - u_{xx}$, $\rho = v - v_{xx}$ and $v = \bar{\rho} - \bar{\rho}_0$, we can rewrite the MCH2 system (1.1) as follows:

$$\begin{cases} y_t + uy_x + 2yu_x = -g(v - v_{xx})v_x, & x \in \mathbb{R}, t > 0; \\ (v - v_{xx})_t + ((v - v_{xx})u)_x = 0, & x \in \mathbb{R}, t > 0, \end{cases} \tag{1.2}$$

which takes an equivalent form of a quasi-linear evolution equation of hyperbolic type:

$$\begin{cases} u_t + uu_x = -\partial_x \left(G * \left(u^2 + \frac{1}{2}u_x^2 + \frac{g}{2}v^2 - \frac{g}{2}v_x^2 \right) \right), & x \in \mathbb{R}, t > 0; \\ v_t + uv_x = -G * ((u_x v_x)_x + u_x v), & x \in \mathbb{R}, t > 0, \end{cases} \tag{1.3}$$

where the sign $*$ denotes the spatial convolution, G is the associated Green's function of the operator $(1 - \partial_x^2)^{-1}$.

What interests us for the MCH2 system is to investigate further formation of singularities of solutions to (1.3) with the case of $g = 1$, just for simplicity mathematically.

We now finish this Introduction by outlining the rest of this paper. In Section 2, we recall some preliminary results on well-posedness and blow-up scenario. In Section 3, the detailed blow-up criteria are presented.

2. Preliminaries

In this section, for completeness, we recall some elementary results and skip their proofs. Local well-posedness for the MCH2 system can be obtained by Kato's semi-group theory [31]. In [28], the authors gave a detailed description on the well-posedness theorem.

Theorem 2.1 (*[28]*). *Given $X_0 = (u_0, v_0)^T \in H^s \times H^{s-1}$, $s \geq 5/2$, there exists a maximal $T = T(\|X_0\|_{H^s \times H^{s-1}}) > 0$, and a unique solution $X = (u, v)^T$ to system (1.3) such that*

$$X = X(\cdot, X_0) \in C\left([0, T); H^s \times H^{s-1}\right) \cap C^1\left([0, T); H^{s-1} \times H^{s-2}\right).$$

Moreover, the solution depends continuously on the initial data, i.e., the mapping

$$X \to X(\cdot, X_0) : H^s \times H^{s-1} \to C\left([0, T); H^s \times H^{s-1}\right) \cap C^1\left([0, T); H^{s-1} \times H^{s-2}\right).$$

is continuous.

The next result describes the precise blow-up scenario for sufficiently regular solutions to system (1.3).

Theorem 2.2 (*[28]*). *Let $X_0 = (u_0, v_0)^T \in H^s \times H^{s-1}$, $s \geq 5/2$, and let T be the maximal existence time of the solution $X = (u, v)^T$ to system (1.3) with the initial data X_0. Then the corresponding solution blows up in finite time if and only if*

$$\lim_{t \to T} \inf_{x \in \mathbb{R}} \{u_x(x, t)\} = -\infty.$$

We also need to introduce the classical particle trajectory method for later use. Consider the following initial value problem

$$\begin{cases} q_t = u(q,t), & x \in \mathbb{R}, 0 < t < T; \\ q(x,0) = x, & x \in \mathbb{R}, \end{cases}$$

where T is the lifespan of the solution, then q is a diffeomorphism of the line. Moreover, we know that the map $q(\cdot,t)$ is an increasing diffeomorphism of $\mathbb{R}$ with

$$q_x(x,t) = \exp\left(\int_0^t u_x(q,s)ds\right) > 0, \quad (x,t) \in \mathbb{R} \times [0,T). \tag{2.1}$$

After local well-posedness of a strong solution (see Theorem 2.1) is established, a natural question is whether this local solution can exist globally. If the solution exists only in finite time, under what conditions does MCH2 system admit blow-up solutions? On the other hand, to find sufficient conditions to guarantee the finite time singularity or global existence is of great interest, especially for sufficient conditions added on certain initial data. The following results will give positive answers.

3. Blow-up phenomena

In this section, we pay more attention to the formation of singularities for strong solutions of (1.3). In some previous work, blow-up criteria were usually discussed under some conditions added on the slope of the initial velocity at certain points, the negativity of the initial slope is needed. The following theorems will show that wave breaking is the one way that singularities arise in smooth solutions, and they are different from the previous results for the MCH2 system. We start this section with the following useful lemma.

Lemma 3.1. *Let $X_0 = (u_0, v_0)^T \in H^s \times H^{s-1}$, $s \geq 2$. T is assumed to be the maximal existence time of the solution $X = (u, v)^T$ to system (1.3) corresponding to the initial data X_0. Then for all $t \in [0,T)$, we have the following conservation law*

$$E(t) = \int_{\mathbb{R}} (u^2 + u_x^2 + v^2 + v_x^2)\mathrm{d}x.$$

Proof. We will prove that $E(t)$ is a conserved quantity with respect to time variable. Here we use the classical energy method. Multiplying the first equation in (1.1) by $u(x,t)$ and integrating by parts, we obtain

$$\int_{\mathbb{R}} uu_t\mathrm{d}x + \int_{\mathbb{R}} u_xu_{xt}\mathrm{d}x = -\int_{\mathbb{R}} uvv_x\mathrm{d}x + \int_{\mathbb{R}} uv_xv_{xx}\mathrm{d}x.$$

Similarly, we have the following identity for the second equation in (1.1)

$$\int_{\mathbb{R}} vv_t\mathrm{d}x - \int_{\mathbb{R}} vv_{xxt}\mathrm{d}x = -\int_{\mathbb{R}} vd((v - v_{xx})u) = \int_{\mathbb{R}} (v - v_{xx})uv_x\mathrm{d}x.$$

This implies that

$$\int_{\mathbb{R}} vv_t\mathrm{d}x + \int_{\mathbb{R}} v_xv_{xt}\mathrm{d}x = \int_{\mathbb{R}} uvv_x\mathrm{d}x - \int_{\mathbb{R}} uv_xv_{xx}\mathrm{d}x.$$

Combining the above equalities, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} (u^2 + u_x^2 + v^2 + v_x^2)\mathrm{d}x = 2\int_{\mathbb{R}} (uu_t + u_xu_{xt} + vv_t + v_xv_{xt})\mathrm{d}x = 0.$$

Therefore, $E(t)$ is conserved. $\square$

Using this conservation law, we obtain

$$\begin{aligned} \|u(\cdot,t)\|^2_{L^\infty(\mathbb{R})} + \|v(\cdot,t)\|^2_{L^\infty(\mathbb{R})} &\leq \frac{1}{2}\|u\|^2_{H^1(\mathbb{R})} + \frac{1}{2}\|v\|^2_{H^1(\mathbb{R})} \\ &= \frac{1}{2}\left(\|u_0\|^2_{H^1(\mathbb{R})} + \|v_0\|^2_{H^1(\mathbb{R})}\right) = \frac{1}{2}E_0, \end{aligned}$$

for all $t \in [0,T)$, where E_0 is the initial value of $E(t)$.

We state our first criterion as follows.

Theorem 3.2. *Let $X_0 = (u_0, v_0)^T \in H^s \times H^{s-1}$, $s \geq 5/2$. Assume that the following inequality holds*

$$\int_{\mathbb{R}} u_{0x}^3\mathrm{d}x < -\frac{\sqrt{30E_0^3}}{4}.$$

Then the corresponding solution to (1.3) blows up in finite time.

Proof. Differentiating the first equation of (1.3) with respect to x, we obtain

$$u_{xt}+u_x^2+uu_{xx}+\partial_x^2 G*\left(u^2+\frac{u_x^2}{2}+\frac{v^2}{2}-\frac{v_x^2}{2}\right)=0. \tag{3.1}$$

Applying the relation $\partial_x^2(G*f)=G*f-f$ to (3.1) yields

$$u_{xt}+\frac{1}{2}u_x^2+uu_{xx}+G*\left(u^2+\frac{u_x^2}{2}+\frac{v^2}{2}-\frac{v_x^2}{2}\right)-u^2-\frac{v^2}{2}+\frac{v_x^2}{2}=0. \tag{3.2}$$

Let

$$M(t)=\int_{\mathbb{R}}u_x^3(x)\mathrm{d}x,\quad t\geq 0.$$

Multiplying (3.2) with u_x^2 and integrating by parts subsequently, we get the equation for $M(t)$ as

$$\begin{aligned}\frac{1}{3}\frac{\mathrm{d}M(t)}{\mathrm{d}t}&=-\frac{1}{2}\int_{\mathbb{R}}u_x^4\mathrm{d}x+\frac{1}{3}\int_{\mathbb{R}}u_x^4\mathrm{d}x-\int_{\mathbb{R}}u_x^2G*\left(u^2+\frac{u_x^2}{2}+\frac{v^2}{2}-\frac{v_x^2}{2}\right)\mathrm{d}x+\int_{\mathbb{R}}u_x^2\left(u^2+\frac{v^2}{2}-\frac{v_x^2}{2}\right)\mathrm{d}x\\&\leq-\frac{1}{6}\int_{\mathbb{R}}u_x^4\mathrm{d}x+\frac{1}{2}\int_{\mathbb{R}}u^2u_x^2\mathrm{d}x+\frac{1}{2}\int_{\mathbb{R}}u_x^2G*v_x^2\mathrm{d}x+\frac{1}{2}\int_{\mathbb{R}}u_x^2v^2\mathrm{d}x\\&\leq-\frac{1}{6}\int_{\mathbb{R}}u_x^4\mathrm{d}x+\frac{1}{2}\int_{\mathbb{R}}u_x^2(u^2+v^2)\mathrm{d}x+\frac{1}{4}\|v_x^2\|_{L^1}\int_{\mathbb{R}}u_x^2\mathrm{d}x,\end{aligned}$$

where the last step used

$$\|G*v_x^2\|_{L^\infty}\leq\|G\|_{L^\infty}\|v_x^2\|_{L^1}\leq\frac{1}{2}\|v_x^2\|_{L^1}.$$

In view of the invariant property of $E(t)$, we find that

$$\frac{3}{2}\int_{\mathbb{R}}u_x^2(u^2+v^2)\mathrm{d}x+\frac{3}{4}\|v_x\|_{L^1}^2\int_{\mathbb{R}}u_x^2\mathrm{d}x\leq\frac{3E_0^2}{4}+\frac{3E_0^2}{16}=\frac{15E_0^2}{16}.$$

On the other hand, the Cauchy–Schwarz inequality implies that

$$\left|\int_{\mathbb{R}}u_x^3\mathrm{d}x\right|\leq\left(\int_{\mathbb{R}}u_x^4\mathrm{d}x\right)^{1/2}\left(\int_{\mathbb{R}}u_x^2\mathrm{d}x\right)^{1/2},$$

hence

$$\int_{\mathbb{R}}u_x^4\mathrm{d}x\geq\frac{1}{E_0}\left(\int_{\mathbb{R}}u_x^3\mathrm{d}x\right)^2.$$

Thus we can deduce from the above that

$$\frac{\mathrm{d}M(t)}{\mathrm{d}t}\leq-\frac{M^2}{2E_0}+\frac{15E_0^2}{16}.$$

The hypothesis of this theorem and the standard argument on the Riccati type equation ensure that there exists a time T such that

$$\lim_{t\to T}\int_{\mathbb{R}}u_x^3\mathrm{d}x=-\infty.$$

Since

$$\int_{\mathbb{R}}u_x^3\mathrm{d}x\geq\inf u_x(x,t)\int_{\mathbb{R}}u_x^2\mathrm{d}x\geq\inf u_x(x,t)E_0,$$

which shows that

$$\lim_{t\to T}\inf_{x\in\mathbb{R}}u_x(x,t)=-\infty.$$

We complete the proof. □

Motivated by Zhou's recent work [15], we give our second criterion via the associated initial potential.

Theorem 3.3. *Suppose $X_0 = (u_0, v_0)^T \in H^s \times H^{s-1}$, $s \geq 5/2$, $\rho_0(x_0) = v_0(x_0) - v_{0xx}(x_0)$ and $y_0(x_0) = u_0(x_0) - u_{0xx}(x_0)$ satisfy $\rho_0(x_0) = 0$ and $y_0(x_0) = 0$ respectively. Furthermore,*

$$\int_{-\infty}^{x_0} e^{\xi} y_0(\xi)\mathrm{d}\xi > \frac{\sqrt{6E_0}}{2} \quad \text{and} \quad \int_{x_0}^{\infty} e^{-\xi} y_0(\xi)\mathrm{d}\xi < -\frac{\sqrt{6E_0}}{2},$$

for some point $x_0 \in \mathbb{R}$. Then the solution to system (1.3) with the initial value X_0 blows up in finite time.

Proof. Eq. (3.2) in combination with (2.1) gives

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} u_x\left(q(x_0, t), t\right) &= \left(u_{xt} + u u_{xx}\right)\left(q(x_0, t), t\right) \\
&\leq \frac{1}{2}u^2\left(q(x_0, t), t\right) - \frac{1}{2}u_x^2\left(q(x_0, t), t\right) + \frac{1}{2}v^2\left(q(x_0, t), t\right) + \frac{1}{2}\left(G * v_x^2\right)\left(q(x_0, t), t\right) \\
&\leq \frac{1}{2}u^2\left(q(x_0, t), t\right) - \frac{1}{2}u_x^2\left(q(x_0, t), t\right) + \frac{1}{4}\|v\|_{H^1}^2 + \frac{1}{4}\|v_x^2\|_{L^1} \\
&\leq \frac{1}{2}u^2\left(q(x_0, t), t\right) - \frac{1}{2}u_x^2\left(q(x_0, t), t\right) + \frac{1}{2}E_0,
\end{aligned}$$

where we used the fact

$$G * \left(u^2 + \frac{1}{2}u_x^2\right) \geq \frac{1}{2}u^2(x).$$

In order to arrive at our result, we need the following two claims.

Claim 1. $y\left(q(x_0, t), t\right) = 0$ for all t in its lifespan.

It is worth noting the equivalent form of the first equation in (1.2) in what follows

$$y_t + 2yu_x + y_x u + \rho v_x = 0. \tag{3.3}$$

Applying the particle trajectory method to (3.3) and the second equation in (1.1), we obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(y\left(q(x, t), t\right) q_x^2(x, t)\right) &= \left(y_t + 2yu_x + y_x u\right)\left(q(x, t), t\right) q_x^2(x, t) \\
&= -\rho\left(q(x, t), t\right) v_x\left(q(x, t), t\right) q_x^2(x, t),
\end{aligned}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\rho\left(q(x, t), t\right) q_x(x, t)\right) = 0,$$

which implies that

$$\rho\left(q(x, t), t\right) q_x(x, t) = \rho_0(x).$$

Obviously, $\rho\left(q(x_0, t), t\right) = 0$ since $\rho_0\left(x_0\right) = 0$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(y\left(q(x_0, t), t\right) q_x^2(x_0, t)\right) = 0.$$

Thus, $y\left(q(x_0, t), t\right) q_x^2(x_0, t)$ is independent on time t. We get by taking $t = 0$ without loss of generality

$$y\left(q(x_0, t), t\right) q_x^2(x_0, t) = y_0(x_0) = 0.$$

Therefore, thanks to (2.1) we obtain $y\left(q(x_0, t), t\right) = 0$, for all t in its lifespan.

Claim 2. $u^2\left(q(x_0, t), t\right) - u_x^2\left(q(x_0, t), t\right) + E_0 < 0$, on $[0, T)$. Furthermore, $u_x\left(q(x_0, t), t\right) < 0$ and is strictly decreasing with respect to time, where T is the maximal existence time of the solution.

We will prove it by the method of contradiction. Suppose not, there exists a t_0 such that $u^2\left(q(x_0, t), t\right) + 3E_0/2 < u_x^2\left(q(x_0, t), t\right)$ on $[0, t_0)$ but $u^2\left(q(x_0, t_0), t_0\right) + 3E_0/2 \geq u_x^2\left(q(x_0, t_0), t_0\right)$.

Since $y = u - u_{xx}$, then $u(x, t)$ can be given by the convolution $u = G * y$ with $G = \frac{1}{2}e^{-|x|}$, for $x \in \mathbb{R}$, and therefore

$$u(x, t) = \frac{1}{2}e^{-x}\int_{-\infty}^{x} e^{\xi} y(\xi, t)\mathrm{d}\xi + \frac{1}{2}e^{x}\int_{x}^{\infty} e^{-\xi} y(\xi, t)\mathrm{d}\xi,$$

from which we get

$$u_x(x, t) = -\frac{1}{2}e^{-x}\int_{-\infty}^{x} e^{\xi} y(\xi, t)\mathrm{d}\xi + \frac{1}{2}e^{x}\int_{x}^{\infty} e^{-\xi} y(\xi, t)\mathrm{d}\xi.$$

Now we consider the problem at point $(q(x_0,t),t)$, for simplicity let

$$I=\frac{1}{2}e^{-q(x_0,t)}\int_{-\infty}^{q(x_0,t)}e^{\xi}y(\xi,t)\mathrm{d}\xi.$$

Then

$$\frac{\mathrm{d}I}{\mathrm{d}t}=-\frac{1}{2}u\left(q(x_0,t),t\right)e^{-q(x_0,t)}\int_{-\infty}^{q(x_0,t)}e^{\xi}y(\xi,t)\mathrm{d}\xi+\frac{1}{2}e^{-q(x_0,t)}\int_{-\infty}^{q(x_0,t)}e^{\xi}y_t(\xi,t)\mathrm{d}\xi. \tag{3.4}$$

The first term of (3.4) on the right-hand side yields

$$\begin{aligned}
&-\frac{1}{2}u\left(q(x_0,t),t\right)e^{-q(x_0,t)}\int_{-\infty}^{q(x_0,t)}e^{\xi}y(\xi,t)\mathrm{d}\xi\\
&=-\frac{1}{2}u\left(q(x_0,t),t\right)e^{-q(x_0,t)}\left[\int_{-\infty}^{q(x_0,t)}e^{\xi}(u+u_x)\mathrm{d}\xi-e^{q(x_0,t)}u_x\left(q(x_0,t),t\right)\right]\\
&=-\frac{1}{2}u^2\left(q(x_0,t),t\right)+\frac{1}{2}u\left(q(x_0,t),t\right)u_x\left(q(x_0,t),t\right).
\end{aligned}$$

The second term of (3.4) on the right-hand side in combination with the first equation in (1.2) gives

$$\begin{aligned}
&\frac{1}{2}e^{-q(x_0,t)}\int_{-\infty}^{q(x_0,t)}e^{\xi}y_t(\xi,t)\mathrm{d}\xi\\
&=-\frac{1}{2}e^{-q(x_0,t)}\int_{-\infty}^{q(x_0,t)}e^{\xi}\left(yu_x+(yu)_x+\rho v_x\right)(\xi,t)\mathrm{d}\xi\\
&=-\frac{1}{2}u\left(q(x_0,t),t\right)u_x\left(q(x_0,t),t\right)+\frac{1}{2}e^{-q(x_0,t)}\int_{-\infty}^{q(x_0,t)}e^{\xi}\left(u^2+\frac{1}{2}u_x^2\right)\mathrm{d}\xi\\
&\quad+\frac{1}{4}u_x^2\left(q(x_0,t),t\right)-\frac{1}{2}e^{-q(x_0,t)}\int_{-\infty}^{q(x_0,t)}e^{\xi}\rho v_x\mathrm{d}\xi\\
&=-\frac{1}{2}u\left(q(x_0,t),t\right)u_x\left(q(x_0,t),t\right)+\frac{1}{2}e^{-q(x_0,t)}\int_{-\infty}^{q(x_0,t)}e^{\xi}\left(u^2+\frac{1}{2}u_x^2\right)\mathrm{d}\xi\\
&\quad+\frac{1}{4}u_x^2\left(q(x_0,t),t\right)-\frac{1}{4}v^2(q(x_0,t),t)+\frac{1}{4}v_x^2(q(x_0,t),t)\\
&\quad+\frac{1}{4}e^{-q(x_0,t)}\int_{-\infty}^{q(x_0,t)}e^{\xi}v^2(\xi,t)\mathrm{d}\xi-\frac{1}{4}e^{-q(x_0,t)}\int_{-\infty}^{q(x_0,t)}e^{\xi}v_x^2(\xi,t)\mathrm{d}\xi.
\end{aligned}$$

Therefore,

$$\frac{\mathrm{d}I}{\mathrm{d}t}\geq\frac{1}{4}u_x^2\left(q(x_0,t),t\right)-\frac{1}{4}u^2\left(q(x_0,t),t\right)-\frac{3E_0}{8},$$

where we have used the fact

$$\begin{aligned}
&\int_{-\infty}^{q(x_0,t)}e^{\xi}\left(u^2+u_x^2\right)\mathrm{d}\xi\geq\int_{-\infty}^{q(x_0,t)}e^{\xi}du^2=e^{q(x_0,t)}u^2-\int_{-\infty}^{q(x_0,t)}e^{\xi}u^2\mathrm{d}\xi,\\
&\int_{-\infty}^{q(x_0,t)}e^{\xi}\left(u^2+\frac{1}{2}u_x^2\right)\mathrm{d}\xi\geq\frac{1}{2}e^{q(x_0,t)}u^2\left(q(x_0,t),t\right).
\end{aligned}$$

Hence combining these inequalities and our hypothesis together, (3.4) reads

$$\frac{\mathrm{d}I}{\mathrm{d}t}\geq\frac{1}{4}u_x^2\left(q(x_0,t),t\right)-\frac{1}{4}u^2\left(q(x_0,t),t\right)-\frac{3E_0}{8}>0,\quad\text{on }[0,t_0),$$

which implies that I is strictly increasing on time on $[0,t_0)$, we will easily get from the continuity property that

$$e^{-q(x_0,t_0)}\int_{-\infty}^{q(x_0,t_0)}e^{\xi}y(\xi,t_0)\mathrm{d}\xi>e^{-x_0}\int_{-\infty}^{x_0}e^{\xi}y_0(\xi)\mathrm{d}\xi>e^{-x_0}\frac{\sqrt{6E_0}}{2}. \tag{3.5}$$

In the following, Let

$$II=\frac{1}{2}e^{q(x_0,t)}\int_{q(x_0,t)}^{\infty}e^{-\xi}y(\xi,t)\mathrm{d}\xi,$$

a very similar way as I for II, it follows that

$$\frac{\mathrm{d}II}{\mathrm{d}t} \le \frac{1}{4}u^2\left(q(x_0,t),t\right) - \frac{1}{4}u_x^2\left(q(x_0,t),t\right) + \frac{3E_0}{8} < 0, \quad \text{on } [0,t_0).$$

Therefore, continuity property yields

$$e^{q(x_0,t_0)}\int_{q(x_0,t_0)}^{\infty} e^{-\xi}y(\xi,t_0)\mathrm{d}\xi < e^{x_0}\int_{x_0}^{\infty} e^{-\xi}y_0(\xi)\mathrm{d}\xi < -e^{x_0}\frac{\sqrt{6E_0}}{2}. \tag{3.6}$$

Summarizing (3.5) and (3.6), we obtain

$$\begin{aligned} u_x^2\left(q(x_0,t_0),t_0\right) - u^2\left(q(x_0,t_0),t_0\right) &= \left(e^{-q(x_0,t)}\int_{-\infty}^{q(x_0,t_0)} e^{\xi}y(\xi,t_0)\mathrm{d}\xi\right)\left(-e^{q(x_0,t)}\int_{q(x_0,t_0)}^{\infty} e^{-\xi}y(\xi,t_0)\mathrm{d}\xi\right) \\ &> \left(e^{-x_0}\int_{-\infty}^{x_0} e^{\xi}y_0(\xi)\mathrm{d}\xi\right)\left(-e^{x_0}\int_{x_0}^{\infty} e^{-\xi}y_0(\xi)\mathrm{d}\xi\right) \\ &= u_x^2(x_0,0) - u^2(x_0,0) > \frac{3E_0}{2}. \end{aligned}$$

This is an obvious contradiction. Therefore, we have

$$u^2\left(q(x_0,t),t\right) - u_x^2\left(q(x_0,t),t\right) + E_0 < u^2\left(q(x_0,t),t\right) - u_x^2\left(q(x_0,t),t\right) + 3E_0/2 < 0, \quad \text{for all } t \in [0,T).$$

Thus, $u_x\left(q(x_0,t),t\right)$ is strictly decreasing. On the other hand

$$\begin{aligned} u_x(q(x_0,t),t) &= -\frac{1}{2}e^{-q(x_0,t)}\int_{-\infty}^{q(x_0,t)} e^{\xi}y(\xi,t)\mathrm{d}\xi + \frac{1}{2}e^{q(x_0,t)}\int_{q(x_0,t)}^{\infty} e^{-\xi}y(\xi,t)\mathrm{d}\xi \\ &< -\frac{1}{2}e^{-x_0}\int_{-\infty}^{x_0} e^{\xi}y_0(\xi)\mathrm{d}\xi + \frac{1}{2}e^{x_0}\int_{x_0}^{\infty} e^{-\xi}y_0(\xi)\mathrm{d}\xi. \end{aligned}$$

Then the initial assumption makes $u_x\left(q(x_0,t),t\right) < 0$ to be obvious. So our claim is proved.

Now let us denote $W(t) = u_x\left(q(x_0,t),t\right)$ for $t \ge 0$, then

$$\begin{aligned} \frac{\mathrm{d}W(t)}{\mathrm{d}t} &\le \frac{1}{2}u^2\left(q(x_0,t),t\right) - \frac{1}{2}u_x^2\left(q(x_0,t),t\right) + \frac{E_0}{2} \\ &\le \frac{1}{2}\left(u_0^2(x_0) - u_{0x}^2(x_0) + E_0\right) < 0. \end{aligned} \tag{3.7}$$

Suppose the corresponding solution exists globally in time. Since $W(t)$ is strictly decreasing with the initial assumption $W(0) < 0$, there exists a t_1 such that for all $t > t_1$, we have $W(t) < -\sqrt{3E_0} < 0$. Thus, (3.7) becomes

$$\begin{aligned} \frac{\mathrm{d}W(t)}{\mathrm{d}t} &\le -\frac{1}{2}W^2(t) + \frac{1}{2}u^2\left(q(x_0,t),t\right) + \frac{E_0}{2} \\ &\le -\frac{1}{2}W^2(t) + \frac{3E_0}{4} \le -\frac{1}{4}W^2(t), \quad \text{for } t \in (t_1,\infty). \end{aligned}$$

Solving the above inequality directly, one gets

$$W(t) \le \frac{4}{\frac{4}{W(t_1)} + (t - t_1)}.$$

It is easy to observe that $W(t) \to -\infty$ as t goes to $t_1 - \frac{4}{W(t_1)}$. This fact implies that the solution does not exist globally, i.e., wave breaking occurs.

Finally, as for time t_1, we have the following choice. Let us go back to (3.7),

$$\frac{\mathrm{d}W(t)}{\mathrm{d}t} \le -\frac{1}{2}W^2(t) + \frac{3E_0}{2}.$$

If $W(0) < -\sqrt{3E_0}$, we take $t_1 = 0$. Otherwise, suppose $W(0) > -\sqrt{3E_0}$, and $W(t_1) = -\sqrt{3E_0}$ we obtain by integrating (3.7) from 0 to t_1,

$$-\sqrt{3E_0} - W(0) \le \frac{1}{2}\left(u_0^2(x_0) - u_{0x}^2(x_0) + E_0\right)t_1.$$

Consequently, $t_1 \leq \frac{-2\left(\sqrt{3E_0}+W(0)\right)}{u_0^2(x_0)-u_{0x}^2(x_0)+E_0}$. Therefore, we can choose

$$t_1 = \frac{-2\left(\sqrt{3E_0}+W(0)\right)}{u_0^2(x_0)-u_{0x}^2(x_0)+E_0}.$$

This completes the proof of the theorem. □

Throughout this paper, let $\mathbb{S} := \mathbb{R}/\mathbb{Z}$ be the unit circle. Now our attention is drew to show wave breaking may occur for the periodic case. The definition of periodicity is the same to the Camassa–Holm equation; here we ignore its detailed description. We investigate sufficient conditions which guarantee the existence of finite time singularities of strong solutions for the periodic case.

Theorem 3.4. *Suppose $X_0 = (u_0, v_0)^T \in H^s \times H^{s-1}$, $s \geq 5/2$ is the initial data. Suppose that there exists $x_0 \in \mathbb{S}$, such that*

$$u_0'(x_0) < -\sqrt{2}\left((2-C_0)C_1E_0\right)^{1/2},$$

where

$$C_0 = \frac{1}{2} + \frac{\arctan(\sinh(1/2))}{2\sinh(1/2) + 2\arctan(\sinh(1/2))\sinh^2(1/2)},$$
$$C_1 = \frac{\cosh(1/2)}{2\sinh(1/2)}.$$

Then the corresponding solution to system (1.3) with the initial data X_0 blows up in finite time, while the solution itself still remains uniformly bounded in its lifespan.

Proof. Let

$$m(t) := \inf_{x\in\mathbb{S}} u_x(x,t) = u_x(\xi(t),t), \quad t \in [0,T).$$

Clearly, $u_{xx}(\xi(t),t) = 0$, and from (3.2) we find

$$\begin{aligned}\frac{\mathrm{d}m(t)}{\mathrm{d}t} &= -\frac{1}{2}m^2(t) + u^2(\xi(t),t) - \left(G * \left(u^2 + \frac{u_x^2}{2}\right)\right)(\xi(t),t) - \frac{1}{2}\left(G * v^2\right)(\xi(t),t)\\ &\quad + \frac{1}{2}\left(G * v_x^2\right)(\xi(t),t) + \frac{1}{2}v^2(\xi(t),t) - \frac{1}{2}v_x^2(\xi(t),t), \quad a.e.(0,T).\end{aligned}$$

First, we have the Sobolev inequality due to [32]

$$G * \left(u^2 + \frac{u_x^2}{2}\right) \geq C_0 u^2(x),$$

where C_0 is the optimal constant and the function $G(x)$ is given by

$$G(x) = \frac{\cosh(x - [x] - 1/2)}{2\sinh(1/2)}.$$

Furthermore, in [16], Zhou has proved that

$$\|u\|_{L^\infty(\mathbb{S})}^2 \leq C_1\|u\|_{H^1(\mathbb{S})}^2.$$

Combining the above inequalities, we obtain

$$\begin{aligned}\frac{\mathrm{d}m(t)}{\mathrm{d}t} &\leq -\frac{1}{2}m^2(t) + (1-C_0)u^2(\xi(t),t) + \frac{1}{2}\left(G * v_x^2\right)(\xi(t),t) + \frac{1}{2}v^2(\xi(t),t)\\ &\leq -\frac{1}{2}m^2(t) + (1-C_0)C_1\|u\|_{H^1(\mathbb{S})}^2 + \frac{1}{2}\|G\|_{L^\infty(\mathbb{S})}\|v_x\|_{L^1(\mathbb{S})}^2 + \frac{1}{2}\|v\|_{L^\infty(\mathbb{S})}^2\\ &\leq -\frac{1}{2}m^2(t) + (1-C_0)C_1\|u\|_{H^1(\mathbb{S})}^2 + \frac{1}{2}C_1E_0 + \frac{1}{2}C_1E_0\\ &\leq -\frac{1}{2}m^2(t) + \left((2-C_0)C_1\right)E_0,\end{aligned} \tag{3.8}$$

where we have used the fact

$$\frac{1}{2\sinh(1/2)} \leq G(x) \leq \frac{\cosh(1/2)}{2\sinh(1/2)}.$$

Using the notation

$$C_2 = 2\left((2 - C_0)C_1\right)E_0,$$

we have

$$\frac{\mathrm{d}m(t)}{\mathrm{d}t} \le -\frac{1}{2}\left(m^2(t) - C_2\right).$$

In view of the initial condition, it is not difficult to obtain $\frac{\mathrm{d}m(t)}{\mathrm{d}t} \le \frac{\delta-1}{2}m^2(t)$, with $0 < \delta < 1$ determined by $\delta m^2(0) = C_2$. Then, by using standard arguments for this type of equations, it is easy to conclude that the lifespan of the solution is finite. Furthermore, the lifespan T can be estimated as

$$T \le \frac{2}{m(0)(\delta - 1)}.$$

Therefore, we complete the proof. □

Theorem 3.5. *Assume that $X_0 = (u_0, v_0)^T \in H^s \times H^{s-1}$, $s \ge 5/2$ is the initial data. Suppose that there exists $x_0 \in \mathbb{S}$, such that*

$$u_0'(x_0) < -\sqrt{2}\left(\left(\frac{3}{2} + \frac{C_1}{2} - C_0\right)K_0\right)^{1/2},$$

where the constants C_0 and C_1 are the same as above and

$$K_0 = \left(\int_{\mathbb{S}} u_0(x)\mathrm{d}x\right)^2 + E_0.$$

Then the corresponding solution to system (1.3) only exists in finite time, i.e., wave breaking occurs.

Proof. It is easy to know that $\int_{\mathbb{S}} u(x,t)\mathrm{d}x$ and $\int_{\mathbb{S}} v(x,t)\mathrm{d}x$ are also invariants with respect to time t. We apply the known results in [16] to the estimates for $u(x,t)$ and $v(x,t)$. There it was proved that

$$\|f(x)\|^2_{L^\infty(\mathbb{S})} \le \left(\int_{\mathbb{S}} f(x)\mathrm{d}x\right)^2 + \|f(x)\|^2_{H^1(\mathbb{S})},$$

for any function $f \in H^2(\mathbb{S})$. Then we have

$$\begin{aligned}\|u(x,t)\|^2_{L^\infty(\mathbb{S})} &\le \left(\int_{\mathbb{S}} u(x,t)\mathrm{d}x\right)^2 + \|u(x)\|^2_{H^1(\mathbb{S})} \\ &< \left(\int_{\mathbb{S}} u_0(x)\mathrm{d}x\right)^2 + E_0 = K_0 \end{aligned} \tag{3.9}$$

and

$$\|v(x,t)\|^2_{L^\infty(\mathbb{S})} < K_0 \tag{3.10}$$

Therefore, Theorem 3.5 can be easily proved by using (3.9) and (3.10) in the first inequality of (3.8) instead of $\|u(x,t)\|^2_{L^\infty(\mathbb{S})} \le C_1\|u(x,t)\|^2_{H^1(\mathbb{S})} < C_1E_0$. □

Remark 3.6. We applied the different estimates to $u(x,t)$ and $v(x,t)$ in Theorems 3.4 and 3.5. Very similar to [8], we can also take some interesting examples to show the application of the above two theorems, the ultimate purpose is to show that they make sense and are also different. In order to make this paper concision, we are not going to repeat them, the readers who are interested in it are referred to [8] for details.

Recently, Henry [20] proved the infinite propagation speed by establishing a detailed description on the profile of the corresponding solution with compactly supported initial datum for the CH2 system. However, for the MCH2 system, this modification can't allow us to apply the method in [13] to show the propagation speed, and we also do not know whether or not the solutions can exist globally in time. Therefore, these problems are still open and worthy of being investigated in the future.

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