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Global synchronization for directed complex networks

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ABSTRACT

Synchronization for generic directed networks is investigated in this paper. Based on Lyapunov stability theory, a sufficient condition for synchronization of directed networks is derived. Compared with other approaches, where weighted path-lengths are required, or the complex eigenvalues of an asymmetric matrix, Jacobian matrix or Kronecker product need to be calculated, the sufficient condition in this paper is simple and convenient to use. Two typical examples of directed networks with chaotic nodes are finally demonstrated to verify the theoretical results and the effectiveness of the proposed synchronization scheme.

1. Introduce

Complex networks are currently being studied across many fields of science [1,2]. In particular, synchronization of complex networks of dynamical systems has received a great deal of attention from the nonlinear dynamics community. In the past decade, special attention has been focused on the synchronization of chaotic dynamical systems, particularly large-scale and complex networks of chaotic oscillators [3–5]. Recently, synchronization in different small-world and scale-free dynamical network models has also been carefully studied [6–8]. These studies may shed some new light on the synchronization phenomenon in various real-world complex networks. However, in these studies, the networks are recognized as unidirected, i.e. the coupling matrix is symmetric. Recently, I. Belykh, V. Belykh and M. Hasler have presented a method of synchronization in asymmetric coupled networks with "node balance" [9]. The study only considers the situation of "node balance", which means that the sum of the coupling coefficients of all edges directed to a node equals the sum of the coupling coefficients of all edges directed outward from the node.

In this work, we study a more generic situation [10–12]. A sufficient condition for the synchronization of an arbitrary directed network is obtained by using the Lyapunov stability theorem, and this condition is simply given in terms of the network coupling matrix, and therefore is very convenient to use.

2. Mathematical preliminaries

Throughout this paper, the following notations will be used.

(1) R denotes the set of real numbers, $R^n = \underbrace{R \times R \times \cdots \times R}_{n}$.

(2) I_n is the identity matrix of order n.

(3) $A > (\geq)\ 0$ shows that the symmetric matrix A is positive (semi-positive) definite and $A < (\leq)\ 0$ that the symmetric matrix A is negative (semi-negative) definite.

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3. Complex dynamical network model

Consider a dynamical network consisting of N identical nodes with diffusive couplings, in which each node is an n-dimensional dynamical system. The state equations of the network are

$$\dot{x}_i = f(x_i) + c\sum_{j=1}^{N} a_{ij}\Gamma x_j, \quad i = 1, 2, \dots, N \tag{1}$$

where $x_i = (x_{i1}, x_{i2}, \dots, x_{in})^{\mathrm{T}} \in R^n$ represents the state vector of the ith node, $f : R^n \times R^+ \to R^n$ is a smooth nonlinear vector-valued function, the constant $c > 0$ represents the coupling strength, the constant matrix $\Gamma = \mathrm{diag}(r_1, r_2, \dots, r_n)$ is a diagonal matrix with $r_i = 1$ for the ith subsystem and $r_j = 0$ for $j \neq i$, which means that two coupled nodes are linked through their ith state variables. In other words, Γ determines which components are used in the connections. The connectivity matrix A with entries a_{ij} is an $n \times n$ matrix with zero row sums and non-negative off-diagonal elements such that $\sum_{j=1}^{N} a_{ij} = 0$ and $a_{ii} = -\sum_{\substack{j=1\\ j\neq i}}^{N} a_{ij},\ i = 1, 2, \dots, N$. Matrix A may be symmetric or asymmetric, which is defined as a directed network. If there is a connection from node j to node i $(j \neq i)$, then $a_{ij} \neq 0$; otherwise, $a_{ij} = 0$. In this paper, the network is assumed to be *strongly connected*; thus there is at least one entry $a_{ij} \neq 0$ for all rows, and all the diagonal elements $a_{ii} \neq 0,\ i = 1, 2, \dots, N$.

The dynamical network (1) is said to achieve (asymptotical) synchronization if

$$x_1(t) = x_2(t) = \cdots = x_N(t), \quad \text{as } t \to \infty. \tag{2}$$

4. Main theoretical results

An arbitrary connectivity matrix (symmetric or asymmetric) $A = \{a_{ij}\}$ can be decomposed into two $N \times N$ matrices E and Δ:

$$A = E + \Delta \tag{3}$$

where matrix E is symmetric

$$E = \left\{\varepsilon_{ij}\right\} = \begin{cases} \varepsilon_{ij} = \varepsilon_{ji} = \dfrac{1}{2}(a_{ij} + a_{ji}), & \text{for } j \neq i, \\ \varepsilon_{ii} = -\dfrac{1}{2}\displaystyle\sum_{k=1;k\neq i}^{N}(a_{ik} + a_{ki}), & \text{for } j = i \end{cases} \tag{4}$$

and matrix Δ is

$$\Delta = \left\{\delta_{ij}\right\} = \begin{cases} \delta_{ij} = -\delta_{ji} = \dfrac{1}{2}(a_{ij} - a_{ji}), & \text{for } j \neq i, \\ \delta_{ii} = -\dfrac{1}{2}\displaystyle\sum_{k=1;k\neq i}^{N}(a_{ik} - a_{ki}), & \text{for } j = i. \end{cases} \tag{5}$$

Considering only off-diagonal elements, the matrices E and Δ may be thought of as the symmetric and anti-symmetric connectivity matrices, respectively [9]. Both the matrices E and Δ have zero row sums: $\sum_{j=1}^{N} \varepsilon_{ij} = 0$ and $\sum_{j=1}^{N} \delta_{ij} = 0$, respectively. When the connectivity matrix A is symmetric, the matrix $\Delta = 0$.

Using the decomposition (3), we can rewrite Eq. (1) in the form

$$\dot{x}_i = f(x_i) + c\sum_{j=1}^{N} \varepsilon_{ij}\Gamma x_j + c\sum_{j=1}^{N} \delta_{ij}\Gamma x_j, \quad i = 1, 2, \dots, N. \tag{6}$$

Next, let $\bar{x}(t) = (1/N)\sum_{i=1}^{N} x_i(t)$, $e_i(t) = x_i(t) - \bar{x}(t),\ i = 1, 2, \dots, N$, and $e(t) = (e_1, e_2, \dots, e_N)$. Since $\sum_{j=1}^{N} \varepsilon_{jl} = \sum_{j=1}^{N} \varepsilon_{lj} = 0$, one has $\sum_{j=1}^{N}\sum_{l=1}^{N} \varepsilon_{jl}\Gamma x_l = \sum_{l=1}^{N}(\sum_{j=1}^{N} \varepsilon_{jl})\Gamma x_l = 0$. Then the following dynamical error equation can be obtained:

$$\begin{aligned} \dot{e}_i &= f(x_i) - \bar{f} + c\sum_{j=1}^{N} \varepsilon_{ij}\Gamma e_j + c\sum_{j=1}^{N} \delta_{ij}\Gamma e_j - \frac{c}{N}\sum_{j=1}^{N}\sum_{l=1}^{N} \varepsilon_{jl}\Gamma x_l - \frac{c}{N}\sum_{j=1}^{N}\sum_{l=1}^{N} \delta_{jl}\Gamma e_l \\ &= f(x_i) - \bar{f} + c\sum_{j=1}^{N} \varepsilon_{ij}\Gamma e_j + c\sum_{j=1}^{N} \delta_{ij}\Gamma e_j - \frac{c}{N}\sum_{j=1}^{N}\sum_{l=1}^{N} \delta_{jl}\Gamma e_l \end{aligned} \tag{7}$$

where $i = 1, 2, \dots, N$ and $\bar{f} = \frac{1}{N}\sum_{l=1}^{N} f(x_l)$.

If $e_i(t) \to 0$ for all $i = 1, 2, \dots, N$, then objective (2) will be achieved.

Assumption 1. Assume that Assumption 1 in [9] is true; in other words, there exists a critical value $a^* > 0$ such that

$$[f(x_j) - f(x_i)]^{\mathrm{T}} X_{ij} - a^* X_{ij}^{\mathrm{T}} \Gamma X_{ij} \leq 0, \quad \mathrm{X}_{ij} \neq 0 \tag{8}$$

where $X_{ij} = x_j - x_i$.

This assumption is not always true. The article [9] specifies some examples in which (8) is satisfied. It indicates that all double-scrolls, Hindmarsh–Rose neuron models, Lorenz systems and so on are conformable to the condition (8). For example, when the critical value $a^* = [b(b+1)(r+\sigma)^2]/[16(b-1)] - \sigma$, the condition (8) is true for two unidirected x-coupled Lorenz system oscillators.

Lemma 1. *If Assumption 1 is true, then*

$$\sum_{i=1}^{N} [(f(x_i) - \bar{f})^{\mathrm{T}} e_i + e_i^{\mathrm{T}} (f(x_i) - \bar{f})] - 2a \sum_{i=1}^{N} e_i^{\mathrm{T}} \Gamma e_i < 0 \tag{9}$$

when $a > a^*$.

Proof. Since $(f(x_i) - \bar{f})^{\mathrm{T}} e_i$ is a scalar, one has $(f(x_i) - \bar{f})^T e_i = e_i^{\mathrm{T}} (f(x_i) - \bar{f})$. The summation of the series is

$$\sum_{i=1}^{N} \left[(f(x_i) - \bar{f})^{\mathrm{T}} e_i\right] = \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} \left[f(x_i) - f(x_j)\right]^{\mathrm{T}} e_i.$$

Replace the summation index i by j and vice versa, the summation of the series could be written in the following form.

$$\sum_{i=1}^{N} \left[(f(x_i) - \bar{f})^{\mathrm{T}} e_i\right] = \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} \left[f(x_j) - f(x_i)\right]^{\mathrm{T}} e_j.$$

Therefore, we can get

$$\begin{aligned}
\sum_{i=1}^{N} \left[(f(x_i) - \bar{f})^{\mathrm{T}} e_i + e_i^{\mathrm{T}} (f(x_i) - \bar{f})\right] &= \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} \left[f(x_i) - f(x_j)\right]^{\mathrm{T}} e_i + \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} \left[f(x_j) - f(x_i)\right]^{\mathrm{T}} e_j \\
&= \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} \left[f(x_i) - f(x_j)\right]^{\mathrm{T}} (e_i - e_j) \\
&= \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} \left[f(x_i) - f(x_j)\right]^{\mathrm{T}} X_{ij}.
\end{aligned}$$

From (8), we get

$$\sum_{i=1}^{N} \left[(f(x_i) - \bar{f})^{\mathrm{T}} e_i + e_i^{\mathrm{T}} (f(x_i) - \bar{f})\right] \leq \frac{a^*}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} X_{ij}^{\mathrm{T}} \Gamma X_{ij} < \frac{a}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} X_{ij}^{\mathrm{T}} \Gamma X_{ij}. \tag{10}$$

From $X_{ij} = x_j - x_i = e_j - e_i$, we get

$$\frac{a}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} X_{ij}^{\mathrm{T}} \Gamma X_{ij} = 2a \sum_{i=1}^{N} e_i^{\mathrm{T}} \Gamma e_i - \frac{2a}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} e_i^{\mathrm{T}} \Gamma e_j. \tag{11}$$

Since $\sum_{i=1}^{N} e_i = 0$, one has $\sum_{i=1}^{N} \sum_{j=1}^{N} e_i^{\mathrm{T}} \Gamma e_j = 0$. Thus

$$\frac{a}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} X_{ij}^{\mathrm{T}} \Gamma X_{ij} = 2a \sum_{i=1}^{N} e_i^{\mathrm{T}} \Gamma e_i. \tag{12}$$

Therefore

$$\sum_{i=1}^{N} \left[(f(x_i) - \bar{f})^{\mathrm{T}} e_i + e_i^{\mathrm{T}} (f(x_i) - \bar{f})\right] - 2a \sum_{i=1}^{N} e_i^{\mathrm{T}} \Gamma e_i < 0. \tag{13}$$

The proof is completed. □

From Assumption 1 and Lemma 1, we get following theorem.

Theorem 1. *Deposing the coupling matrix into two matrices according to (3) and (4), if the second largest eigenvalue of the symmetric matrix E satisfies*

$$c(\lambda_2(E)+\delta)\leq -a \tag{14}$$

then network (1) synchronizes in the sense of (2) under Assumption 1. Here $\delta=\max_{1\leq i\leq N}\{\delta_{ii}\}$.

Proof. For system (7), choose a Lyapunov function as

$$V=\sum_{i=1}^{N}e_i^{\mathrm{T}}e_i. \tag{15}$$

The time derivative of (10) along the trajectory of system (7) gives

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}t}&=\sum_{i=1}^{N}\left[f(x_i)-\bar f+c\sum_{j=1}^{N}\varepsilon_{ij}\Gamma e_j+c\sum_{j=1}^{N}\delta_{ij}\Gamma e_j-\frac{c}{N}\sum_{j=1}^{N}\sum_{l=1}^{N}\delta_{jl}\Gamma e_l\right]^{\mathrm{T}}e_i\\&\quad+\sum_{i=1}^{N}e_i^{\mathrm{T}}\left[f(x_i)-\bar f+c\sum_{j=1}^{N}\varepsilon_{ij}\Gamma e_j+c\sum_{j=1}^{N}\delta_{ij}\Gamma e_j-\frac{c}{N}\sum_{j=1}^{N}\sum_{l=1}^{N}\delta_{jl}\Gamma e_l\right]\\&=\sum_{i=1}^{N}\left[\left(f(x_i)-\bar f\right)^{\mathrm{T}}e_i+e_i^{\mathrm{T}}\left(f(x_i)-\bar f\right)\right]+c\left[\sum_{i=1}^{N}\sum_{j=1}^{N}\varepsilon_{ij}e_j^{\mathrm{T}}\Gamma e_i+\sum_{i=1}^{N}\sum_{j=1}^{N}\varepsilon_{ji}e_i^{\mathrm{T}}\Gamma e_j\right]\\&\quad+c\left[\sum_{i=1}^{N}\sum_{j=1}^{N}\delta_{ij}e_j^{\mathrm{T}}\Gamma e_i+\sum_{i=1}^{N}\sum_{j=1}^{N}\delta_{ji}e_i^{\mathrm{T}}\Gamma e_j\right]-\frac{c}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{l=1}^{N}\delta_{jl}e_l^{\mathrm{T}}\Gamma e_i-\frac{c}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{l=1}^{N}\delta_{jl}e_i^{\mathrm{T}}\Gamma e_l.\end{aligned}$$

Since $\sum_{i=1}^{N}e_i=0$, we have $\frac{c}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{l=1}^{N}\delta_{jl}e_l^{\mathrm{T}}\Gamma e_i=0$ and $\frac{c}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{l=1}^{N}\delta_{jl}e_i^{\mathrm{T}}\Gamma e_l=0$; therefore

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}t}&=\sum_{i=1}^{N}\left[\left(f(x_i)-\bar f\right)^{\mathrm{T}}e_i+e_i^{\mathrm{T}}\left(f(x_i)-\bar f\right)\right]+c\left[\sum_{i=1}^{N}\sum_{j=1}^{N}\varepsilon_{ij}e_j^{\mathrm{T}}\Gamma e_i+\sum_{i=1}^{N}\sum_{j=1}^{N}\varepsilon_{ji}e_i^{\mathrm{T}}\Gamma e_j\right]\\&\quad+c\left[\sum_{i=1}^{N}\sum_{j=1}^{N}\delta_{ij}e_j^{\mathrm{T}}\Gamma e_i+\sum_{i=1}^{N}\sum_{j=1}^{N}\delta_{ji}e_i^{\mathrm{T}}\Gamma e_j\right].\end{aligned} \tag{16}$$

From (9), we have

$$\frac{\mathrm{d}V}{\mathrm{d}t}<2a\sum_{i=1}^{N}e_i^{\mathrm{T}}\Gamma e_i+c\left[\sum_{i=1}^{N}\sum_{j=1}^{N}\varepsilon_{ij}e_j^{\mathrm{T}}\Gamma e_i+\sum_{i=1}^{N}\sum_{j=1}^{N}\varepsilon_{ji}e_i^{\mathrm{T}}\Gamma e_j\right]+c\left[\sum_{i=1}^{N}\sum_{j=1}^{N}\delta_{ij}e_j^{\mathrm{T}}\Gamma e_i+\sum_{i=1}^{N}\sum_{j=1}^{N}\delta_{ji}e_i^{\mathrm{T}}\Gamma e_j\right]. \tag{17}$$

Consider the second term,

$$S_2=c\left[\sum_{i=1}^{N}\sum_{j=1}^{N}\varepsilon_{ij}e_j^{\mathrm{T}}\Gamma e_i+\sum_{i=1}^{N}\sum_{j=1}^{N}\varepsilon_{ji}e_i^{\mathrm{T}}\Gamma e_j\right]. \tag{18}$$

Replace in the second term of S_2 the summation index i by j and vice versa, using the symmetry of ε, we can rewrite S_2 in the form

$$S_2=2c\sum_{i=1}^{N}\sum_{j=1}^{N}\varepsilon_{ij}e_i^{\mathrm{T}}\Gamma e_j. \tag{19}$$

Consider the third term of (17):

$$S_3=c\left[\sum_{i=1}^{N}\sum_{j=1}^{N}\delta_{ij}e_j^{\mathrm{T}}\Gamma e_i+\sum_{i=1}^{N}\sum_{j=1}^{N}\delta_{ji}e_i^{\mathrm{T}}\Gamma e_j\right]. \tag{20}$$

Since $e_i^{\mathrm{T}}\Gamma e_j=e_j^{\mathrm{T}}\Gamma e_i$, one has $\sum_{i=1}^{N}\sum_{j=1}^{N}\delta_{ij}e_j^{\mathrm{T}}\Gamma e_i=\sum_{i=1}^{N}\sum_{j=1}^{N}\delta_{ij}e_i^{\mathrm{T}}\Gamma e_j$. Using $\delta_{ij}+\delta_{ji}=0$ (for all $i\neq j$), we get

$$S_3=c\sum_{i=1}^{N}\sum_{j=1}^{N}\left(\delta_{ij}+\delta_{ji}\right)e_i^{\mathrm{T}}\Gamma e_j=2c\sum_{i=1}^{N}\delta_{ii}e_i^{\mathrm{T}}\Gamma e_i. \tag{21}$$

Substituting (19) and (21) into (17) gives

$$\frac{\mathrm{d}V}{\mathrm{d}t} < 2a\sum_{i=1}^{N} e_i^{\mathrm{T}}\Gamma e_i + 2c\sum_{i=1}^{N}\sum_{j=1}^{N}\varepsilon_{ij}e_i^{\mathrm{T}}\Gamma e_j + 2c\sum_{i=1}^{N}\delta_{ii}e_i^{\mathrm{T}}\Gamma e_i$$
$$\leq 2a\sum_{k=1}^{n} r_k e^{(k)}\left(e^{(k)}\right)^{\mathrm{T}} + 2c\sum_{k=1}^{n} r_k e^{(k)}E\left(e^{(k)}\right)^{\mathrm{T}} + 2c\delta\sum_{k=1}^{n} r_k e^{(k)}\left(e^{(k)}\right)^{\mathrm{T}} \tag{22}$$

where $e^{(k)} = (e_{k1}, e_{k2}, \dots, e_{kN}) \in R^N$, $k = 1, 2, \dots, n$.

Since $E = E^{\mathrm{T}}$ is a real symmetric matrix, by $\sum_{j=1}^{N}\varepsilon_{ij} = 0$, $\varepsilon_{ii} = -\sum_{j=1;j\neq i}^{N}\varepsilon_{ij}$, $i, j = 1, 2, \dots, N$ and the *Gershgorin circle theorem*, there exists an unitary matrix $U = (u_1, u_2, \dots, u_N)$ such that

$$E = E^{\mathrm{T}} = U\Lambda U^{\mathrm{T}} \tag{23}$$

where $U^{\mathrm{T}}U = I$, $\Lambda = \mathrm{diag}\,(\lambda_1, \lambda_2, \dots, \lambda_N)$, $0 = \lambda_1 > \lambda_2 \geq \cdots \geq \lambda_N$ [13].

Introducing a transformation $y = eU$, we get

$$\frac{\mathrm{d}V}{\mathrm{d}t} < 2a\sum_{k=1}^{n} r_k y^{(k)}\left(y^{(k)}\right)^{\mathrm{T}} + 2c\sum_{k=1}^{n} r_k y^{(k)}\Lambda\left(y^{(k)}\right)^{\mathrm{T}} + 2c\delta\sum_{k=1}^{n} r_k y^{(k)}\left(y^{(k)}\right)^{\mathrm{T}}$$
$$= \sum_{k=1}^{n} r_k\left[2a\sum_{i=1}^{N}\left(y_i^{(k)}\right)^2 + 2c\sum_{i=1}^{N}\lambda_i\left(y_i^{(k)}\right)^2 + 2c\delta\sum_{i=1}^{N}\left(y_i^{(k)}\right)^2\right] \tag{24}$$

where $y\,(t) = (y_1, y_2, \dots, y_N)\,, y_k \in R^n$, r_k is the kth diagonal element of Γ, and $y^{(k)} = (y_{k1}, y_{k2}, \dots, y_{kN}) \in R^N$, $k = 1, 2, \dots, n$.

Note that $\lambda_1 = 0$ is an eigenvalue of the coupling matrix A and its corresponding eigenvector is $u_1 = \left(1/\sqrt{N}, 1/\sqrt{N}, \dots, 1/\sqrt{N}\right)^{\mathrm{T}}$. Since $\sum_{i=1}^{N} e_i = 0$, we have

$$y_1 = eu_1 = 0. \tag{25}$$

Therefore

$$\frac{\mathrm{d}V}{\mathrm{d}t} < \sum_{k=1}^{n}\sum_{i=2}^{N} r_k\left[2a\left(y_i^{(k)}\right)^2 + 2c\lambda_i\left(y_i^{(k)}\right)^2 + 2c\delta\left(y_i^{(k)}\right)^2\right]. \tag{26}$$

From (14), we get $\frac{\mathrm{d}V}{\mathrm{d}t} < 0$. Hence, the solutions of system (7) are globally asymptotically stable. Namely, $\lim_{t\to\infty}\|x_i\,(t) - x_j\,(t)\| = 0$, $i, j = 1, 2, \dots, N$. The proof is thus completed. □

Next consider the case of $\Gamma = I_n$.

Assumption 2. Assume that there exist nonnegative constants γ_{ij}, $i, j = 1, 2, \dots, N$, such that

$$\left\|f\,(x_i) - f\left(x_j\right)\right\| \leq \gamma_{ij}\left\|x_i - x_j\right\|, \quad i \neq j, i, j = 1, 2, \dots, N. \tag{27}$$

This assumption is as same as the Assumption 1 of [13].

Lemma 2. *Let S_1 be the first term of (16), namely*

$$S_1 = \sum_{i=1}^{N}\left[\left(f\,(x_i) - \bar{f}\right)^{\mathrm{T}} e_i + e_i^{\mathrm{T}}\left(f\,(x_i) - \bar{f}\right)\right]; \tag{28}$$

then

$$S_1 \leq 2\gamma\sum_{i=1}^{N} e_i^{\mathrm{T}} e_i \tag{29}$$

where $\gamma = \sum_{i=1}^{N}\gamma_i$, $\gamma_i = \sum_{j=1;j\neq i}^{N}\gamma_{ij}$.

Proof. From Assumption 2, one has

$$
\begin{aligned}
S_1 &\le \frac{2}{N}\sum_{i=1}^{N}\sum_{l=1;l\neq i}^{N}\gamma_{il}\left(\|e_i\|+\|e_l\|\right)\|e_i\| \\
&\le \frac{2}{N}\sum_{i=1}^{N}\left(\sum_{l=1;l\neq i}^{N}\gamma_{il}\right)\left(\|e_i\|+\sum_{j=1;j\neq i}^{N}\|e_j\|\right)\|e_i\| \\
&= \frac{2}{N}\sum_{i=1}^{N}\gamma_i\left(\|e_i\|^2+\sum_{j=1;j\neq i}^{N}\|e_i\|\,\|e_j\|\right) \\
&\le \frac{2\gamma}{N}\left(\sum_{i=1}^{N}\|e_i\|^2+\sum_{i=1}^{N}\sum_{j=1;j\neq i}^{N}\|e_j\|\,\|e_i\|\right) \\
&\le \frac{2\gamma}{N}\sum_{i=1}^{N}\|e_i\|^2+\frac{\gamma}{N}\left(\sum_{i=1}^{N}\sum_{j=1;j\neq i}^{N}\|e_i\|^2+\sum_{i=1}^{N}\sum_{j=1;j\neq i}^{N}\|e_j\|^2\right) \\
&= \frac{2\gamma}{N}\sum_{i=1}^{N}\|e_i\|^2+\frac{2\gamma\,(N-1)}{N}\sum_{i=1}^{N}\|e_i\|^2 \\
&= 2\gamma\sum_{i=1}^{N}e_i^{\mathrm{T}}e_i.
\end{aligned}
\tag{30}
$$

From Lemma 2, we know that Assumption 1 is satisfied if Assumption 2 is satisfied and $a^* \ge 2\gamma$ when $\Gamma = I_n$. Therefore, we easily get the following theorem. □

Theorem 2. *If the condition of (14) is satisfied with*

$$a > 2\gamma, \tag{31}$$

then network (1) synchronizes globally in the sense of (2) under Assumption 2 when $\Gamma = I_n$.

Remark. (1) The theorems can be regarded as improvements of those in [13]. Actually, if the coupling matrix A is symmetric, then the matrix $\Delta = 0$. Therefore the condition (14) becomes $c\lambda_2 \le -a$, which is essentially the same as that of a undirected dynamical network. It should be noticed that the value of a in [13] must be selected as $a > \frac{4\gamma(N-1)}{N}$. From (31), we know the condition of the presented approach is more relaxed.

(2) If the network has the property of "node balance", i.e. all nodes in the network have equal input and output weight sums, $\delta = 0$. Then the condition (14) becomes $c\lambda_2 \le -a$, which is the same as that of a symmetric network.

(3) From (14) and $c > 0$, we have

$$\lambda_2(E) + \delta < 0. \tag{32}$$

In other words, if only the condition (32) is satisfied, then there always exists a positive constant $c < +\infty$ such that the condition (14) is satisfied.

5. Numerical simulation and discussion

To demonstrate the theoretical results obtained above, a chaotic Chen system is used as nodes of network [14]. A single Chen system is described by [13,14]

$$
\begin{aligned}
\dot{x}_1 &= \sigma\,(x_2 - x_1) \\
\dot{x}_2 &= (r-\sigma)\,x_1 + rx_2 - x_1x_3 \\
\dot{x}_3 &= -bx_3 + x_1x_2
\end{aligned}
\tag{33}
$$

which is chaotic when the parameters $\sigma = 35$, $b = 3$, $r = 28$. Furthermore, when $\Gamma = I_n$, the state equations of the entire network are given by

$$
\begin{pmatrix}\dot{x}_{i1}\\ \dot{x}_{i2}\\ \dot{x}_{i3}\end{pmatrix} = \begin{pmatrix}\sigma\,(x_{i2}-x_{i1})\\ (r-\sigma)\,x_{i1}+rx_{i2}-x_{i1}x_{i3}\\ -bx_{i3}+x_{i1}x_{i2}\end{pmatrix} + c\sum_{j=1}^{N}a_{ij}x_j, \quad i = 1, 2, \ldots, N.
\tag{34}
$$

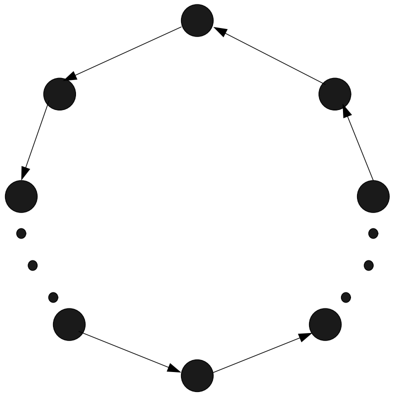

Fig. 1. The unidirectional circular coupling network.

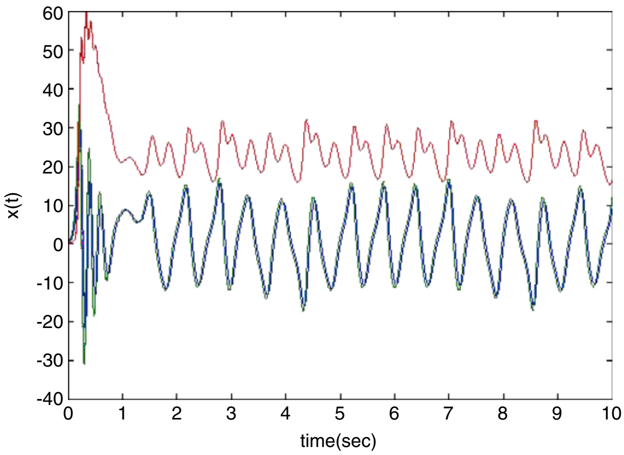

Fig. 2. States of network (34): $x_i(t),\ i = 1, 2, \ldots, 50$, with unidirectional circular coupling.

5.1. Unidirectional circular network

Consider the unidirectional coupled ring of dynamical systems (Fig. 1). At each node of the graph, one edge enters and one leaves. Thus, the network has the property of "node balance". The coupling matrix is

$$A = \begin{pmatrix} -1 & 0 & 0 & \cdots & 1 \\ 1 & -1 & 0 & \cdots & 0 \\ \vdots & & & \cdots & \\ 0 & & & \cdots & 1 \\ 0 & 0 & \cdots & 1 & -1 \end{pmatrix}. \tag{35}$$

According to (4) and (5), we have

$$E = \begin{pmatrix} -1 & 0.5 & 0 & \cdots & 0.5 \\ 0.5 & -1 & 0.5 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & & \cdots & 0.5 \\ 0.5 & 0 & \cdots & 0.5 & -1 \end{pmatrix}, \quad \text{and} \quad \Delta = \begin{pmatrix} 0 & -0.5 & 0 & \cdots & 0.5 \\ 0.5 & 0 & -0.5 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & & \cdots & -0.5 \\ -0.5 & 0 & \cdots & 0.5 & 0 \end{pmatrix}. \tag{36}$$

The symmetric matrix E is similar to the coupling matrix of a bidirectional or undirected nearest neighbor network. Its eigenvalues are $0 = \lambda_1 > \lambda_i = -2\sin^2(\frac{i\pi}{N}),\ i = 1, 2, \ldots, N$, so $\lambda_2(E) < 0$ is always satisfied when the number of the nodes $N < +\infty$. From (36) (or Remark 2), one has $\delta = 0$. Therefore the condition (32) is always satisfied and the condition (14) becomes $c > -a/\lambda_2(E)$. This condition is very simple and is equivalent to that of an undirected circular network [9,13]. In the simulation, we choose $N = 50$ and $\gamma = 40$. The simulation results are shown in Figs. 2 and 3.

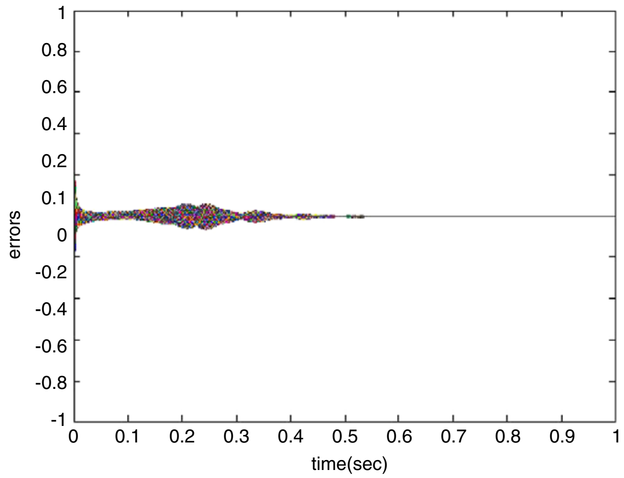

Fig. 3. Synchronization errors between node i and the node $i+1$: $x_i(t) - x_{i+1}(t)$, $i = 1, 2, \dots, 49$, with unidirectional circular coupling.

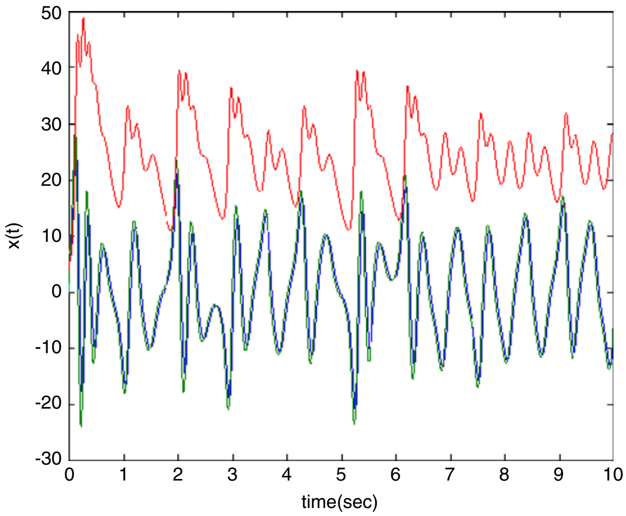

Fig. 4. States of network (34): $x_i(t)$, $i = 1, 2, \dots, 50$, with irregular directed coupling.

5.2. Irregular directed network

Consider the directed network with irregular coupling of N nodes. In the simulation, we still choose $N = 50$ and $\gamma = 40$. The coupling of the nodes is stochastic, and the probability of coupling is 0.3: i.e. for arbitrary nodes i and j ($j \neq i$), if $rand \leq 0.3$, then $a_{ij} = 1$; otherwise $a_{ij} = 0$. After decomposing the coupling matrix according to (4) and (5), we get $\lambda_2(E) = -11.2988$, $\delta = 5.5$. Then (32) is satisfied and the sufficient condition (14) becomes $c > a/5.7988 \approx 27$. It is obvious that the estimation of the synchronization condition is very simple. The simulation results are shown in Figs. 4 and 5 with $c = 30$.

If we use connection graph stability [10] to estimate the synchronization condition, the path-lengths of every two nodes would be required. However, calculation of weighted path-lengths is quite a laborious task for networks with large complicated coupling schemes. The methods of articles [11,12] are a little simpler, but complex eigenvalues of the asymmetric matrix [11], Jacobian matrix and Kronecker product [12] are required, which are more complicated than using the method in this paper.

6. Conclusion

This paper gives a sufficient condition for global synchronization in an arbitrary network. Differing from most studies where the coupling matrix of the network is symmetric, this paper studies the synchronization of an arbitrary directed network where the coupling matrix may be asymmetric. Based on Lyapunov stability theory, we have extended the method of a symmetric network in [13] to a directed one. In other words, the method of [13] is only a special case of this paper.

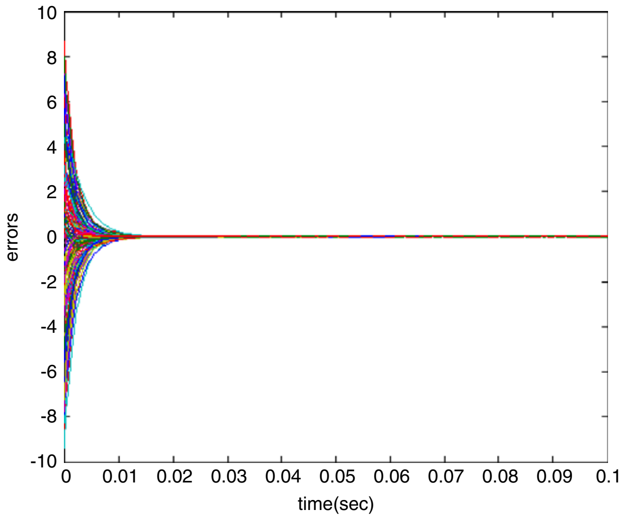

Fig. 5. Synchronization errors between node i and node $i+1$: $x_i(t) - x_{i+1}(t)$, $i = 1, 2, \ldots, 49$, with irregular directed coupling.

The method is very simple and fast, since the most laborious task is nothing more than calculating the eigenvalues of an asymmetric matrix, unlike other approaches, where the calculation of weighted path-lengths is required, or complex eigenvalues of the asymmetric, Jacobian matrix and Kronecker product have to be calculated. Finally, two typical examples of a directed network with chaotic nodes were simulated, to verify the theoretical results.

In fact, the condition of global synchronization in this paper is only a sufficient condition. That dissatisfaction of the condition does not mean the network cannot synchronize. In this paper, the condition (32) is the precondition of estimating the synchronization condition. Actually, when the asymmetric degree of a single node is high, for example, perhaps δ is sequentially large so that (32) cannot be satisfied, the synchronization of the network could not be judged. Whether the condition could be relaxed is our future research.

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